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STRONGLY BOUNDED REPRESENTING MEASURES AND CONVERGENCE THEOREMS

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Abstract. Let K be a compact Hausdorff space, X a Banach space and $C(K, X)$ the Banach space of all continuous functions $f : K \to X$ endowed with the supremum norm. In this paper we study weakly precompact operators defined on $C(K, X)$.

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1. Introduction. Suppose that X and Y are real Banach spaces, K is a compact Hausdorff space, $C(K, X)$ is the Banach space of all continuous X-valued functions defined on K (with the supremum norm) and $T : C(K, X) \to Y$ is an operator with representing measure $m : \Sigma \to L(X, Y^{**})$, where Σ is the σ-algebra of subsets of K, Y^{**} is the bidual of Y and $L(X, Y^{**})$ is the Banach space of all operators $T : X \to Y^{**}$ [**3**]. Denote the semivariation of m by $\tilde{m}$. The operator T (or the measure m) is said to be strongly bounded if $(\tilde{m}(A_i)) \to 0$ whenever (A_i) is a pairwise disjoint sequence from Σ. By Theorem 4.4 of [**14**], a strongly bounded representing measure takes its values in $L(X, Y)$. It is well known that if T is unconditionally converging, then m is strongly bounded [**3**, **19**, **28**].

The Riesz Representation Theorem in this setting asserts that to each operator $T : C(K, X) \to Y$ there corresponds a unique representing measure $m : \Sigma \to L(X, Y^{**})$ with finite semivariation so that $T(f) = \int_K f \, dm$ and $\|T\| = \tilde{m}(K)$. This correspondence between T and m will be denoted by $m \leftrightarrow T$. We note that [**14**] and Chapter 3 of [**18**] contain a detailed discussion of this setting. (The reader should note that for $f \in C(K, X)$, $\int_K f \, dm \in Y$ even if m is not $L(X, Y)$-valued.)

Let χ_A denote the characteristic function of a set A, and $B(\Sigma, X)$ denote the space of totally measurable functions on Σ with values in X. Certainly $C(K, X)$ is contained isometrically in $B(\Sigma, X)$. Further, $B(\Sigma, X)$ embeds isometrically in $C(K, X)^{**}$; e.g. see [**14**]. The reader should note that if $m \leftrightarrow T$, then $m(A)x = T^{**}(\chi_A x)$, for each $A \in \Sigma$, $x \in X$. If $f \in B(\Sigma, X)$, then f is the uniform limit of X-valued simple functions, $\int_K f \, dm$ is well defined, which defines an extension $\hat{T}$ of T; e.g. see [**18**]. Theorem 2 of [**7**] shows that $\hat{T}$ maps $B(\Sigma, X)$ into Y if and only if the representing measure m of T is $L(X, Y)$-valued. If $T : C(K, X) \to Y$ is strongly bounded, then m is $L(X, Y)$-valued [**14**], and thus $\hat{T} : B(\Sigma, X) \to Y$. Since $\hat{T}$ is the restriction to $B(\Sigma, X)$ of the operator T^{**}, it is clear that an operator $T : C(K, X) \to Y$ is compact (resp. weakly compact) if and only if its extension $\hat{T} : B(\Sigma, X) \to Y$ is compact (resp. weakly compact). Several authors have found the study of $\hat{T}$ to be quite helpful. We mention the work of Batt and Berg [**7**], Bombal and Cembranos [**13**] and Bombal and Porras [**11**]. In these papers it has been proved that if m is strongly bounded, then $T : C(K, X) \to Y$ is weakly compact, compact, Dunford-Pettis, Dieudonné, unconditionally converging, strictly singular or strictly cosingular if and only if its extension $\hat{T} : B(\Sigma, X) \to Y$ has the same property. Our results will be concerned with relating properties of the operator T to properties of its representing measure in the case of weakly precompact operators and operators with weakly precompact adjoints. An operator $T : X \to Y$ is called weakly precompact (or almost weakly compact) if every sequence in the image of a bounded set has a weakly Cauchy subsequence.

The general bilinear integral of Bartle [**4**] can be used in the context of strongly bounded representing measures to establish convergence results which unify several approaches and have numerous applications and corollaries. Although the convergence theorems in [**4**] are similar to some of the conclusions in our first theorem, it is not clear that [**4**] can be used to obtain the specific results we desire. For this reason, as well as for the convenience of the reader, we include a brief description of the bilinear integral we shall use and a proof of the convergence results we need. In the process, the technique and the results in [**21**] are extended.

Suppose that m is a strongly bounded representing measure with control measure λ, i.e. $0 \leq \lambda \in rca(\Sigma)$ and $\tilde{m}(A) \to 0$ as $\lambda(A) \to 0$. If $g \in L^1(\lambda, X)$ and g is pointwise bounded, choose a uniformly pointwise bounded sequence of X-valued simple functions (s_n) so that $s_n(t) \to g(t)$ a.e.- λ (see [**20**], p. 117). The standard approach in Section 7, pp. 106–108, of Dinculeanu [**18**] is used to define the integral of an X-valued simple function with respect to an $L(X, Y)$-valued measure with finite semivariation, i.e. if $s = \sum \chi_{A_i} x_i$ and $m : \Sigma \to L(X, Y)$ is finitely additive and has finite semivariation, then $\int s\, dm$ is defined to be $\sum m(A_i)x_i$. Egoroff's theorem guarantees that $(\int_K s_n\, dm)$ converges. Define $\int_K g\, dm$ to be $\lim_n \int_K s_n\, dm$. It is not difficult to check that $\int_K g\, dm$ is well defined.

2. Main results.

THEOREM 1. *Suppose that $m \leftrightarrow T : C(K, X) \to Y$ is strongly bounded and λ is a control measure for m.*
(i) *If (g_n) is a uniformly pointwise bounded sequence and $(g_n) \to 0$ in $L^1(\lambda, X)$, then $(\int_K g_n\, dm) \to 0$ in Y. Consequently, if (g_n) is uniformly pointwise bounded and $(g_n) \stackrel{w}{\to} 0$, then $(\int_K g_n\, dm) \stackrel{w}{\to} 0$.*
(ii) *If (h_n) is a uniformly pointwise bounded sequence in $L^1(\lambda, X)$ and $(h_n(t))$ is weakly Cauchy for each $t \in K$, then $(\int_K h_n\, dm)$ is weakly Cauchy in Y.*
(iii) *Suppose that H is a bounded set in $C(K, X)$. If (f_n) is a sequence in H and $f : K \to X$ is a function such that $f_n(t) \to f(t)$ for each $t \in K$, then $(\int_K f_n\, dm) \to \int_K f\, dm$.*

Proof. (i) Without loss of generality, suppose that $\|g_n(t)\| < 1$ for all $n \in \mathbb{N}$ and all $t \in K$. Since $(\int_K \|g_n\|\, d\lambda) \to 0$, we may suppose without loss of generality that $(g_n(t)) \to 0$ for almost all $t \in K$. Let $\epsilon > 0$ and choose $E \in \Sigma$ such that $\tilde{m}(K \setminus E) < \epsilon$ and $(g_n) \to 0$ uniformly on E. Choose $n_0 \in \mathbb{N}$ so that if $n \geq n_0$, then $\|g_n(t)\| \leq \epsilon$, $t \in E$.

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The definition of $\int_E g_n\,dm$ and $\int_{K\setminus E} g_n\,dm$ show that for $n \geq n_0$,

$$\left\| \int_K g_n\,dm \right\| = \left\| \int_E g_n\,dm + \int_{K\setminus E} g_n\,dm \right\| \leq \epsilon\,\tilde{m}(E) + \epsilon.$$

We claim that $(\int_K g_n\,dm) \xrightarrow{w} 0$, when (g_n) is uniformly pointwise bounded and $(g_n) \xrightarrow{w} 0$ in $L^1(\lambda, X)$. Indeed, if (g_{n_i}) is an arbitrary subsequence of (g_n), then $0 \in \overline{co}\{g_{n_i} : i \geq 1\}$ (since $(g_n) \xrightarrow{w} 0$). Thus, $0 \in \overline{co}\{\int_K g_{n_i}\,dm : i \geq 1\}$. This implies that $(\int_K g_n\,dm) \xrightarrow{w} 0$. Otherwise, one can strictly separate 0 from the closed convex hull of some subsequence of $(\int_K g_n\,dm)$, a contradiction.

(ii) Without loss of generality, suppose $\|h_n(t)\| < 1$ for all $n \in \mathbb{N}$ and $t \in K$. Let $\epsilon > 0$. Using the existence of a control measure for m and Lusin's theorem, we can find a compact subset K_0 of K such that $\tilde{m}(K \setminus K_0) < \epsilon/2$ and $\phi_n = h_n|_{K_0}$ is continuous for each $n \in \mathbb{N}$. Let $H = [\phi_n]$ be the closed linear span of (ϕ_n) in $C(K_0, X)$ and $S : H \to C(K, X)$ be the isometric extension operator given by Theorem 1 of [**13**]. Let $\psi_n = S(\phi_n)$, $n \in \mathbb{N}$. Since $(\phi_n(t))$ is weakly Cauchy for each $t \in K_0$, the sequence (ϕ_n) is weakly Cauchy in $C(K_0, X)$ (Theorem 9 of [**19**], Lemma 3.2 of [**3**]). Then (ψ_n) is weakly Cauchy in $C(K, X)$ and $(T(\psi_n))$ is weakly Cauchy. For each $n \in \mathbb{N}$,

$$\left\| \int_K h_n\,dm - \int_K \psi_n\,dm \right\| = \left\| \int_{K\setminus K_0} (h_n - \psi_n)\,dm \right\| \leq 2\,\tilde{m}(K \setminus K_0) < \epsilon.$$

Then $(\int_K h_n\,dm)$ is weakly Cauchy.

(iii) Let H be the unit ball of $C(K, X)$, (f_n) be a sequence in H and let $f : K \to X$ be a function such that $f_n(t) \to f(t)$ for each $t \in K$. Without loss of generality suppose that $\|f(t)\| \leq 1$, $t \in K$. Then f is strongly measurable (by the Pettis measurability theorem) and $\int_K f\,dm$ exists. Let $\epsilon > 0$. Use Lusin's theorem and the existence of the control measure to choose a compact subset K_0 of K such that $g = f|_{K_0}$ is continuous and $\tilde{m}(K \setminus K_0) < \epsilon/2$. Let $g_n = f_n|_{K_0}$, $n \in \mathbb{N}$. Use Egoroff's theorem to choose a compact subset K_1 of K_0 such that $\tilde{m}(K_0 \setminus K_1) < \epsilon/2$ and $(g_n - g) \to 0$ uniformly on K_1. Let $n_0 \in \mathbb{N}$ so that if $n \geq n_0$, then $\|g_n(t) - g(t)\| \leq \epsilon$, $t \in K_1$. For $n \geq n_0$, we have

$$\begin{aligned}\left\| \int_K (f_n - f)\,dm \right\| &= \left\| \int_{K_1} (f_n - f)\,dm + \int_{K_0\setminus K_1} (f_n - f)\,dm + \int_{K\setminus K_0} (f_n - f)\,dm \right\| \\ &\leq \sup_{t\in K_1} \|g_n(t) - g(t)\|\,\tilde{m}(K_1) + 2\,\tilde{m}(K_0 \setminus K_1) + 2\,\tilde{m}(K \setminus K_0) \\ &\leq \epsilon\,\tilde{m}(K) + 2\epsilon.\end{aligned}$$

□

Abbott [**1**] gave an example of a pair $m \leftrightarrow T$ such that T is weakly precompact and m is not strongly bounded. The following corollary is related to results in [**32**].

COROLLARY 2. *If $\ell_1 \not\hookrightarrow X$, then every strongly bounded operator $T : C(K, X) \to Y$ is weakly precompact.*

Proof. Suppose that $T : C(K, X) \to Y$ is a strongly bounded operator with representing measure m and control measure λ. We have

$$T(f) = \int_K f\,dm\,, f \in C(K, X).$$

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Let (f_n) be a sequence in the unit ball of $C(K, X)$. Then (f_n) is uniformly integrable in $L^1(\lambda, X)$. Since $\ell_1 \not\hookrightarrow X$, (f_n) is weakly precompact in $L^1(\lambda, X)$ [**12**]. Without loss of generality, suppose (f_n) is weakly Cauchy in $L^1(\lambda, X)$. By results of Talagrand [**30**], we can write $f_n = g_n + h_n$ a.e. in K, where (g_n) and (h_n) are sequences in $L^1(\lambda, X)$ such that (g_n) is weakly null in $L^1(\lambda, X)$ and $(h_n(t))$ is weakly Cauchy for each $t \in K$. Further, we have (g_n) and (h_n) uniformly pointwise bounded. By Theorem 1, $(\int_K g_n\, dm) \stackrel{w}{\to} 0$ and $(\int_K h_n\, dm)$ is weakly Cauchy. Hence $(\int_K f_n\, dm)$ is weakly Cauchy, and thus T is weakly precompact. □

3. Applications. An operator $T : X \to Y$ is called a Dieudonné (or weakly completely continuous) operator if T maps weakly Cauchy sequences in X to weakly convergent sequences in Y, and X is said to have the Dieudonné property if every Dieudonné operator with domain X is weakly compact [**25**]. If X is a $C(K)$-space or if $\ell_1 \not\hookrightarrow X$, then X has the Dieudonné property.

Corollary 3. ([21, 26]) *If $\ell_1 \not\hookrightarrow X$, then $C(K, X)$ has the Dieudonné property.*

Proof. If $m \leftrightarrow T : C(K, X) \to Y$ is a Dieudonné operator, then T is unconditionally converging and m is strongly bounded [**3**, **19**, **28**]. Let (f_n) be a sequence in the unit ball of $C(K, X)$. Using the arguments in Corollary 2 and Theorem 1, we obtain sequences (g_n), (h_n) and (ψ_n) so that

$$T(f_n) = \int_K f_n\, dm = \int_K g_n\, dm + \int_K h_n\, dm,$$

$(\int_K g_n\, dm) \stackrel{w}{\to} 0$, (ψ_n) is weakly Cauchy in $C(K, X)$ and $\| \int_K h_n\, dm - T(\psi_n)\| \to 0$. Let $y \in Y$ such that $(T(\psi_n)) \stackrel{w}{\to} y$. Then $(\int_K h_n\, dm) \stackrel{w}{\to} y$, and thus $(T(f_n)) \stackrel{w}{\to} y$. □

A Banach space X has property (u) if for every weakly Cauchy sequence (x_n) in X, there is a weakly unconditionally converging series $\sum y_n$ in X such that $(x_n - \sum_{i=1}^n y_i) \stackrel{w}{\to} 0$. A Banach space X has property (V) if every unconditionally converging operator T from X to any Banach space Y is weakly compact [**27**].

Corollary 4. (i) ([14, 32]) *If X is reflexive, then every strongly bounded operator $T : C(K, X) \to Y$ is weakly compact.*
(ii) ([**27**]) *If X is reflexive, then $C(K, X)$ has property (V).*
(iii) ([15, 32]) *If $\ell_1 \not\hookrightarrow X$ and X has property (u), then $C(K, X)$ has property (V).*

Proof. (i) Let $m \leftrightarrow T : C(K, X) \to Y$ be a strongly bounded operator and (f_n) be a sequence in the unit ball of $C(K, X)$. Repeating the construction in Corollary 2, we obtain uniformly pointwise bounded sequences (g_n), (h_n) in $L^1(\lambda, X)$ so that $f_n = g_n + h_n$ a.e. in K, (g_n) is weakly null in $L^1(\lambda, X)$ and $(h_n(t))$ is weakly Cauchy for each $t \in K$. Let $\epsilon > 0$. Repeating the construction in Theorem 1, we obtain a compact subset K_0 of K and a sequence (ϕ_n) so that $\tilde{m}(K \setminus K_0) < \epsilon/2$ and $\phi_n = h_n|_{K_0}$ is continuous for each $n \in \mathbb{N}$; further, $(\phi_n(t))$ is weakly Cauchy for each $t \in K_0$, $(\int_K g_n\, dm) \stackrel{w}{\to} 0$ and

$$T(f_n) = \int_K f_n\, dm = \int_K g_n\, dm + \int_K h_n\, dm.$$

Let $\phi : K_0 \to X$ be a function so that $(\phi_n(t)) \stackrel{w}{\to} \phi(t)$, $t \in K_0$ (the reflexivity of X assures the existence of ϕ). Then ϕ is bounded, and since for each n, ϕ_n is continuous, ϕ

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is separably valued and weakly measurable. By Pettis's measurability theorem, ϕ is strongly measurable. Use Lusin's theorem and the existence of the control measure to choose a compact subset K_1 of K_0 such that $h = \phi|_{K_1}$ is continuous and $\tilde{m}(K_0 \setminus K_1) < \epsilon/2$. Hence $(\phi_n) \xrightarrow{w} h$ in $C(K_1, X)$.

Let $H = [\phi_n]$ be the closed linear span of (ϕ_n) in $C(K_1, X)$ and $S : H \to C(K, X)$ be the isometric extension operator given by Theorem 1 of [**13**]. Let $\psi_n = S(\phi_n)$, $n \in \mathbb{N}$ and $\psi = S(h)$. Since $(\phi_n) \xrightarrow{w} h$ in $C(K_1, X)$, it follows that $(TS(\phi_n)) \xrightarrow{w} TS(h)$; i.e. $(T(\psi_n)) \xrightarrow{w} T(\psi) := y$ in Y. Further, for each $n \in \mathbb{N}$,

$$\left\| \int_K h_n \, dm - \int_K \psi_n \, dm \| = \| \int_{K\setminus K_1} (h_n - \psi_n) \, dm \right\| \leq 2\,\tilde{m}(K \setminus K_1) < 2\epsilon.$$

Then $(\int_K h_n \, dm) \xrightarrow{w} y$, hence $(\int_K f_n \, dm) \xrightarrow{w} y$.

(ii) Every unconditionally converging operator on $C(K, X)$ is strongly bounded [**3**, **19**, **28**], and thus weakly compact.

(iii) If $T : C(K, X) \to Y$ is an unconditionally converging operator, then T is a Dieudonné operator, since X has property (u) [**32**]. By Corollary 3, T is weakly compact. □

Gamlen [**23**] proved that if X^* has the Radon–Nikodym property and Y is weakly sequentially complete, then any operator $T : C(K, X) \to Y$ is weakly compact. Bello [**8**] generalized this result to the case of X not containing copies of ℓ_1. The following result contains Theorem 12 [**8**].

Corollary 5. *Suppose that* $\ell_1 \not\hookrightarrow X$.
(i) *If* $c_0 \not\hookrightarrow Y$, *then every operator* $T : C(K, X) \to Y$ *is weakly precompact.*
(ii) *If* Y *is weakly sequentially complete, then every operator* $T : C(K, X) \to Y$ *is weakly compact.*
(iii) *If* Y *has the Schur property, then every operator* $T : C(K, X) \to Y$ *is compact.*

Proof. (i) Suppose $T : C(K, X) \to Y$ is an operator. Since $c_0 \not\hookrightarrow Y$, T is unconditionally converging, and thus strongly bounded [**3**, **19**, **28**]. By Corollary 2, T is weakly precompact.
(ii) Since Y is weakly sequentially complete, T is weakly compact.
(iii) Since Y has the Schur property, T is compact. □

We remark that if $c_0 \not\hookrightarrow Y$ and $T : C(K, X) \to Y$ is an operator with representing measure m, then m is countably additive. To see this, note that T is unconditionally converging, m is strongly bounded, and thus countably additive [**3**, **14**].

Corollary 6. (i) *If* X^* *has the Radon–Nikodym property, then every strongly bounded operator* $T : C(K, X) \to Y$ *is weakly precompact.*
(ii) *If* X^* *is separable, then every strongly bounded operator* $T : C(K, X) \to Y$ *is weakly precompact.*

Proof. (i) If X^* has the Radon–Nikodym property, then $\ell_1 \not\hookrightarrow X$ [**17**]. Apply Corollary 2. (ii) If X^* is separable, then X^* has the Radon–Nikodym property. □

Corollary 7. *Suppose that* X *is a Banach space such that for every compact Hausdorff space* K *and every Banach space* Y, *an operator* $m \leftrightarrow T : C(K, X) \to Y$ *is weakly precompact whenever* m *satisfies the following conditions:*

(i) *m is strongly bounded and*
(ii) *$m(A) : X \to Y$ is weakly precompact for each $A \in \Sigma$.*
Then ℓ_1 is not complemented in X.

Proof. Suppose that ℓ_1 is complemented in X. If $P : X \to \ell_1$ is a projection, then P is not compact. By Theorem 2.2 of [**2**], there is a compact space Δ and a continuous linear surjection $m \leftrightarrow T : C(\Delta, X) \to \ell_1$ so that m is strongly bounded and $m(A) : X \to Y$ is compact for each $A \in \Sigma$. Since T is a surjection onto ℓ_1, T is not weakly precompact. □

Corollary 8. *If $\ell_1 \not\hookrightarrow X^*$ and $T : C(K, X) \to Y$ is strongly bounded, then T and T^* are weakly precompact.*

Proof. T^* is weakly precompact by Theorem 9 [**6**]. Since $\ell_1 \not\hookrightarrow X^*$, $\ell_1 \not\hookrightarrow X$ ([**16**], p. 211). Apply Corollary 2. □

Corollary 9. *Suppose that $\ell_1 \not\hookrightarrow X^*$ and $T : C(K, X) \to Y$ is an operator. Then the following are equivalent:*
(i) *T is strongly bounded.*
(ii) *T^* is weakly precompact.*
(iii) *T is unconditionally converging.*

Proof. (i) implies (ii). If $T : C(K, X) \to Y$ is a strongly bounded operator, then T^* is weakly precompact by Theorem 9 [**6**].
(ii) implies (iii). If T^* is weakly precompact, then T is unconditionally converging by Corollary 2 [**6**].
(iii) implies (i). Every unconditionally converging operator on $C(K, X)$ is strongly bounded [**3**, **19**, **28**]. □

Corollary 10. ([**14**]) *If $c_0 \not\hookrightarrow X$ and $T : C(K, X) \to Y$ is a strongly bounded operator, then T is unconditionally converging.*

Proof. It is enough to show that if $\sum f_n$ is weakly unconditionally converging in $C(K, X)$, then $\|T(f_n)\| \to 0$. Suppose that $\sum f_n$ is weakly unconditionally converging. Then for each $t \in K$, $\sum f_n(t)$ is weakly unconditionally converging, and thus unconditionally converging in X (since $c_0 \not\hookrightarrow X$). Hence $\|f_n(t)\| \to 0$ for each $t \in K$, and $(T(f_n)) \to 0$ by Theorem 1. □

An operator $T : X \to Y$ is called completely continuous (or Dunford–Pettis) if T maps weakly Cauchy sequences to norm convergent sequences. The Banach space X has the Dunford–Pettis property (DPP) if every weakly compact operator with domain X is completely continuous. Talagrand showed that there is a Banach space X such that X^* has the Schur property (hence X has the DPP), but neither $C(K, X)$ nor $L^1(X^*)$ has the DPP [**31**].

Corollary 11. *Suppose that X has the Schur property. Then the following assertions hold:*
(i) *Every strongly bounded operator $T : C(K, X) \to Y$ is completely continuous.*

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(ii) ([**19**]) *$C(K, X)$ has the DPP.*
(iii) ([**14**]) *If $c_0 \not\hookrightarrow Y$, then every operator $T : C(K, X) \to Y$ is completely continuous.*
(iv) *If $T : C(K, X) \to Y$ is an operator with a weakly precompact adjoint, then T is completely continuous.*
(v) *If $T : C(K, X) \to Y$ is an operator, then T is a Dieudonné operator if and only if T is completely continuous.*

Proof. (i) Let (f_n) be a weakly null sequence in the unit ball of $C(K, X)$ and $T : C(K, X) \to Y$ be a strongly bounded operator. Since $(f_n(t))$ is weakly null in X, and X has the Schur property, $\|f_n(t)\| \to 0$ for each $t \in K$. By Theorem 1, $(T(f_n)) \to 0$, and thus T is completely continuous.

(ii) Every weakly compact operator $T : C(K, X) \to Y$ is strongly bounded [**14**]. Then T is completely continuous, and thus $C(K, X)$ has the DPP.

(iii) If $c_0 \not\hookrightarrow Y$ and $T : C(K, X) \to Y$ is an operator, then T is unconditionally converging, and thus strongly bounded [**3**, **19**, **28**]. By part (i), T is completely continuous.

(iv) If $T^* : Y^* \to C(K, X)^*$ is weakly precompact, then $T : C(K, X) \to Y$ is unconditionally converging (Corollary 2 in [**6**]), and thus strongly bounded. Apply (i).

(v) If $T : C(K, X) \to Y$ is a Dieudonné operator, then T is unconditionally converging, hence strongly bounded. Apply (i). The converse is clear. □

The next result establishes a connection between weakly precompact operators and unconditionally converging adjoints. It is known that if $T : X \to Y$ is an operator, then $T(B_X)$ is a V^* -subset of Y if and only if $T^* : Y^* \to X^*$ is unconditionally converging [5, 24].

THEOREM 12. *If $T : X \to Y$ is weakly precompact, then $T^* : Y^* \to X^*$ is unconditionally converging.*

Proof. Suppose $T : X \to Y$ is weakly precompact. Then $T(B_X)$ is weakly precompact, and thus a V^*-subset of Y [**27**]. It follows that T^* is unconditionally converging. □

We remark that the converse of this theorem is not true. Specifically, let X be a Banach space such that $\ell_1 \hookrightarrow X$ and $\ell_1 \overset{c}{\not\hookrightarrow} X$. Let $T : \ell_1 \to X$ be an isomorphic embedding. Then $T^* : X^* \to \ell_\infty$ is unconditionally converging (since $c_0 \not\hookrightarrow X^*$) and T is not weakly precompact (since it is an isomorphism on ℓ_1).

If $\ell_1 \not\hookrightarrow X$, then every strongly bounded operator $T : C(K, X) \to Y$ is weakly precompact and has an unconditionally converging adjoint (by Corollary 2 and Theorem 12). This observation gives the following result.

COROLLARY 13. *If $\ell_1 \not\hookrightarrow X$, then every unconditionally converging (resp. completely continuous) operator $T : C(K, X) \to Y$ is weakly precompact and has an unconditionally converging adjoint.*

Proof. If $T : C(K, X) \to Y$ is an unconditionally converging operator, then T is strongly bounded. Since every completely continuous operator is unconditionally converging, every completely continuous operator $T : C(K, X) \to Y$ is strongly bounded. □

THEOREM 14. *Suppose that $\ell_1 \overset{c}{\not\hookrightarrow} X$. Then every operator $T : C(K, X) \to Y$ has an unconditionally converging adjoint.*

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Proof. Suppose $T : C(K, X) \to Y$ is an operator and $T^* : Y^* \to C(K, X)^*$ is not unconditionally converging. Using **[9]** or problem 8, p. 54, of **[16]**, one obtains an isomorphic copy U of c_0 in Y^* on which T^* acts as an isomorphism. If $L : c_0 \to U \subset Y^*$ is an isomorphic embedding, $T^*L : c_0 \to C(K, X)^*$ is an isomorphism. Then $c_0 \hookrightarrow C(K, X)^*$, and thus $\ell_1 \overset{c}{\hookrightarrow} C(K, X)$ **[9]**. The main result in **[29]** implies that $\ell_1 \overset{c}{\hookrightarrow} X$, a contradiction which concludes the proof. □

The Banach space X has property (V^*) (resp. (wV^*)) if every V^*-subset of X is relatively weakly compact (resp. weakly precompact) **[10, 27]**. The following result contains Theorem 1.6 of **[22]**.

Corollary 15. *Suppose that $\ell_1 \overset{c}{\not\hookrightarrow} X$ and Y has property (V^*) (resp. (wV^*)). Then every operator $T : C(K, X) \to Y$ is weakly compact (resp. weakly precompact).*

Proof. Suppose that $T : C(K, X) \to Y$ is an operator and Y has property (V^*) (resp. (wV^*)). By the previous result, $T^* : Y^* \to C(K, X)^*$ is unconditionally converging. Apply Theorem 3.10 of **[24]** to obtain that T is weakly compact (resp. weakly precompact). □

Corollary 2 of **[6]** shows that if $T^* : Y^* \to X^*$ is weakly precompact, then $T : X \to Y$ is unconditionally converging and weakly precompact. It follows that if $T : C(K, X) \to Y$ has a weakly precompact adjoint, then T is strongly bounded (since it is unconditionally converging) and T^* is unconditionally converging (by Theorem 12).

Suppose that $T : C(K, X) \to Y$ is an operator and $\hat{T} : B(\Sigma, X) \to Y^{**}$ is its extension to $B(\Sigma, X)$. We remark that if $m \leftrightarrow T : C(K, X) \to Y$ is strongly bounded, then m is $L(X, Y)$-valued **[14]** and $\hat{T}$ maps $B(\Sigma, X)$ into Y (as noted in the Introduction).

Theorem 16. *Suppose that $T : C(K, X) \to Y$ is a strongly bounded operator. Then T is weakly precompact if and only if its extension $\hat{T} : B(\Sigma, X) \to Y$ is weakly precompact.*

Proof. Suppose that $T : C(K, X) \to Y$ is weakly precompact and $\hat{T}$ is not weakly precompact. Let $\epsilon > 0$, $y^* \in B_{Y^*}$ and (f_n) be a sequence in the unit ball of $B(\Sigma, X)$ such that $|\langle y^*, \hat{T}(f_n - f_m)\rangle| > \epsilon$, for $n \neq m$.

Using the existence of a control measure for m and Lusin's theorem, one can find a compact subset K_0 of K such that $\tilde{m}(K \setminus K_0) < \epsilon/8$ and $g_n = f_n|_{K_0}$ is continuous for each $n \in \mathbb{N}$. Let $H = [g_n]$ be the closed linear subspace spanned by (g_n) in $C(K_0, X)$ and $S : H \to C(K, X)$ be the isometric extension operator given by Theorem 1 of **[13]**. If $h_n = S(g_n)$, $n \in \mathbb{N}$, then (h_n) is in the unit ball of $C(K, X)$, and for $n \neq m$,

$$\begin{aligned}
|\langle y^*, T(h_n - h_m)\rangle| &\geq \left|\left\langle y^*, \int_{K_0} (h_n - h_m)\, dm\right\rangle\right| - \left|\left\langle y^*, \int_{K\setminus K_0} (h_n - h_m)\, dm\right\rangle\right| \\
&\geq \left|\left\langle y^*, \int_{K_0} (f_n - f_m)\, dm\right\rangle\right| - \epsilon/4 \\
&\geq \left|\left\langle y^*, \int_{K} (f_n - f_m)\, dm\right\rangle\right| - \left|\left\langle y^*, \int_{K\setminus K_0} (f_n - f_m)\, dm\right\rangle\right| - \epsilon/4 \\
&\geq |\langle y^*, \hat{T}(f_n - f_m)\rangle| - \epsilon/2 > \epsilon/2.
\end{aligned}$$

This is a contradiction, since T is weakly precompact. □

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COROLLARY 17. *Suppose that* $m \leftrightarrow T : C(K, X) \to Y$ *is a strongly bounded operator. If* T *is weakly precompact, then* $m(A) : X \to Y$ *is weakly precompact for each* $A \in \Sigma$.

Proof. If $A \in \Sigma$, $A \neq \emptyset$, define $\theta_A : X \to B(\Sigma, X)$ by $\theta_A(x) = \chi_A x$. Then θ_A is an isomorphic isometric embedding of X into $B(\Sigma, X)$ and $\hat{T}\theta_A = m(A)$. By Theorem 16, $\hat{T}$ is weakly precompact, and thus $m(A)$ is weakly precompact. □

A Banach space X is a Grothendieck space if *weak** and weak convergence of sequences in X^* coincide.

COROLLARY 18. *Suppose that* $C(K)$ *is a Grothendieck space. If* $m \leftrightarrow T : C(K, X) \to Y$ *is a weakly precompact operator, then* m *is* $L(X, Y)$*-valued and* $m(A) : X \to Y$ *is weakly precompact for each* $A \in \Sigma$.

Proof. Suppose $m \leftrightarrow T : C(K, X) \to Y$ is a weakly precompact operator. For each $x \in X$, define an operator $T_x : C(K) \to Y$ by $T_x(f) = T(f \cdot x), f \in C(K)$. Then T_x is weakly precompact. By Corollary 6 of [**6**], T_x^* is weakly precompact. Hence T_x is an unconditionally converging operator on a C(K)-space, and every unconditionally converging operator on a C(K)-space is weakly compact [14, 27]. If m_x is the representing measure of T_x, then m_x is Y-valued ([**17**], p. 105). Since $m_x(A) = m(A)x$, m is $L(X, Y)$-valued. An application of Corollary 6 of [**1**] concludes the proof. □

If S is a subspace of X and $T : X \to Y$ is an operator, let T_S denote the restriction of T to S. A closed operator ideal $\mathcal{O}$ is said to be separably determined provided that for each pair of Banach spaces X and Y, an operator $T : X \to Y$ belongs to $\mathcal{O}(X, Y)$ if and only if $T_S \in \mathcal{O}(S, Y)$ for each separable subspace S of X.

THEOREM 19. *Suppose that* $T : C(K, X) \to Y$ *is an operator and* $\hat{T} : B(\Sigma, X) \to Y^{**}$ *is its extension to* $B(\Sigma, X)$. *Then* T^* *is weakly precompact if and only if* $\hat{T}^*$ *is weakly precompact.*

Proof. If $\hat{T}^*$ is weakly precompact, then $\hat{T}$ is unconditionally converging and weakly precompact [**6**]. Hence T is strongly bounded and $\hat{T} : B(\Sigma, X) \to Y$. Apply Theorem 4 of [**6**] to obtain a subspace Z of $C(K, X)$ and an operator $S : Y \to \ell_\infty$ so that $ST(Z) = c_0$. Since $\hat{T}$ is an extension of T, there is a subspace W of $B(\Sigma, X)$ so that $S\hat{T}(W) = c_0$. Thus by Theorem 4 of [**6**], $\hat{T}^*$ is not weakly precompact, and we have a contradiction.

Conversely, suppose that T^* is weakly precompact. Then T is strongly bounded and $\hat{T} : B(\Sigma, X) \to Y$. Let $\mathcal{O} = \{L : X \to Y \mid L^* \text{ is weakly precompact}\}$. By Proposition 8 of [**6**], $\mathcal{O}$ is a closed separably determined operator ideal. Apply Proposition 4.1 of [**2**] to conclude that $\hat{T}$ is an element of $\mathcal{O}$, i.e. $\hat{T}^*$ is weakly precompact. □

COROLLARY 20. ([**6**]) *Suppose that* $m \leftrightarrow T : C(K, X) \to Y$ *is an operator. If* T^* *is weakly precompact, then* $m(A)^* : Y^* \to X^*$ *is weakly precompact for each* $A \in \Sigma$.

Proof. For $A \in \Sigma$, $A \neq \emptyset$, define $\theta_A : X \to B(\Sigma, X)$ by $\theta_A(x) = \chi_A x$. Then $\hat{T}\theta_A = m(A)$, $\hat{T}^*$ is weakly precompact (by Theorem 19), and thus $m(A)^*$ is weakly precompact. □

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