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# QUANTUM TANAKA FORMULA IN TERMS OF QUANTUM BROWNIAN MOTION

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**Abstract**

A quantum local time, which is a generalized operator-valued process, is defined for quantum Brownian motion, and a quantum analogue of the classical Tanaka formula is then established.

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## 1. Introduction

In classical stochastic analysis, we have the following Tanaka formula for Brownian motion $\{B_t, t \geq 0\}$:

$$|B_t - a| = |a| + \int_0^t \operatorname{sgn}(B_s - a)\, dB_s + \int_0^t \delta_a(B_s)\, ds \quad t > 0, a \in \mathbb{R} \tag{1.1}$$

where $\int_0^t \delta_a(B_s)\, ds$ is known as the local time of $\{B_t, t \geq 0\}$ at $a \in \mathbb{R}$. It is proved that the process

$$\left\{ L_t^a = \int_0^t \delta_a(B_s)\, ds, t > 0, a \in \mathbb{R} \right\}$$

is the density of the measure $A \mapsto \int_0^t 1_A(B_s)\, ds$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and a continuous Markovian process with respect to $(t, a)$ on $\mathbb{R}_+ \times \mathbb{R}$; moreover, Chung and Williams [2] applied local time to study the Brownian motion with a rejection wall. Furthermore, the Itô formula was extended to convex functions applying local time.

On the other hand, white noise analysis initiated by Hida [3] is essentially an infinite-dimensional analogue of Schwartz's distribution theory, which has important

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applications in many fields, including stochastic analysis and quantum physics [7, 9, 10]. The mathematical framework of white noise analysis is the Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*$$

over $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$, where $(E)$ and $(E)^*$ are known as Hida's testing and generalized functional spaces, respectively. Let $\mathcal{L} = \mathcal{L}[(E), (E)^*]$ be the space of continuous linear operators from $(E)$ to $(E)^*$. Elements of $\mathcal{L}$ are called generalized operators (GOs), which are significant generalizations of bounded operators on Hilbert space $(L^2)$.

Quantum Brownian motion, which is an observable of Schwartz class and $\delta$-composable [12, 14], is a very important operator-valued process in white noise analysis. The quantum Itô formula holds only for the product of two quantum semi-martingales, hence it is meaningful to investigate the quantum local time and quantum Tanaka formula of quantum Brownian motion. As we will see, the quantum Tanaka formula has a similar form to the classical one.

The paper is organized as follows. In Section 2 we recall some notions, notation and lemmas in white noise analysis. In Section 3 the quantum local time of $\{Q_t \mid t > 0\}$ is defined, which is the density of a generalized operator-valued measure (GOVM) from $\mathcal{B}(\mathbb{R})$ to $\mathcal{L}$, and we therefore name it the *quantum local time*. In the final section, the Tanaka formula is established.

## 2. Preliminaries

In this section, we briefly recall some notions, notation and facts in white noise analysis. For details, see [1, 4–6, 8, 11, 13, 15, 16].

Let $H = L^2(\mathbb{R}, dt; \mathbb{R})$ be the Hilbert space of real-valued square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and inner product $\langle\cdot, \cdot\rangle$. The space $E = S(\mathbb{R})$ (respectively $E^* = S^*(\mathbb{R})$) is the space of Schwartz rapidly decreasing (respectively generalized) functionals. $E$ is a nuclear space and we have a Gel'fand triple $E \subset H \subset E^*$. The canonical bilinear form on $E^* \times E$ denoted by $\langle\cdot, \cdot\rangle$ coincides with the inner product of $H$.

Let $\mu$ be the standard Gaussian measure on $E^*$ and let $(E^*, \mu)$ be the *white noise space*. By the method of second quantization, we have a complex Gel'fand triple,

$$(E) \subset (L^2) \subset (E)^*,$$

which is known as the *canonical framework of white noise analysis* and $(L^2) = L^2(E^*, \mu; \mathbb{C})$. The canonical bilinear form on $(E) \times (E)^*$ is denoted by $\langle\!\langle\cdot, \cdot\rangle\!\rangle$.

DEFINITION 2.1. For $x \in E^*_{\mathbb{C}}$, the Wick product of $x$ is defined inductively as follows:

$$\begin{aligned} &{:}x^{\otimes 0}{:} = 1, \\ &{:}x^{\otimes 1}{:} = x, \\ &{:}x^{\otimes n}{:} = x \mathbin{\hat{\otimes}} {:}x^{\otimes n-1}{:} - (n-1)\tau \mathbin{\hat{\otimes}} {:}x^{\otimes n-2}{:} \quad \text{for } n \geq 2 \end{aligned}$$

where $\tau \in E^* \mathbin{\hat{\otimes}} E^*$ satisfies $\langle\tau, f \otimes g\rangle = \langle f, g\rangle$, $f, g \in E$.

For $\xi \in E_{\mathbb{C}}$, the exponential functional $\phi_\xi$ associated with $\xi$ is defined as

$$\phi_\xi(x) = e^{\langle x,\xi\rangle - \langle \xi,\xi\rangle/2} = \sum_{n=0}^{+\infty} \frac{1}{n!}\langle :x^{\otimes n}:, \xi^{\otimes n}\rangle, \qquad x \in E_{\mathbb{C}}^*.$$

The set $\{\phi_\xi, \xi \in E_{\mathbb{C}}^*\}$ is total in $(E)$. For $T \in \mathcal{L}[(E), (E)^*]$, its *symbol* $\hat{T} : E_{\mathbb{C}}^* \times E_{\mathbb{C}}^* \to \mathbb{C}$ is defined as

$$\hat{T}(\xi, \eta) = \langle\langle T\phi_\xi, \phi_\eta\rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

The self-adjoint operator from $(L^2)$ to $(L^2)$ is called an *observable*.

LEMMA 2.2. *Suppose that* $\{X(t), t \geq 0\} \subset \mathcal{L}$, $M \in \mathcal{B}(\mathbb{R}_+)$. *If* $\{X(t), t \geq 0\}$ *satisfies the conditions:*

(1) *for any* $\xi, \eta \in E_{\mathbb{C}}$, $\widehat{X(\cdot)}(\xi, \eta) : M \mapsto \mathbb{C}$ *is measurable;*
(2) *there exist a constant* $K$, $p \geq 0$, *and a nonnegative measurable function* $c(t)$ *in* $\mathbb{R}_+$ *integrable in* $M$, *such that*

$$|\widehat{X(t)}(\xi, \eta)| \leq c(t)\exp\{K(|\xi|_p^2 + |\eta|_p^2)\} \quad \xi, \eta \in E_{\mathbb{C}},$$

*then for any* $l, m \in \mathbb{N}$, *the integral of quantum white noise of* $\{X(t)\}$ *in* $M$ *with respect to* $W_{l,m}(dt)$ *exists, denoted by*

$$\int_M X(t)\, dW_{l,m}(t) = \int_M X(t) \diamond \partial_t^{*l}\partial_t^m\, dt \in \mathcal{L}$$

*and*

$$\int_M \widehat{X(t)\, dW_{l,m}}(t)(\xi, \eta) = \int_M \widehat{X(t)}(\xi, \eta)\xi^l(t)\eta^m(t)\, dt \quad \xi, \eta \in E_{\mathbb{C}}.$$

REMARK 2.3. $:W_{l,m}(dt) = \partial_t^{*l}\partial_t^m\, dt$ is the quantum white noise measure, where $\partial_t, \partial_t^*$ are the *annihilation and creation* operators.

DEFINITION 2.4. An observable $T$ in $(L^2)$ is called an *observable of Schwartz class* if, for any $\xi, \eta \in E$, there exists a function $\rho_{\xi,\eta}^T \in E_{\mathbb{C}}$ such that

$$\langle\langle P_T(S)\phi_\xi, \phi_\eta\rangle\rangle = \int_S \rho_{\xi,\eta}^T(\lambda)\, d\lambda, \qquad S \in \mathcal{B}(\mathbb{R}),$$

where $P_T$ is the spectral measure of $T$. The function $\rho_{\xi,\eta}^T$ is called the *spectral density* of $T$ corresponding to $\xi, \eta$.

LEMMA 2.5. *Let* $T$ *be an observable of Schwartz class with spectral density* $\rho_{\xi,\eta}^T$. *Then for any bounded Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$, *we have* $f(T) \in \mathcal{L}[(L^2)]$ *and*

$$\widehat{f(T)}(\xi, \eta) = \langle f, \rho_{\xi,\eta}^T\rangle.$$

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PROPOSITION 2.6. *Let $T$ be an observable of Schwartz class with spectral density $\rho^T_{\xi,\eta}$, and let $\Phi_T(\xi, \eta) = \rho^T_{\xi,\eta}(0)$ for $\xi, \eta \in E_{\mathbb{C}}$. Then $T$ is $\delta$-composable if and only if $\Phi_T$ satisfies the following two conditions:*

(1) *for any $\xi, \xi', \eta, \eta' \in E$, $(s, t) \mapsto \Phi_T(\xi + s\xi', \eta + t\eta')$, $s, t \in \mathbb{R}$, has an entire analytic extension to $\mathbb{C} \times \mathbb{C}$;*

(2) *there exist constants $C, K, p \geq 0$ such that*

$$|\Phi_T(\xi, \eta)| \leq C \exp\{K(|\xi|_p^2 + |\eta|_p^2)\} \quad \xi, \eta \in E_{\mathbb{C}}.$$

*In this case, $\widehat{\delta(T)}(\xi, \eta) = \rho^T_{\xi,\eta}(0)$.*

Let $(\xi, \eta)_0 = \langle \xi, \bar{\eta} \rangle$ for $\xi, \eta \in H_{\mathbb{C}}$. The Weyl operator $W(u)$ on $(L^2)$ is defined as

$$W(u)\phi_\xi = \exp\left\{-(\xi, u)_0 - \frac{|u|_0^2}{2}\right\}\phi_{\xi+u}.$$

Obviously $W(u)$ is a unitary operator on $(L^2)$. For $t > 0$, $u_t = 1_{[0,t]} \in H_{\mathbb{C}}$, let $U_s = W(isu_t)$, $s \in \mathbb{R}$. Then $\{U_s \mid s \in \mathbb{R}\}$ forms a strongly continuous semigroup of unitary operators. By Stone's theorem, there exists a self-adjoint operator $Q_t$ such that $W(isu_t) = e^{isQ_t}$. Then $\{Q_t \mid t > 0\}$ is called the quantum Brownian motion, $dQ_t = (\partial_t + \partial_t^*)\, dt$ and $dO_t \cdot dQ_t = dt$.

## 3. Quantum local time

In this section, we define the quantum local time of $\{Q_t \mid t > 0\}$ and show that it is the density of a measure on the Borel $\sigma$-field $\mathcal{B}(\mathbb{R})$. Let $\delta$ be the Dirac $\delta$-function; then $\delta$ is a Schwartz generalized function.

As the self-adjoint operator, $Q_t$ has the spectral decomposition

$$Q_t = \int_{\mathbb{R}} \lambda P_t\,(d\lambda),$$

where $P_t$ is the orthogonal projective operator on $(L^2)$ corresponding to $Q_t$ and $U_s = \int_{\mathbb{R}} e^{is\lambda} P_t(d\lambda)$, $s \in \mathbb{R}$.

PROPOSITION 3.1. *For any $t \geq 0$, $Q_t$ is an observable of Schwartz class with spectral density*

$$\rho^t_{\xi,\eta}(\lambda) = \frac{1}{\sqrt{2\pi t}} \exp\left\{(\xi, \eta)_0 - \frac{1}{2t}(\lambda - (u_t, \xi + \eta)_0)^2\right\}.$$

*Moreover, $\int_0^t \delta(Q_s)\, ds \in \mathcal{L}$ and for any $\xi, \eta \in E_{\mathbb{C}}$,*

$$\int_0^t \widehat{\delta(Q_s)}\, ds(\xi, \eta) = \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi, \eta)_0 - \frac{1}{2s}(u_s, \xi + \eta)_0^2\right\} ds.$$

PROOF. Let $\xi, \eta \in E_{\mathbb{C}}$. Then $\langle\langle P_t(\cdot)\phi_\xi, \phi_\eta\rangle\rangle : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ is a Borel measure. For $s \in \mathbb{R}$,

$$\begin{aligned}\int_R e^{is\lambda}\langle\langle P_t(d\lambda)\phi_\xi, \phi_\eta\rangle\rangle &= \left\langle\!\!\left\langle \int_R e^{is\lambda} P_t(d\lambda)\phi_\xi, \phi_\eta \right\rangle\!\!\right\rangle = \langle\langle W(isu_t)\phi_\xi, \phi_\eta\rangle\rangle \\ &= \exp\left\{-(\xi, isu_t)_0 - \frac{|isu_t|_0^2}{2}\right\}\langle\langle \phi_{\xi+isu_t}, \phi_\eta\rangle\rangle \\ &= \exp\left\{-(\xi+\eta, isu_t)_0 - \frac{s^2 t}{2} + (\xi, \eta)_0\right\} \\ &= \int_R e^{is\lambda}\frac{1}{\sqrt{2\pi t}}\exp\left\{(\xi, \eta)_0 - \frac{(\lambda - (u_t, \xi+\eta)_0^2)}{2t}\right\} d\lambda\end{aligned}$$

which implies

$$\langle\langle P_t(d\lambda)\phi_\xi, \phi_\eta\rangle\rangle = \frac{1}{\sqrt{2\pi t}}\exp\left\{(\xi, \eta)_0 - \frac{(\lambda - (u_t, \xi+\eta)_0^2)}{2t}\right\} d\lambda,$$

thus, $Q_t$ is an observable of Schwartz class with spectral density

$$\rho^t_{\xi,\eta}(\lambda) = \frac{1}{\sqrt{2\pi t}}\exp\left\{(\xi, \eta)_0 - \frac{1}{2t}(\lambda - (u_t, \xi+\eta)_0)^2\right\}.$$

For $\xi, \xi', \eta, \eta' \in E$, the function

$$\begin{aligned}(s, r) &\mapsto \rho^t_{\xi+s\xi', \eta+r\eta'}(\lambda) \\ &= \frac{1}{\sqrt{2\pi t}}\exp\left\{(\xi+s\xi', \eta+r\eta')_0 - \frac{1}{2t}(\lambda - (u_t, \xi+s\xi'+\eta+r\eta')_0)^2\right\}\end{aligned}$$

has an entire analytic extension to $\mathbb{C} \times \mathbb{C}$ and

$$|\rho^t_{\xi,\eta}(0)| \le \frac{1}{\sqrt{2\pi t}}\exp\left\{\frac{3}{2}(|\xi|_p^2 + |\eta|_p^2)\right\}.$$

Hence by Proposition 2.6, for each $t > 0$, $Q_t$ is $\delta$-composable, and $\delta(Q_t) \in \mathcal{L}$,

$$\widehat{\delta(Q_t)}(\xi, \eta) = \langle \delta, \rho^t_{\xi,\eta}\rangle = \rho^t_{\xi,\eta}(0) = \frac{1}{\sqrt{2\pi t}}\exp\left\{(\xi, \eta)_0 - \frac{1}{2}(u_t, \xi+\eta)_0^2\right\}.$$

By Lemma 2.2, $\int_0^t \delta(Q_s)\, ds \in \mathcal{L}$ with $c(s) = 1/\sqrt{2\pi s}$, $K = 1/2$ and

$$\begin{aligned}\int_0^t \widehat{\delta(Q_s)\, ds}(\xi, \eta) &= \int_0^t \widehat{\delta(Q_s)}(\xi, \eta)\, ds \\ &= \int_0^t \frac{1}{\sqrt{2\pi s}}\exp\left\{(\xi, \eta)_0 - \frac{1}{2}(u_s, \xi+\eta)_0^2\right\} ds.\end{aligned}$$

This concludes the proof. □

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COROLLARY 3.2. *For any $a \in \mathbb{R}$ and $t > 0$, we have $\int_0^t \delta_a(Q_s)\,ds \in \mathcal{L}$, and for any $\xi, \eta \in E_{\mathbb{C}}$,*

$$\int_0^t \widehat{\delta_a(Q_s)}\,ds(\xi,\eta) = \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{1}{2s}(a - (u_s, \xi+\eta)_0)^2\Big\}\,ds.$$

Next, we define a *generalized operator valued measure* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with GO-valued density.

THEOREM 3.3. *For $B \in \mathcal{B}(\mathbb{R})$, $t > 0$, $\int_0^t 1_B(Q_s)\,ds \in \mathcal{L}$ and the map $B \mapsto \int_0^t 1_B(Q_s)\,ds \in \mathcal{L}$ from $\mathcal{B}(\mathbb{R})$ to $\mathcal{L}$ defines a GOVM with density $\int_0^t \delta_a(Q_s)\,ds$.*

PROOF. For any $B \in \mathcal{B}(\mathbb{R})$, $1_B$ is a bounded Borel function, by Lemma 2.5, $1_B(Q_s) \in \mathcal{L}$ and for $\xi, \eta \in E_{\mathbb{C}}$,

$$\widehat{1_B(Q_s)}(\xi,\eta) = \int_B \rho^s_{\xi,\eta}(\lambda)\,d\lambda = \int_B \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\Big\}\,d\lambda$$

which satisfies the conditions in Lemma 2.2, hence $\int_0^t 1_B(Q_s)\,ds \in \mathcal{L}$ and

$$\begin{aligned}
\int_0^t \widehat{1_B(Q_s)}\,ds(\xi,\eta) &= \int_0^t \widehat{1_B(Q_s)}(\xi,\eta)\,ds \\
&= \int_0^t \int_B \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\Big\}\,d\lambda\,ds \\
&= \int_B \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\Big\}\,ds\,d\lambda.
\end{aligned}$$

It is easy to see that the complex-valued function $B \mapsto \int_0^t \widehat{1_B(Q_s)}\,ds(\xi,\eta)$ is $\sigma$-additive on $\mathcal{B}(\mathbb{R})$. In fact, for any $\{A_n\} \subset \mathcal{B}(\mathbb{R})$, $A_n \cap A_m = \emptyset$ with $n \neq m$,

$$\begin{aligned}
&\int_0^t \widehat{1_{\Sigma_n A_n}(Q_s)}\,ds(\xi,\eta) \\
&\quad= \int_0^t \widehat{1_{\Sigma_n A_n}(Q_s)}(\xi,\eta)\,ds = \int_0^t \int_{\Sigma_n A_n} \rho^s_{\xi,\eta}(\lambda)\,d\lambda\,ds \\
&\quad= \int_0^t \int_{\Sigma_n A_n} \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\Big\}\,d\lambda\,ds \\
&\quad= \Sigma_n \int_0^t \int_{A_n} \frac{1}{\sqrt{2\pi s}} \exp\Big\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\Big\}\,d\lambda\,ds \\
&\quad= \Sigma_n \int_0^t \widehat{1_{A_n}(Q_s)}(\xi,\eta)\,ds = \Sigma_n \int_0^t \widehat{1_{A_n}(Q_s)}\,ds(\xi,\eta)
\end{aligned}$$

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which shows that $\int_0^t 1_{(\cdot)}(Q_s)\,ds$ is a GOVM on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and

$$\begin{aligned}
\int_0^t \widehat{1_B(Q_s)}\,ds(\xi,\eta) &= \int_0^t \widehat{1_B(Q_s)}(\xi,\eta)\,ds \\
&= \int_0^t \int_B \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi,\eta)_0 - \frac{(\lambda-(u_s,\xi+\eta)_0)^2}{2s}\right\} d\lambda\,ds \\
&= \int_B \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi,\eta)_0 - \frac{(\lambda-(u_s,\xi+\eta)_0)^2}{2s}\right\} ds\,d\lambda \\
&= \int_B \int_0^t \widehat{\delta_\lambda(Q_s)}(\xi,\eta)\,ds\,d\lambda = \int_B \int_0^t \widehat{\delta_\lambda(Q_s)}\,ds\,d\lambda(\xi,\eta),
\end{aligned}$$

hence

$$\int_0^t 1_B(Q_s)\,ds = \int_B \int_0^t \delta_\lambda(Q_s)\,ds\,d\lambda$$

and the GOVM $B \mapsto \int_0^t 1_B(Q_s)\,ds$ has density $\int_0^t \delta_\lambda(Q_s)\,ds \in \mathcal{L}$. □

DEFINITION 3.4. For $t > 0$ and $a \in \mathbb{R}$ the generalized operator $\int_0^t \delta_a(Q_s)\,ds$ is called the quantum local time of quantum Brownian motion $\{Q_t, t \geq 0\}$ at $a \in \mathbb{R}$, denoted by $L_t^a$. The GO-valued process $\{L_t^a \mid t > 0, a \in \mathbb{R}\}$ is called the *quantum local time process*.

Next, we will interpret the meaning of the quantum local time of $L_t^a$. For any $t \geq 0$, $a \in \mathbb{R}$ and $\xi, \eta \in E_{\mathbb{C}}$,

$$\begin{aligned}
\widehat{L_t^a}(\xi,\eta) &= \int_0^t \widehat{\delta_a(Q_s)}\,ds(\xi,\eta) \\
&= \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi,\eta)_0 - \frac{(a-(u_s,\xi+\eta)_0)^2}{2s}\right\} ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi,\eta)_0 - \frac{(\lambda-(u_s,\xi+\eta)_0)^2}{2s}\right\} d\lambda\,ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} \widehat{\delta_\lambda(Q_s)}(\xi,\eta)\,d\lambda\,ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} \langle\!\langle \delta_\lambda(Q_s)\phi_\xi, \phi_\eta \rangle\!\rangle\,d\lambda\,ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \left\langle\!\!\left\langle \int_{a-\epsilon}^{a+\epsilon} \delta_\lambda(Q_s)\,d\lambda\phi_\xi, \phi_\eta \right\rangle\!\!\right\rangle ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} \widehat{\delta_\lambda(Q_s)}\,d\lambda(\xi,\eta)\,ds \\
&= \int_0^t \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \widehat{1_{(a-\epsilon,a+\epsilon)}(Q_s)}\,ds(\xi,\eta).
\end{aligned}$$

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In the sense of GO symbols,

$$L_t^a = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t 1_{(a-\epsilon,a+\epsilon)}(Q_s)\,ds$$

and

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t 1_{(a-\epsilon,a+\epsilon)}(Q_s)\,ds = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}L\{s;\ s\in[0,t],\ Q_s=\lambda I,\ \lambda\in(a-\epsilon,a+\epsilon)\}$$

where $L$ denotes the Lebesgue measure on $\mathbb{R}$.

## 4. Quantum Tanaka formula

In this section the quantum Tanaka formula for the quantum Brownian motion $\{Q_t \mid t>0\}$ is established; its form is similar to the classical case. In this part all quantities are to be interpreted in the sense of GO symbols.

PROPOSITION 4.1. *For $t\geq 0$, $\int_0^t \mathrm{sgn}(Q_s)(\partial_s+\partial_s^*)\,ds\in\mathcal{L}$ and for any $\xi,\eta\in E_{\mathbb{C}}$,*

$$\begin{aligned}\int_0^t \widehat{\mathrm{sgn}(Q_s)(\partial_s+\partial_s^*)\,ds}(\xi,\eta) &= \int_0^t(\xi(s)+\eta(s))\int_{\mathbb{R}}\mathrm{sgn}(\lambda)\frac{1}{\sqrt{2\pi s}} \\ &\quad\times\exp\Big\{(\xi,\eta)_0-\frac{(\lambda-(u_s,\xi+\eta)_0)^2}{2s}\Big\}\,d\lambda\,ds.\end{aligned}\tag{4.1}$$

PROOF. $\mathrm{sgn}(\cdot)$ is bounded on $\mathbb{R}$, by Lemma 2.5, $\mathrm{sgn}(Q_s)\in\mathcal{L}$ and for any $\xi,\eta\in E_{\mathbb{C}}$,

$$\begin{aligned}\widehat{\mathrm{sgn}(Q_s)}(\xi,\eta) &= \int_{\mathbb{R}}\mathrm{sgn}(\lambda)\rho^s_{\xi,\eta}(\lambda)\,d\lambda \\ &= \int_{\mathbb{R}}\mathrm{sgn}(\lambda)\frac{1}{\sqrt{2\pi s}}\exp\Big\{(\xi,\eta)_0-\frac{(\lambda-(u_s,\xi+\eta)_0)^2}{2s}\Big\}\,d\lambda.\end{aligned}\tag{4.2}$$

It is measurable in $s$ and

$$|\widehat{\mathrm{sgn}(Q_s)}(\xi,\eta)|\leq\exp\{\tfrac{1}{2}(|\xi|_0^2+|\eta|_0^2)\};$$

by Lemma 2.2, $\mathrm{sgn}(Q_s)$ is integrable in $(0,t]$ with respect to $W_{0,1}(ds)+W_{1,0}(ds)$. Also

$$\begin{aligned}\int_0^t \widehat{\mathrm{sgn}(Q_s)(\partial_s+\partial_s^*)\,ds}(\xi,\eta) &= \int_0^t \mathrm{sgn}(Q_s)\widehat{(\partial_s+\partial_s^*)}\,ds(\xi,\eta) \\ &= \int_0^t \widehat{\mathrm{sgn}(Q_s)}\partial_s\,ds(\xi,\eta)+\int_0^t \widehat{\mathrm{sgn}(Q_s)}\partial_s^*\,ds(\xi,\eta)\end{aligned}$$

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$$= \int_0^t \xi(s)\widehat{\operatorname{sgn}(Q_s)}(\xi,\eta)\,ds + \int_0^t \eta(s)\widehat{\operatorname{sgn}(Q_s)}(\xi,\eta)\,ds$$
$$= \int_0^t (\xi(s)+\eta(s)) \int_{\mathbb{R}} \operatorname{sgn}(\lambda)\frac{1}{\sqrt{2\pi s}} \times \exp\left\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\right\} d\lambda\, ds.$$

This concludes the proof. □

COROLLARY 4.2. *For $t > 0$, $a \in \mathbb{R}$, we have $\int_0^t \operatorname{sgn}(Q_s - aI)(\partial_s + \partial_s^*)\,ds \in \mathcal{L}$ and*

$$\int_0^t \widehat{\operatorname{sgn}(Q_s - aI)(\partial_s + \partial_s^*)\,ds}(\xi,\eta) = \int_0^t (\xi(s)+\eta(s)) \int_{\mathbb{R}} \operatorname{sgn}(\lambda - a)\frac{1}{\sqrt{2\pi s}} \times \exp\left\{(\xi,\eta)_0 - \frac{(\lambda - (u_s, \xi+\eta)_0)^2}{2s}\right\} d\lambda\, ds. \tag{4.3}$$

PROOF. The result follows by a simple computation. □

THEOREM 4.3. *For any $t > 0$,*

$$|Q_t| = \int_0^t \operatorname{sgn}(Q_s)(\partial_s + \partial_s^*)\,ds + \int_0^t \delta(Q_s)\,ds.$$

PROOF. By Proposition 3.1, $Q_t$ is an observable of Schwartz class:

$$Q_t = \int_{\mathbb{R}} \lambda P_t(d\lambda) \quad \text{and} \quad \widehat{Q}_t(\xi,\eta) = \int_{\mathbb{R}} \lambda \rho^t_{\xi,\eta}(d\lambda), \quad \xi,\eta \in E_{\mathbb{C}}.$$

Applying Lemma 2.5 for $f(x) = |x|$ we have

$$\widehat{|Q_t|}(\xi,\eta) = \int_{\mathbb{R}} |\lambda| \rho^t_{\xi,\eta}(\lambda)\,d\lambda$$
$$= \int_{\mathbb{R}} |\lambda| \frac{1}{\sqrt{2\pi t}} \exp\left\{(\xi,\eta)_0 - \frac{(\lambda - (u_t,\xi+\eta)_0)^2}{2t}\right\} d\lambda$$
$$= \int_{\mathbb{R}} |\sqrt{t}\lambda + (u_t,\xi+\eta)_0| \frac{1}{\sqrt{2\pi}} \exp\left\{(\xi,\eta)_0 - \frac{\lambda^2}{2}\right\} d\lambda$$
$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left\{(\xi,\eta)_0 - \frac{\lambda^2}{2}\right\} \times \int_0^t \operatorname{sgn}(\sqrt{s}\lambda + (u_s,\xi+\eta)_0)\left(\frac{\lambda}{2\sqrt{s}} + (\xi(s)+\eta(s))\right) ds\,d\lambda$$

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$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_0^t \frac{\lambda}{2\sqrt{2\pi s}} \operatorname{sgn}(\sqrt{s}\lambda + (u_s, \xi + \eta)_0) \exp\left\{(\xi, \eta)_0 - \frac{\lambda^2}{2}\right\} ds\, d\lambda \\
&\quad + \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left\{(\xi, \eta)_0 - \frac{\lambda^2}{2}\right\} \\
&\quad \times \int_0^t \operatorname{sgn}(\sqrt{s}\lambda + (u_s, \xi + \eta)_0)(\xi(s) + \eta(s))\, ds\, d\lambda \\
&= -\int_0^t \frac{1}{2\sqrt{2\pi s}} \int_{\mathbb{R}} \operatorname{sgn}(\sqrt{s}\lambda + (u_s, \xi + \eta)_0)\, d\left(\exp\left\{(\xi, \eta)_0 - \frac{\lambda^2}{2}\right\}\right) ds \\
&\quad + \int_0^t (\xi(s) + \eta(s)) \int_{\mathbb{R}} \operatorname{sgn}(\lambda + (u_s, \xi + \eta)_0) \frac{1}{\sqrt{2\pi}} \\
&\quad \times \exp\left\{(\xi, \eta)_0 - \frac{\lambda^2}{2}\right\} d\lambda\, ds \\
&= \frac{1}{2} \int_0^t \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \exp\left\{(\xi, \eta)_0 - \frac{\lambda^2}{2}\right\} \cdot 2\delta(\sqrt{s}\lambda + (u_s, \xi + \eta)_0)\, d\lambda\, ds \\
&\quad + \int_0^t (\xi(s) + \eta(s)) \int_{\mathbb{R}} \operatorname{sgn}(\lambda) \frac{1}{\sqrt{2\pi s}} \\
&\quad \times \exp\left\{(\xi, \eta)_0 - \frac{(\lambda - (u_s, \xi + \eta)_0)^2}{2s}\right\} d\lambda\, ds \\
&= \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi, \eta)_0 - \frac{(u_s, \xi + \eta)_0^2}{2s}\right\} ds \\
&\quad + \int_0^t (\xi(s) + \eta(s)) \langle \operatorname{sgn}, \rho^s_{\xi,\eta} \rangle\, ds \\
&= \int_0^t \frac{1}{\sqrt{2\pi s}} \exp\left\{(\xi, \eta)_0 - \frac{(u_s, \xi + \eta)_0^2}{2s}\right\} ds \\
&\quad + \int_0^t (\xi(s) + \eta(s)) \widehat{\operatorname{sgn}(Q_s)}(\xi, \eta)\, ds \\
&= \int_0^t \widehat{\delta(Q_s)}\, ds(\xi, \eta) + \int_0^t \widehat{\operatorname{sgn}(Q_s)(\partial_s + \partial_s^*)}\, ds(\xi, \eta).
\end{aligned}
$$

Hence $|Q_t| = \int_0^t \operatorname{sgn}(Q_s)(\partial_s + \partial_s^*)\, ds + \int_0^t \delta(Q_s)\, ds$. □

THEOREM 4.4. *For any $t > 0$ and $a \in \mathbb{R}$,*

$$
|Q_t - aI| = |a| I + \int_0^t \operatorname{sgn}(Q_s - aI)(\partial_s + \partial_s^*)\, ds + \int_0^t \delta_a(Q_s)\, ds
$$

*where $I : (L^2) \to (L^2)$ is the identity operator.*

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PROOF. By the spectral decomposition of $Q_t \in \mathcal{L}$,

$$Q_t - aI = \int_{\mathbb{R}} (\lambda - a) P_t(d\lambda) \quad \text{and} \quad (\widehat{Q_t - aI})(\xi, \eta) = \int_{\mathbb{R}} (\lambda - a) \rho^t_{\xi,\eta}(d\lambda).$$

Then

$$|Q_t - aI| = \int_{\mathbb{R}} |\lambda - a| P_t(d\lambda) \quad \text{and} \quad |\widehat{Q_t - aI}|(\xi, \eta) = \int_{\mathbb{R}} |\lambda - a| \rho^t_{\xi,\eta}(d\lambda).$$

Applying Lemma 2.5 for $f(x) = |x - a|$, and with a similar computation in Theorem 4.3, Corollary 3.2 as well as Corollary 4.2, the equality

$$|Q_t - aI| = |a|I + \int_0^t \operatorname{sgn}(Q_s - aI)(\partial_s + \partial_s^*)\, ds + \int_0^t \delta_a(Q_s)\, ds$$

holds. □

DEFINITION 4.5. For $t > 0$ and $a \in \mathbb{R}$, the equality in Theorem 4.4,

$$|Q_t - aI| = |a|I + \int_0^t \operatorname{sgn}(Q_s - aI)(\partial_s + \partial_s^*)\, ds + \int_0^t \delta_a(Q_s)\, ds,$$

is called the *quantum Tanaka formula* of $\{Q_t \mid t > 0\}$ at $a$.

REMARK 4.6. The last term in the *quantum Tanaka formula* is the quantum local time $L_t^a$ defined in Section 3.

COROLLARY 4.7. *For $t > 0$ and $a \in \mathbb{R}$,*

$$(Q_t - aI)^+ = \int_0^t 1_{(a,+\infty)}(Q_s)(\partial_s + \partial_s^*)\, ds + \frac{1}{2} L_t^a,$$
$$(Q_t - aI)^- = \int_0^t 1_{(-\infty,a)}(Q_s)(\partial_s + \partial_s^*)\, ds + \frac{1}{2} L_t^a.$$

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