Canad. Math. Bull.Vol. **35** (3), 1992 pp. 371–375

A METHOD OF SOLVING A CLASS OF CIV BOUNDARY VALUE PROBLEMS

NEZAM IRANIPARAST

ABSTRACT. A method will be introduced to solve problems $u_{tt} - u_{ss} = h(s,t)$, $u(t,t) = u(1+t, 1-t)$, $u(s,0) = g(s)$, $u(1,1) = 0$ and $u_{tt} - u_{ss} = h(s,t)$, $\frac{du}{d\sigma}(t,t) = \frac{du}{d\tau}(1+t, 1-t)$, $u_t(s,0) = u(1,1)$, for (s,t) in the characteristic triangle $R = \{(s,t) : t \leq s \leq 2-t, 0 \leq t \leq 1\}$. Here $\frac{du}{d\sigma}$ and $\frac{du}{d\tau}$ represent the directional derivatives of u in the characteristic directions $e_1 = (-1,-1)$ and $e_2 = (1,-1)$, respectively. The method produces the symmetric Green's function of Kreith [1] in both cases.

1. **Introduction.** A successful attempt by Kreith [1] to generalize the eigenvalue problem

$$\frac{d^2u}{dt^2} + \lambda p(t)u = 0,$$
$$u(0) = 0 = u(1),$$

to the case of vibrations of a finite string governed by

$$u_{tt} - u_{ss} + \lambda p(s,t)u = 0, \tag{1.1}$$

in the characteristic triangle

$$R = \{(s,t) : t \leq s \leq 2-t, 0 \leq t \leq 1\},$$

and subject to certain boundary conditions, has resulted in the establishment of eigenvalues and eigenfunctions. The technique of [1] furnishes a symmetric Green's function for the eigenequation (1.1) subject to the characteristic boundary and initial conditions

$$u(t,t) = u(1+t, 1-t), \quad 0 \leq t \leq 1, \tag{1.2}$$

$$u(s,0) = u(1,1) = 0, \quad 0 \leq s \leq 2, \tag{1.3}$$

by computing the product of the Green's functions for a pair of operators $\frac{i\partial}{\partial x}, \frac{i\partial}{\partial y}$, $s = x+y$, $t = y-x$ and mixed boundary conditions for which $\frac{\partial}{\partial x}$ is selfadjoint. The symmetric Green's function of (1.1)–(1.3) is then used to construct a Green's function for (1.1) subject to

$$u_t(s,0) = u(1,1) = 0, \quad 0 \leq s \leq 2. \tag{1.4}$$

Received by the editors May 28, 1990.
AMS subject classification: 35L20, (35P05).

https://doi.org/10.4153/CMB-1992-049-2 Published online by Cambridge University Press

However, the behavior of u along the characteristics is not given in this case.

In what follows we will consider the more general problem of solving (1.1) subject to (1.2) and

$$u(s,0) = g(s),\ u(1,1) = 0, \quad 0 \le s \le 2. \tag{1.5}$$

Furthermore we will show that the appropriate characteristic boundary condition for (1.1) subject to (1.4) is

$$\frac{du}{d\sigma}(t,t) = \frac{du}{d\tau}(1+t, 1-t), \tag{1.6}$$

where $\frac{du}{d\sigma}$ and $\frac{du}{d\tau}$ represent the directional derivatives of u in the characteristic directions, $e_1 = (-1,-1)$ and $e_2 = (1,-1)$, respectively.

2. **The first CIV boundary value problem.** For convenience we make the change of independent variables $s = \frac{1}{\sqrt{2}}(\sigma - \tau) + 1,\ t = -\frac{1}{\sqrt{2}}(\sigma + \tau) + 1$. In $\sigma\tau$-coordinates the problem (1.1), (1.2), (1.5) reads

$$w_{\sigma\tau} = -\frac{\lambda}{2} Pw = F(\sigma,\tau), \quad (\sigma,\tau) \text{ in } R', \tag{2.1}$$

$$w(\sigma,0) = w(0,\sqrt{2}-\sigma), \quad 0 \le \sigma \le \sqrt{2}, \tag{2.2}$$

$$w(\sigma,\sqrt{2}-\sigma) = g(\sqrt{2} \quad \sigma) = G(\sigma), \quad 0 \le \sigma \le \sqrt{2}, \tag{2.3}$$

$$w(0,0) = 0 = G(0) = G(\sqrt{2}), \tag{2.4}$$

where w, P are u and p at (σ,τ), F, G are defined to be $-\frac{\lambda}{2}Pw$ and $g(\sqrt{2}\sigma)$ respectively and R' is the region.

$$R' = \{(\sigma,\tau) : 0 \le \sigma \le \sqrt{2}, 0 \le \tau \le \sqrt{2} - \sigma\}.$$

Integrating (2.1) over the rectangle $Q = [0,\sigma] \times [0,\tau] \subseteq R'$, we obtain

$$w(\sigma,\tau) = w(0,\tau) + w(\sigma,0) - w(0,0) + \int_0^\tau \int_0^\sigma F(\sigma',\tau')\, d\sigma'\, d\tau'. \tag{2.5}$$

Extending the corner (σ,τ) of the rectangle of Q to meet the line $\sigma + \tau = \sqrt{2}$ we have from (2.5)

$$w(\sigma,\sqrt{2}-\sigma) = G(\sigma) = w(0,\sqrt{2}-\sigma) + w(\sigma,0) + \int_0^{\sqrt{2}-\sigma} \int_0^\sigma F(\sigma',\tau')\, d\sigma'\, d\tau',$$

which upon using (2.2) yields

$$w(\sigma,0) = \frac{1}{2}G(\sigma) - \frac{1}{2}\int_0^{\sqrt{2}-\sigma} \int_0^\sigma F(\sigma',\tau')\, d\sigma'\, d\tau', \tag{2.6}$$

and

(2.7) $$w(0,\tau) = w(\sqrt{2}-\tau,0) = \frac{1}{2}G(\sqrt{2}-\tau) - \frac{1}{2}\int_0^\tau \int_0^{\sqrt{2}-\tau} F(\sigma',\tau')\,d\sigma'\,d\tau'.$$

Using (2.6) and (2.7) in (2.5) we have

$$w(\sigma,\tau) = \frac{1}{2}[G(\sigma) + G(\sqrt{2}-\tau)] - \frac{1}{2}\int_0^{\sqrt{2}-\sigma} \int_0^\sigma F(\sigma',\tau')\,d\sigma'\,d\tau'$$
$$- \frac{1}{2}\int_0^\tau \int_0^{\sqrt{2}-\tau} F(\sigma',\tau')\,d\sigma' d\tau' + \int_0^\sigma \int_0^\tau F(\sigma',\tau')\,d\sigma'\,d\tau',$$

or

$$w(\sigma,\tau) = \frac{1}{2}[G(\sigma) + G(\sqrt{2}-\tau)] + \int_{R'}\int K(\sigma,\tau;\sigma',\tau')F(\sigma',\tau')\,d\sigma'\,d\tau',$$

where $K(\sigma,\tau;\sigma',\tau')$ is the symmetric Green's function [1] best described graphically in Figure 2.1.

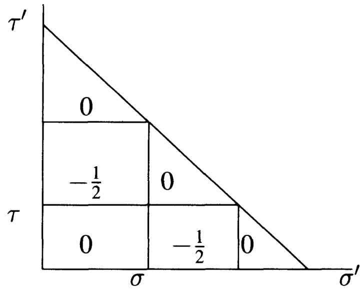

FIGURE 2.1

Therefore in st-coordinates (1.1), (1.2), (1.5) is equivalent to

(2.8) $$u(s,t) = \frac{1}{2}[g(s-t) + g(s+t)] + \lambda\int_R\int M(s,t;s',t')p(s',t')u(s',t')\,ds'\,dt',$$

where $M = -\frac{1}{2}K$.

If we write (2.8) in the form

(2.9) $$u = f + \lambda L[u]$$

where L is the integral operator of (2.8), then in the space of weighted square integrable functions $L_2^p(R)$, for continuous positive p in R, the theory of symmetric completely continuous operators [2] and the result of [1] yields.

THEOREM 2.1. *Let μ_i, $i = 1,2,\ldots$ be the eigenvalues of the homogeneous functional equation $u = \mu L[u]$ with corresponding eigenfunctions ϕ_i. Then*

1) if $\lambda \neq \mu_i$ for all i, (2.9) has a unique solution.

2) if $\lambda = \mu_i$ for some i and f is orthogonal to ϕ_i associated with μ_i, (2.9) has infinitely many solutions.

3) if $\lambda = \mu_i$ for some i and the orthogonality condition of 2 is not satisfied, (2.9) has no solution.

https://doi.org/10.4153/CMB-1992-049-2 Published online by Cambridge University Press

3. **The second CIV boundary value problem.** *Now we consider (1.1), (1.4) and (1.6). Once again the change of variables* $s = \frac{1}{\sqrt{2}}(\sigma - \tau) + 1$ *and* $t = -\frac{1}{\sqrt{2}}(\sigma + \tau) + 1$ *conveniently transforms the problem to*

$$w_{\sigma\tau} = F(\sigma,\tau), \quad (\sigma,\tau) \text{ in } R', \tag{3.1}$$

$$w_\sigma(\sigma,\sqrt{2}-\sigma) + w_\tau(\sigma,\sqrt{2}-\sigma) = 0, \quad 0 \le \sigma \le \sqrt{2}, \tag{3.2}$$

$$w_\sigma(\sigma,0) = w_\tau(0,\sqrt{2}-\sigma), \quad 0 \le \sigma \le \sqrt{2}, \tag{3.3}$$

$$w(0,0) = 0. \tag{3.4}$$

Integrating (3.1) with respect to σ *and* τ *respectively we have*

$$w_\tau(\sigma,\tau) - w_\tau(0,\tau) = \int_0^\sigma F(\sigma',\tau)\,d\sigma' \tag{3.5}$$

$$w_\sigma(\sigma,\tau) - w_\sigma(\sigma,0) = \int_0^\tau F(\sigma,\tau')\,d\tau' \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$w_\tau(\sqrt{2}-\tau,\tau) - w_\tau(0,\tau) = \int_0^{\sqrt{2}-\tau} F(\sigma',\tau)\,d\sigma' \tag{3.7}$$

$$w_\sigma(\sqrt{2}-\tau,\tau) - w_\sigma(\sqrt{2}-\tau,0) = \int_0^\tau F(\sqrt{2}-\tau,\tau')\,d\tau'. \tag{3.8}$$

Adding (3.7) to (3.8) and using condition (3.2) we have

$$w_\tau(0,\tau) + w_\sigma(\sqrt{2}-\tau,0) = -\int_0^{\sqrt{2}-\tau} F(\sigma',\tau)\,d\sigma' - \int_0^\tau F(\sqrt{2}-\tau,\tau')\,d\tau'. \tag{3.9}$$

The condition (3.3) and equality (3.9) now yields

$$w_\tau(0,\tau) = -\frac{1}{2}\left[\int_0^{\sqrt{2}-\tau} F(\sigma',\tau)\,d\sigma' + \int_0^\tau F(\sqrt{2}-\tau,\tau')\,d\tau'\right] = w_\sigma(\sqrt{2}-\tau,0), \tag{3.10}$$

$$w_\sigma(\sigma,0) = -\frac{1}{2}\left[\int_0^\sigma F(\sigma',\sqrt{2}-\sigma)\,d\sigma' + \int_0^{\sqrt{2}-\sigma} F(\sigma,\tau')\,d\tau'\right]. \tag{3.11}$$

Integrating (3.10) and (3.11) over $[0,\tau]$ *and* $[0,\sigma]$ *respectively and using (3.4) results in*

$$w(0,\tau) = -\frac{1}{2}\left[\int_0^\tau\int_0^{\sqrt{2}-\tau'} F(\sigma',\tau')\,d\sigma'\,d\tau' + \int_0^\tau\int_0^{\tau'} F(\sqrt{2}-\tau',\tau'')\,d\tau''\,d\tau'\right], \tag{3.12}$$

$$w(\sigma,0) = -\frac{1}{2}\left[\int_0^\sigma\int_0^{\sigma'} F(\sigma'',\sqrt{2}-\sigma')\,d\sigma''\,d\sigma' + \int_0^\sigma\int_0^{\sqrt{2}-\sigma'} F(\sigma',\tau')\,d\tau'\,d\sigma'\right]. \tag{3.13}$$

Substituting $w(\tau,0)$ *and* $w(0,\sigma)$ *from (3.12) and (3.13) in (2.5) with* $w(0,0)=0$, *we find*

$$\begin{aligned} w(\sigma,\tau) = {} & \int_0^\sigma\int_0^\tau F(\sigma',\tau')\,d\sigma'\,d\tau' - \frac{1}{2}\int_0^\sigma\int_0^{\sigma'} F(\sigma'',\sqrt{2}-\sigma')\,d\sigma''\,d\sigma' \\ & - \frac{1}{2}\int_0^\sigma\int_0^{\sqrt{2}-\sigma'} F(\sigma',\tau')\,d\tau'\,d\sigma' - \frac{1}{2}\int_0^\tau\int_0^{\sqrt{2}-\tau'} F(\sigma',\tau')\,d\sigma'\,d\tau' \\ & - \frac{1}{2}\int_0^\tau\int_0^{\tau'} F(\sqrt{2}-\tau',\tau'')\,d\tau''\,d\tau', \end{aligned}$$

which can be written in the form

$$w(\sigma,\tau) = \lambda\int_R\int N(\sigma,\tau;\sigma',\tau')F(\sigma',\tau')\,d\sigma'\,d\tau'$$

where $N(\sigma,\tau;\sigma',\tau')$ *is the symmetric Green's function whose values in different regions are demonstrated in Figure 3.1.*

Converting the σ,τ variables back to s,t we have

$$u(s,t) = \lambda\int_R\int S(s,t;s',t')p(s',t')u(s',t')\,ds'\,dt',$$

where $S = -\frac{1}{2}N$.

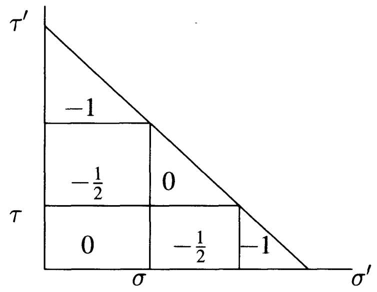

FIGURE 3.1

REFERENCES

1. K. Kreith, *Symmetric Green's Functions for a Class of CIV Boundary Value Problems*, Canad. Math. Bull. (3) **31**(1988), 272–279.
2. F. Riesz and B. Sz. Nagy, *Functional Analysis*, Ungar, New York, 1955.

Western Kentucky University
Bowling Green, Kentucky
U.S.A. 42101

https://doi.org/10.4153/CMB-1992-049-2 Published online by Cambridge University Press