Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 985–1001

PSEUDO-MEASURE ENERGY AND SPECTRAL SYNTHESIS

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Introduction. In this paper we develop a natural notion of continuous pseudo-measure and study the Stieltjes integral with respect to a given pseudo-measure. The common feature to these two topics is the essential appearance in both of integrals having the form

$$||f, \tau|| = \left(\int_0^{2\pi} |f(\gamma + \tau) - f(\gamma)|^2 d\gamma \right)^{\frac{1}{2}}.$$

Such integrals come about naturally when one defines the energy of distributions other than measures [**6**]. The reasons to study continuous pseudo-measures are to find properties analogous with those of continuous measures, and to discover more about the structure of pseudo-measures because of their importance in harmonic analysis, and particularly in spectral synthesis (e.g., [**4**; **15**]). The Stieltjes integral with respect to a pseudo-measure is studied because of its intimate relation with spectral synthesis (e.g., § 5); the key observations on this matter were initially made by Beurling [**6**].

Continuous pseudo-measures are defined and characterized in § 1; a norm defined in this context yields the necessary translation invariance to prove a Fejér theorem. Further, this norm is used to introduce a special type of (Riemann) set of uniqueness in § 4.

In § 2 we accumulate some technical information on the growth of $||f, \tau||$; this is useful in § 5. The relation between continuous functions and continuous pseudo-measures is studied in § 3.

We develop the Beurling integral [**6**, pp. 2959–2962] in § 5. Using this integral we give a different proof to the Beurling-Pollard theorem (e.g., [**15**, pp. 61 ff.]) as well as an even stronger statement, but of the same type. We also indicate the limited scope of Beurling's integral for the solution of spectral synthesis problems (cf., Epilogue).

Acknowledgement. I would like to thank Fulvio Ricci and Raymond Johnson for their interest and expertise on many facets of this paper. Further, Professor Johnson's technical assistance has been invaluable especially in § 3.

0. Notation. $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ is the circle group. Haar measure m on $\mathbf{T}$ is

Received March 6, 1973 and in revised form April 5, 1973.

normalized by $m(\mathbf{T}) = 1$ and denoted in integrals as

$$\int_{\mathbf{T}} \ldots d\gamma \quad or \quad \int_0^{2\pi} \ldots d\gamma.$$

$A(\mathbf{T})$ is the space of absolutely convergent Fourier series $\phi(\gamma) = \sum a_n e^{in\gamma}$, with norm $\|\phi\|_A = \sum |a_n|$. The set of zeros of $\phi \in A(\mathbf{T})$ is denoted by $Z\phi \subseteq \mathbf{T}$. $A'(\mathbf{T})$, the space of pseudo-measures, is the dual of $A(\mathbf{T})$ with canonical norm $\| \quad \|_{A'}$. For $E \subseteq \mathbf{T}$ closed,

$$A'(E) = \{T \in A'(T) : \text{supp } T \subseteq E\},$$
$$A_0'(E) = \{T \in A'(E) : \lim_{|n| \to \infty} \hat{T}(n) = 0\}.$$

$M(E)$, the space of Radon measures supported by E, is contained in $A'(E)$ and has total variation norm $\|T\|_1$. Notationally,

$$M_c(E) = \{T \in M(E) : T \text{ is continuous}\},$$
$$M_0(E) = \{T \in M(E) : \lim_{|n| \to \infty} \hat{T}(n) = 0\}.$$

It is well-known that $M_0(E) \subseteq M_c(E)$. Also, in this paper, we assume without loss of generality that $\hat{T}(0) = 0$ for T in any of these spaces. T_τ is the translate of T:

$$T_\tau \sim \sum c_n e^{in(\gamma - \tau)} \quad \text{where} \quad T \sim \sum c_n e^{in\gamma}.$$

The notation

$$f \sim T$$

shall mean that $T \sim \sum c_n e^{in\gamma}$ is in $A'(\mathbf{T})$ and

$$f \sim d_0 + \sum \frac{c_n}{in} e^{in\gamma}$$

(d_0 an arbitrary constant). $f' = T$ distributionally, and, from the Hausdorff-Young theorem, $f \in \bigcap_p L^p(\mathbf{T})$.

$\phi \in A(\mathbf{T})$ (respectively, $T \in A'(\mathbf{T})$) is *synthesizable* if for all $S \in A'(Z\phi)$ (respectively, for all $\psi \in A(\mathbf{T})$ with supp $T \subseteq Z\psi$) we have $\langle S, \phi \rangle = 0$ (respectively, $\langle T, \psi \rangle = 0$). Set

$$A_S'(E) = \{T \in A'(E) : \text{for all } \phi \in A(\mathbf{T}) \text{ with } E \subseteq Z\phi, \langle T, \phi \rangle = 0\}.$$

E is a *Helson set* if $M(E) = A_S'(E)$ and a *uniqueness set* if $A_0'(E) = \{0\}$.

All other notation is completely standard and can be found in [**15**; **25**]. Also, some of our proofs are more general than the context of $A(\mathbf{T})$, $A'(\mathbf{T})$ duality, but we have chosen this latter setting for the obvious reason of unity.

1. Continuous pseudo-measures. Because of Wiener's characterization of continuous measures we say $T \in A'(\mathbf{T})$ is *continuous* ($T \in A_c'(\mathbf{T})$) if

$$(1.1) \quad \lim_{N \to \infty} \frac{1}{2N+1} \sum_{|n| \leq N} |\hat{T}(n)|^2 = 0.$$

We define the norm

$$M(T) = \sup_N \frac{1}{2N+1} \sum_{|n| \leqq N} |\hat{T}(n)|$$

on $A'(\mathbf{T})$. Clearly (by Hölder's inequality), (1.1) holds if and only if

$$\text{(1.2)} \quad \lim_{N \to \infty} \frac{1}{2N+1} \sum_{|n| \leqq N} |\hat{T}(n)| = 0;$$

and it is obvious that

$$\text{for all } T \in A'(\mathbf{T}), \quad M(T) \leqq ||T||_{A'}.$$

From Parseval's formula, if $f \sim T$,

$$\text{(1.3)} \quad ||f, \tau||^2 = 4 \sum{}' \frac{|\hat{T}(n)|^2}{n^2} \sin^2 \frac{n\tau}{2}.$$

Because of Theorem 1.1(b) we prove that

$$f \sim T \text{ implies } ||f, \tau||^2 = O(\tau), \quad \tau \to 0.$$

In fact, from (1.3),

$$||f, \tau||^2 \leqq K \sum_1 \frac{1}{n^2} (1 - \cos n\tau),$$

and the Fourier series on the right hand side represents the even function $\phi(\tau) = \pi\tau - \tau^2/2$ on $[0, \pi)$.

With this order condition on pseudo-measures in terms of $||f, \tau||$, the following characterization of $T \in A_c'(T)$ (particularly, Theorem 1.1(b)) is interesting.

THEOREM 1.1. *The following are equivalent for* $T \in A'(\mathbf{T})$:
(a) $T \in A_c'(\mathbf{T})$;
(b) $||f, \tau||^2 = o(\tau), \quad \tau \to 0,$ *where* $f \sim T$;
(c) $M(T - T_\gamma) \to 0$ *as* $\gamma \to 0$.

Remark. Theorem 1.1 is proved using Wiener's original computation [**24**] to characterize continuous measures. In fact, the calculations for (a) $\Leftrightarrow$ (b) are precisely Wiener's (e.g., Zygmund's first edition, p. 221); whereas, those for (a) $\Leftrightarrow$ (c) stem from [**11**]. There are, of course, more elegant proofs of Wiener's result (e.g., [**25**, I, pp. 107–108; **16**, Chapter 1]), but these are not as adaptable to our generalization for pseudo-measures.

We know from Wiener's theorem that $M_0(\mathbf{T}) \subseteq M_c(\mathbf{T})$. From Theorem 1.1 we have

PROPOSITION 1.1. (a) *Let* $T \in A'(\mathbf{T})$. $T \in A_0'(\mathbf{T}) \Leftrightarrow ||T - T_\gamma||_{A'} \to 0$ *as* $\gamma \to 0$.
(b) $A_0'(\mathbf{T}) \subseteq A_c'(\mathbf{T})$.

Example 1.1. Goes (1967) proved that if $\{n_k\}$ is strictly increasing to infinity then $T \sim \sum_{k=1} \sin n_k\gamma$ is not a measure. Choose such a sequence $\{n_k\}$ with the property that $\lim_k k/n_k = 0$. Then

$$\frac{1}{N}\sum_1^N |\hat{T}(n)| \leqq \frac{1}{N_k}\sum_1^{N_k} |\hat{T}(n)| = \frac{k}{N_k}$$

where, given N, $N_k \leqq N$ is the largest integer with $|\hat{T}(N_k)| = 1$. Consequently $T \in A_c'(\mathbf{T})\backslash(A_0'(\mathbf{T}) \cup M(\mathbf{T}))$.

Remark. A straightforward calculation first recorded by S. M. Lozinskii for measures [**17**] shows that if $T \in A_c'(\mathbf{T})$ then

$$\lim_{N\to\infty} \frac{1}{\log(2N+1)} \sum_{|n|\leqq N} \left|\frac{\hat{T}(n)}{n}\right| = 0.$$

In light of the continuity of translation property of continuous pseudo-measures (Theorem 1.1) and the fact that such continuity of translation is the key to Fejér's theorem in $L^1(\mathbf{T})$, we ask if T continuous implies $M(T - \sigma_N T) \to 0$, where

$$\sigma_N T(\gamma) = \sum_{|n|\leqq N} \hat{T}(n)\left(1 - \frac{|n|}{N+1}\right) e^{in\gamma}.$$

Remark. Obviously, $||T - \sigma_N T||_{A'} \to 0$ for $T \in A_0'(\mathbf{T})$ (Proposition 1.1(a)), and the result is always false for $T \in A'(\mathbf{T})\backslash A_0'(\mathbf{T})$. Similarly, since $L^1(\mathbf{T}) \subseteq M_0(\mathbf{T})$ is closed in the total variation norm, it never happens that $||\mu - \sigma_N\mu||_1 \to 0$ for $\mu \in M(\mathbf{T})\backslash L^1(\mathbf{T})$. On the other hand if $\mu \in M(\mathbf{T})$ then $\mu \in L^1(\mathbf{T})$ if and only if some subsequence of $\{\sigma_N\mu\}$ converges weakly in $L^1(\mathbf{T})$.

A routine computation yields

THEOREM 1.2 *If* $T \in A_c'(\mathbf{T})$, *then*

$$\lim_N M(T - \sigma_N T) = 0.$$

2. Integrability conditions. Throughout, we shall test functions according to the growth of $||f, \tau||$, but with motives along the lines of Beurling's work [**6**; **7**]. We refer to Herz' classification [**13**] of various Lipschitz (et al) function spaces for other uses of such differences. We now record some routine information on $||f, \tau||$ that will be useful in what follows.

PROPOSITION 2.1. *Let* $f, g \in L^1(\mathbf{T})$ *and assume*

(2.1) $\quad ||f, \tau|| = O(\tau^{1/2}), \quad \tau \to 0.$

(a) *If* $g \in C^1(\mathbf{T})$ *then*

(2.2) $$\int_0^{2\pi} ||g, \tau||\,||f, \tau|| \frac{d\tau}{\tau^2} < \infty.$$

(b) *If* $\sum|\hat{g}(n)|^2|n| < \infty$ *then for any* $\alpha < 2$

$$(2.3)\quad \int_0^{2\pi} ||g, \tau|| \, ||f, \tau|| \frac{d\tau}{\tau^{\alpha}} < \infty.$$

(c) *If* $g \in \Lambda_\alpha$, $\alpha > 1/2$, *then* (2.2) *holds.*

Proof. (a) By hypothesis $||g, \tau|| = O(\tau)$, $\tau \to 0$, and so

$$\int_0^{2\pi} ||g, \tau|| \, ||f, \tau|| \frac{d\tau}{\tau^2} \leqq K \int_0^{2\pi} \tau^{-\frac{1}{2}} d\tau < \infty.$$

(b) By Parseval's formula and our hypothesis

$$\int_0^{2\pi} ||g, \tau|| \, ||f, \tau|| \frac{d\tau}{\tau^{\alpha}} \leqq K \int ||g, \tau|| \tau^{1/2-\alpha} d\tau$$

$$= 2K \int \tau^{3/2-\alpha} \left(\tau^{-2} \sum |\hat{g}(n)|^2 \sin^2 \frac{n\tau}{2} \right)^{1/2} d\tau$$

$$\leqq 2K \left[\int \tau^{3-2\alpha} d\tau \right]^{3/2} \left[\sum |\hat{g}(n)|^2 \int \tau^{-2} \sin^2 \frac{n\tau}{2} d\tau \right]^{1/2}$$

$$\leqq K_\alpha \sum |\hat{g}(n)|^2 |n|.$$

(c) If $g \in \Lambda_\alpha$, $\alpha > 1/2$,

$$\int_0^{2\pi} \omega_\infty(g, \tau)/\tau^{3/2} d\tau < \infty$$

and so

$$\int_0^{2\pi} \omega_2(g, \tau)/\tau^{3/2} d\tau < \infty.$$

(2.2) then follows from (2.1).

Remarks. 1. (2.3) is true for any $\alpha < 3/2$ if in addition to (2.1) we take $g \in L^2(\mathbf{T})$.

2. Clearly, Proposition 2.1 is true for pseudo-measures.

From (1.3) and the fact that

$$\int_0^{2\pi} \tau^{-2} \sin^2 n\tau d\tau = O(|n|), \quad |n| \to \infty,$$

we have

PROPOSITION 2.2. *If* $f \sim T$ *and*

$$(2.4)\quad B_T = \sum \frac{|\hat{T}(n)|^2}{|n|} < \infty$$

then

(2.5) $||f, \tau|| = O(\tau), \quad \tau \to 0.$

Goes [**10**] and Stein [**20**] have proved (for the cases $p = 1, p = 2$, respectively): let $f \in L^2(\mathbf{T})$, $\hat{f}(0) = 0$, and $p \in [1, 2]$; $f \sim T$ if and only if there is M_p such that for any finite sequence of disjoint intervals (a_k, b_k),

$$\left\| \sum \left(f(b_k - x) - f(a_k - x)\right)\right\|_p \leqq M_p.$$

Using Parseval's formula this condition is (for $d_n = O(1)$, $|n| \to \infty$, and $p = 2$)

$$(2.6) \quad \sum{}' \left|\frac{d_n}{n}\right|^2 \left|\sum_k \left(e^{inb_k} - e^{ina_k}\right)\right|^2 \leqq M.$$

The trigonometric sums in (2.6) are reminiscent of the techniques in [**3**].

The condition (2.4) was considered by Beurling [**5**] who proved that for such $f \sim T$ the Fourier series of f diverges only on a set of zero exterior capacity. To fill in the picture, we know that (2.4) implies $T \in A_c'(\mathbf{T})$ (Proposition 2.4). It is an immediate calculation that when (2.4) is satisfied the Dirichlet integral $D(f)$ is given by

$$D(f) \equiv \int_0^1 \int_0^{2\pi} \left|\frac{\partial f(r, \gamma)}{\partial r}\right|^2 r d\gamma dr = \sum{}' \frac{|\hat{T}(n)|^2}{2|n| + 1},$$

where $f(r, \gamma) = \sum \hat{f}(n) r^{|n|} e^{in\gamma}$.

With regard to Proposition 2.1 and (2.4) we use Parseval's formula (twice) and Hölder's inequality to compute

PROPOSITION 2.3. *If* $B_T < \infty$ *for* $f \sim T$ *and* $\sum |\hat{g}(n)|^2 |n| < \infty$ *then* (2.2) *holds.*

There is, of course, no *a priori* relation between $O(\tau)$ and $o(\tau^{1/2})$. On the other hand, from a standard technique we prove

PROPOSITION 2.4. *Let* $f \sim T$ *satisfy* (2.4). *Then* $T \in A_c'(\mathbf{T})$.

Proof. Set $a_n = |\hat{T}(n)|^2$ and $b_n = \sum_{|k| \leqq n} a_k / |k|$. For each $n \geqq 1$, $n(b_n - b_{n-1}) = a_n + a_{-n}$ $(b_0 = 0)$. Thus

$$\sum_{|n| \leqq N} a_n = N b_N - \sum_1^{N-1} b_k$$

and so

$$\frac{1}{N} \sum_{|n| \leqq N} |\hat{T}(n)|^2 = \sum_{|n| \leqq N}{}' \frac{|\hat{T}(n)|^2}{|n|} - \frac{1}{N} \sum_{k=1}^{N-1} \left(\sum_{-k}^{k}{}' \frac{|\hat{T}(n)|^2}{|n|}\right).$$

We are done by (2.4) and the fact that the arithmetic mean on the right hand side will also converge to B_T.

Example 2.1. Let $g = \chi_I$, I an interval. Then $||g, \tau|| = (2\tau)^{1/2}$. To see this let $I = (-\alpha, \alpha)$ and so

$$||g, \tau|| = 4\alpha - 2 \int \chi_I(\gamma + \tau)\chi_I(\gamma)d\gamma = 4\alpha - 2(2\alpha - \tau).$$

Example 2.2. Let g be the de la Vallée–Poussin kernel (trapezoid function): $g = 1$ on $[-b, b]$, $a > b$, $g \geqq 0$, $g = 0$ off $(-a, a)$, and g linear on $[-a, -b]$, $[b, a]$. Then

$$\hat{g}(n) = \frac{2}{n^2} \frac{\cos na - \cos nb}{b - a}$$

and so from Parseval's formula

$$||g, \tau||^2 = \frac{64}{(b-a)^2} \sum{}' \frac{1}{n^4} \sin^2 \frac{n\tau}{2} \sin^2 n \frac{a+b}{2} \sin^2 n \frac{a-b}{2}.$$

Consequently,

$$||g, \tau|| = O(\tau), \quad \tau \to 0.$$

3. $A_c'(\mathbf{T})$ and continuous functions. Suppose $f \sim T$. If $T \in M(\mathbf{T})$ then $T \in M_c(\mathbf{T})$ is characterized by the fact that f is continuous.

(a) We show by example that there are no such implications in the general setting of $f \sim T$ and $T \in A_c'(\mathbf{T})$.

Example 3.1. Consider the Hardy-Littlewood function

$$f(\gamma) \backsim \sum_1^\infty \frac{e^{in \log n}}{n} e^{in\gamma}.$$

f is an element of $\Lambda_{1/2}$ and is not an element of $A(\mathbf{T})$. We see from definition that if $f \backsim T$ then $T \notin A_c'(\mathbf{T})$. In light of Bernstein's theorem it is interesting that we can prove: *if $f \backsim T$ and $f \in A(\mathbf{T})$ then $T \in A_c'(\mathbf{T})$.*

Example 3.2. There are $f \in \bigcap_p L^p(\mathbf{T})\backslash L^\infty(\mathbf{T})$ such that $f \sim T$ and $T \in A_c'(\mathbf{T})$. For example take

$$f(\gamma) \sim \sum_2 \frac{1}{k[\log k]} \exp i\gamma k[\log k].$$

In particular, $f \notin C(\mathbf{T})$. With some extra work f can be taken in $L^\infty(\mathbf{T})$.

(b) Given Example 3.1 we wish to find conditions on continuous f so that $f \sim T$ and $T \in A_c'(\mathbf{T})$.

PROPOSITION 3.1. *Let $f \in C(\mathbf{T})$, $f \sim T$. Assume there is $g \in L^2(\mathbf{T})$ such that*

$$|(f_{-\tau} - f)/\tau| \leqq g \quad a.e. \quad (f_{-\tau}(\gamma) = f(\gamma + \tau)).$$

Then $T \in A_c'(\mathbf{T})$.

https://doi.org/10.4153/CJM-1974-092-3 Published online by Cambridge University Press

Proof. Take $\epsilon > 0$. Since $f_{-\tau} \to f$ pointwise (as $\tau \to 0$) there is A, $mA < \epsilon$, such that $f_{-\tau} \to f$ uniformly on $A^\sim$ (the complement of A). Now

$$\frac{1}{\tau}\|f, \tau\|^2 \leqq \int_{A^\sim} |f_{-\tau} - f| \left|\frac{f_{-\tau} - f}{\tau}\right| d\gamma + \int_A$$

$$\leqq \sup_{\gamma \in A^\sim} |f_{-\tau}(\gamma) - f(\gamma)| \int |g| + 2\|f\|_\infty (mA)^{\frac{1}{2}} \|g\|_2.$$

The last expression yields the condition of Theorem 1.1(b).

(c) Before pursuing the opposite question of finding conditions on $f \sim T$, $T \in A_c'(\mathbf{T})$, to ensure that $f \in C(\mathbf{T})$, it will be convenient to give some explicit non-trivial examples of $f \sim T$.

A function $f \in L^2(\mathbf{T})$ is of *bounded deviation* if for each $(a, b) \subseteq [0, 2\pi)$,

$$\int_a^b f(\gamma) e^{-in\gamma} d\gamma = O\left(\frac{1}{|n|}\right), \quad |n| \to \infty.$$

This notion was introduced by Hadamard (to generalize bounded variation) in his thesis (J. de Math. *8* (1892), 101–186, especially p. 154); and he proved that if $f \in A(\mathbf{T})$ satisfies $\hat{f}(n) = O([|n| \log |n|]^{-1})$, $|n| \to \infty$, then f is of bounded deviation.

The following functions are of bounded deviation:

(3.1) $f(\gamma) = \gamma^\alpha \sin 1/\gamma^\alpha, \quad \alpha > 0,$

(3.2) $f(\gamma) = \sin \log |\gamma|,$

(3.3) $f(\gamma) = (1 - e^{i\gamma})^{-i}[1 - \log (1 - e^{i\gamma})]^{-\alpha}, \quad \alpha \geqq 0$

(defined mod 2π). (3.1) and (3.2) are due to Bray [**8**, pp. 156–157] who obtained them as examples from two general results, respectively. (3.3) is due to Hille [**14**] who also showed that this f is of bounded variation if and only if $\alpha > 0$. Bray, in a Comptes Rendus note (t. *190*, p. 1371) translated some of his results from [**8**] to generalize Hadamard's theorem: $f \in L^\infty(T)$ and $\hat{f}(n) = O([|n| \log |n|]^{-1})$, $|n| \to \infty$, imply f is of bounded deviation. Also, if f is of bounded deviation then $f \in L^\infty(\mathbf{T})$. For perspective, note that there are $f \sim T$ (even with only countable support!) which are not in $L^\infty(\mathbf{T})$ [**3**].

(d) We start to investigate continuity properties of f, given $f \sim T$ and $T \in A_c'(\mathbf{T})$, with the following

PROPOSITION 3.2. *Let $f \sim T$ and $T \in A_c'(\mathbf{T})$.*
(a) *f has no jump discontinuities.*
(b) supp *T is perfect.*

Proof. (a) Suppose x_0 is a jump discontinuity so that $f(x_0 \pm)$ exist, and $|f(x_0+) - f(x_0-)| = 2\alpha > 0$. Take τ_0 so that if $\tau < \tau_0$ and $x \in [x_0 - \tau/2, x_0)$,

$|f(x+\tau) - f(x)| > \alpha$. Thus

$$\frac{1}{\tau} ||f, \tau||^2 \geqslant \alpha^2/2 > 0,$$

and this contradicts Theorem 1.1(b).

(b) If supp T has an isolated point, f must have a jump discontinuity.

Example 3.3. Set $f(x) = x \sin(1/x)$ on $[-\pi, \pi]$. Then f is continuous, $f \sim T$, and $T \in A_c'(\mathbf{T})$. We need only verify Theorem 1.1(b). Note that

$$\frac{1}{\tau} ||f, \tau||^2 = \frac{2}{\tau} \int_{-\pi}^{\pi} x^2 \sin\frac{1}{x}\left(\sin\frac{1}{x} - \sin\frac{1}{x+\tau}\right) dx$$
$$- 2 \int_{-\pi}^{\pi} x \sin\frac{1}{x+\tau} \sin\frac{1}{x}\, dx,$$

so that since

$$\int_{-\pi}^{\pi} x \sin^2\frac{1}{x}\, dx = 0$$

and

$$\int_{-\pi}^{\pi} x^2 \sin\frac{1}{x} \cos\frac{1}{x}\, dx = 0,$$

we are done.

Example 3.4. Beginning with the f of (3.2), take a Cantor set E, $mE > 0$, and define a function f_n on each contiguous interval in terms of f so that $g = \sum f_n$ is not continuous a.e. and $g' \in A'(\mathbf{T})$. We observe that $g' \notin A_c'(\mathbf{T})$. It is enough to prove $f' \notin A_c'(\mathbf{T})$ and this follows using Theorem 1.1(b) and a routine estimate with the mean value theorem.

Some routine calculations yield the following criteria for the continuity of f given information related to the continuity of T, where $f \sim T$.

PROPOSITION 3.3. *Suppose $f \sim T$.*

(a) *If for some $\epsilon > 0$*

$$\frac{1}{\tau} ||f, \tau||^2 = O(\tau^\epsilon), \quad \tau \to 0,$$

then $f \in A(\mathbf{T})$.

(*b*) $$\sum_1^\infty \left(\frac{1}{2N+1} \sum_{|n| \leq N} |\hat{T}(n)|^2\right)^{\frac{1}{2}} < \infty$$

implies $f \in A(\mathbf{T})$.

(*c*) $$\lim_{N \to \infty} \frac{1}{2N+1} \sum_{|n| \leq N} |n\hat{T}(n)| = 0$$

implies $f \in A(\mathbf{T})$.

https://doi.org/10.4153/CJM-1974-092-3 Published online by Cambridge University Press

Part (a) is really a translation of Bernstein's theorem.

Example 3.5. Let $\{a_n\} \subseteq \mathbf{R}$ decrease and assume f has the Fourier series $\sum a_n \sin n\gamma$; then $f \in C(\mathbf{T})$ if and only if $f \sim T$ and $T \in A_c'(\mathbf{T})$.

4. Strong uniqueness sets.

THEOREM 4.1. (a) $A'(\mathbf{T})$, *normed by* M, *is not complete.*
(b) *If* $A'(E)$, *normed by* M, *is complete then* E *is a* U *set.*

Proof. (a) If $A'(\mathbf{T})$ is complete under M then M is equivalent to $|| \quad ||_{A'}$ by the open mapping theorem. Then if $T \in A_c'(\mathbf{T})$ we use Theorem 1.1 and Proposition 1.1 to obtain $T \in A_0'(\mathbf{T})$. This is obviously false generally, even for measures.

(b) Assume E is not U and take a non-zero $T \in A_0'(E)$. Set $S_n = e^{int}T_t$ so that $\hat{S}_n(m) = \hat{T}(m-n)$, $S_n \in A_0'(E)$, and $||T||_{A'} = ||S_n||_{A'}$ for all n. We prove $M(S_n) \rightarrow 0$. This yields the desired contradiction, for, by hypothesis, the M and A' norms are equivalent and so $\{M(S_n)\}$ is bounded away from 0. For notational convenience let $a_k = |\hat{T}(k)|$, so that we show

$$\lim_{k\rightarrow\infty} \sup_N \frac{1}{2N+1} \sum_{|n|\leqq N} a_{n-k} = 0.$$

Let $a_{j_0} = \sup a_j$ and set $b_j = a_{j_0}$ if $|j| \leqq |j_0|$. Let $a_{j_1} = \sup \{a_j : |j| > |j_0|\}$ and set $b_j = a_{j_1}$ for all $|j_1| \geqq |j| > |j_0|$, etc. Hence $\{b_j\}$ is symmetric and monotone decreasing to 0 as $|j| \rightarrow \infty$. Also $b_j \geqq a_j$ for all j and thus we need only prove that

$$\lim_{k\rightarrow\infty} \sup_N \frac{1}{2N+1} \sum_{|n|\leqq N} b_{n-k} = 0.$$

Fix k; then

$$\begin{aligned}
\sup_N \frac{1}{2N+1} \sum_{|n|\leqq N} b_{n-k} &= \frac{1}{2N_k+1} \sum_{-N_k-k}^{N_k-k} b_j \\
&= \frac{1}{2N_k+1} \left(\sum_{-N_k}^{N_k} - \sum_{N_k-k+1}^{N_k} + \sum_{-N_k-k}^{-N_k-1} b_j \right) \\
&\leqq \frac{1}{2N_k+1} \left(\sum_{-N_k}^{N_k} - \sum_{N_k-k+1}^{N_k} + \sum_{-N_k}^{-N_k+k+1} b_j \right) \\
&= \frac{1}{2N_k+1} \sum_{|j|\leqq N_k} b_j.
\end{aligned}$$

The last term tends to 0 as $k \rightarrow \infty$ since $b_j \rightarrow 0$ and by a property of Cesàro sums.

We say that E is a *strong* U *set* if $A'(E)$, normed by M, is complete (i.e., if there is K_E so that for all $T \in A'(E)$, $||T||_{A'} \leqq K_E M(T)$).

Example 4.1. Set $T_k = k^2 e^{ik^3\gamma}$. Then $M(T_k) \to 0$ as $k \to \infty$ whereas $T_k \not\to 0$ in the weak $*$ topology $(\sigma(A'(\mathbf{T}), A(\mathbf{T})))$. In fact, $M(T_k) = k^2/(2k^3 + 1) \to 0$ and for each k, $\sum_n \hat{T}_k(n) a_n = 1$ for

$$a_j = \begin{cases} 0, & j \neq k^3 \\ 1/k^2, & j = k^3. \end{cases}$$

Example 4.2. Every finite set is strong U since, in this case, $A'(E) = \mathbf{C}^n$ and all norms on $\mathbf{C}^n$ are equivalent.

Recall that discrete measures are not in $M_0(\mathbf{T})$ and note, more generally, that if μ is discrete and $\hat{\mu}(n) \to \alpha$ then $\mu = \alpha\delta$.

Remark. From the definition of strong U and the results of § 1 it is interesting to inquire if $M(\mu - \mu_\gamma) \to 0$ implies $||\mu - \mu_\gamma||_{A'} \to 0$ as $\gamma \to 0$ for $\mu \in M_c(E)$ and E strong U. The fact that this phenomenon can not happen follows from Proposition 1.1 and Theorem 4.1 (which prove E is of strict multiplicity when we assume $M(\mu - \mu_\gamma) \to 0$ and E strong U). Observe that if such an implication of "continuity of translation" were true then each strong U set would be countable (for if not take non-0 $\mu \in M_c(E)$, etc.). *A fortiori*, we have the incompatibility of $M(\mu - \mu_\gamma) \to 0$ and $||\mu - \mu_\gamma||_{A'} \to 0$, $\mu \in M_c(E)$, once we know that E contains a perfect Helson set (since such sets are not of strict multiplicity).

We shall not include the details of the following proposition, since, modulo trivialities, the original technique of Kahane and Salem to give examples of sets supporting no true pseudo-measures (e.g., [**4**, **15**]) goes through.

PROPOSITION 4.1. *Let $E \subseteq \mathbf{T}$ be perfect and take $a \in E$. There is a perfect $H \subseteq E$ satisfying $M(H) = A'(H)$ and $a \in H$.*

It is also not difficult to show that every infinite closed set E has a countably infinite Helson subset.

In any case when E is uncountable and closed, $M_c(E) \backslash M_0(E) \neq \emptyset$. More amusing, perhaps, is

PROPOSITION 4.2. *Let E be a perfect totally disconnected set. Then there is $\mu \in M_c(E) \backslash M_0(E)$ such that* supp $\mu = E$.

Proof. Let $\{a_n\} \subseteq E$ be a countable dense subset (of E) of inaccessible points. Choose a perfect H_1, Helson with $a_1 \in H_1$. Suppose a_{n_2} is the "next a" which is outside of H_1. Then there is a perfect open and closed F such that $a_{n_2} \in F$ and $F \cap H_1 = \emptyset$. Choose $H_2 \subseteq F$, Helson with $a_{n_2} \in H_2$. We proceed in this way to form $\{H_j\}$, $\{a_n\} \subseteq \bigcup H_j$, so that $\{H_j\}$ is a disjoint collection and $\overline{\bigcup H_j} = E$. Let $\mu_j \in M_c(H_j)$, supp $\mu_j = H_j$, so that $\mu_j \in M_0(H_j)$ since H_j is Helson. Set

$$x_1 = \overline{\lim_{|n| \to \infty}} |\hat{\mu}_1(n)| > 0$$

and define

$$\mu = \frac{\mu_1}{x_1} + \sum_2^\infty \frac{1}{2^j} \frac{\mu_j}{||\mu_j||_1}.$$

Clearly supp $\mu = E$ and $\mu \in M_c(E)$. Take $n_k \to \infty$ so that $\lim_k |\hat{\mu}_1(n_k)| = x_1$. Then for k large

$$|\hat{\mu}(n_k)| \geqslant \frac{|\hat{\mu}_1(n_k)|}{x_1} - \left| \sum_{j=2}^\infty \frac{1}{2^j} \frac{\hat{\mu}_j(n_k)}{||\mu_j||_1} \right| > \frac{1}{4}$$

since the infinite sum is less than 1/2. Thus $\mu \notin M_0(E)$.

$m(\bigcup H_j) = 0$ since $mH_j = 0$, although, naturally, $m(E)$ need not be 0.

5. Spectral synthesis and integration. Suppose $f \sim T$ and $\phi \in C(\mathbf{T})$. The *Beurling integral* of ϕ with respect to f is

$$(5.1) \quad B(\phi, f) = \frac{1}{\pi} \int_0^\infty \frac{1}{\tau^2} \left(\int_0^{2\pi} (\phi_{-\tau} - \phi)(x) H * (f_{-\tau} - f)(x) dx \right) d\tau$$

where H is the "conjugate distribution"

$$H \sim -i \sum \operatorname{sgn} n e^{inx}.$$

Naturally we define f periodically over $(-\infty, \infty)$.

Remark. The question arises as to the motivation of (5.1). The answer centers about a solution to synthesis. To give more detail let us consider a trivial case where it is known that synthesis holds. Each $\mu \in M(\mathbf{T})$ is synthesizable. Thus for any $\phi \in A(\mathbf{T})$ satisfying supp $\mu \subseteq Z\phi$, we have $\langle \mu, \phi \rangle = 0$. If $\hat{\mu}(0) = 0$ and $f \sim \mu$ then f is of bounded variation and so

$$\int \phi df = \langle \mu, \phi \rangle = 0.$$

The key for us now is to deduce that $\int \phi df = 0$ from first definitions. f is constant on any open interval contiguous to supp μ and so when we write out the Stieltjes integral we see that $\Sigma \phi(\xi_j)(f(x_j) - f(x_{j-1})) = 0$ when x_j, x_{j-1} are in the same contiguous interval. In the limit, the fact that $\phi = 0$ on supp μ assures that the other terms in such sums are 0.

Consequently, and generally, we see that if the inner product $\langle T, \phi \rangle$, $T \in A'(\mathbf{T})$, $\phi \in A(\mathbf{T})$, has a "Stieltjes-integral" representation and $\phi = 0$ on supp T then there is a good chance that synthesis holds, i.e., $\langle T, \phi \rangle = 0$. (Recall that if $\int \phi df$ or $\int f d\phi$ exists then they both exist and $\int \phi df = -\int f d\phi$.)

The first step from this integral representation point of view, then, is to see in what way and when we can write $\langle T, \phi \rangle$ as an integral. We have no choice but to start with Parseval's formula. For $f \sim T$, $\phi \in A(\mathbf{T})$, $B(\phi, f)$ is formally

$$(5.2)\quad \frac{1}{\pi}\int_0^\infty \frac{1}{\tau^2}\left(\sum \hat{\phi}(n)\hat{H}(-n)\hat{f}(-n)(e^{in\tau}-1)(e^{-in\tau}-1)\right)d\tau$$
$$= -i\sum{}' \hat{\phi}(n)\hat{f}(-n)n\left(\frac{4}{|n|\pi}\int_0^\infty \frac{\sin^2(n\tau/2)}{\tau^2}d\tau\right)$$
$$= -i\sum{}' \hat{\phi}(n)\hat{f}(-n)n = \langle T, \phi\rangle.$$

The presence of H is accounted for simply because without it the penultimate term in the previous calculation turns up an $|n|$.

Proposition 5.1. *Let $f \sim T$ and $\phi \in A(\mathbf{T})$. Then $B(\phi, f)$ exists and*

$$(5.3)\quad B(\phi, f) = \langle T, \phi\rangle.$$

Proof. We must verify that we can integrate under the summation sign in the first term of (5.2); but

$$\sum{}' |\hat{\phi}(n)\hat{f}(-n)| \int_0^\infty \frac{\sin^2(n\tau/2)}{\tau^2}d\tau < \infty$$

so that we can use Fubini's theorem.

Clearly, $B(\phi, f)$ will exist for $\phi \in C(\mathbf{T})$ and $f \sim T$ if

$$(5.4)\quad |B|(\phi, f) \equiv \frac{1}{\pi}\int_0^\infty \frac{1}{\tau^2}\left(\int_0^{2\pi} |(\phi_{-\tau}-\phi)(x)H * (f_{-\tau}-f)(x)|dx\right)d\tau < \infty.$$

For any function $\phi \in L^2(\mathbf{T})$ define (with Beurling [**6**; **7**]) the circular contraction

$$\phi_\rho(x) = \begin{cases} \phi(x), & \text{if } |\phi(x)| \leqslant \rho \\ \rho\dfrac{\phi(x)}{|\phi(x)|}, & \text{if } |\phi(x)| \geqslant \rho, \end{cases}$$

$\rho > 0$. A straightforward calculation shows that

$$|\phi_\rho(x+\tau) - \phi_\rho(x)| \leqq |\phi(x+\tau) - \phi(x)|$$

for any $\rho > 0$, x, τ.

Proposition 5.2. *Given $f \sim T$ and $\phi \in C(\mathbf{T})$ for which $|B|(\phi, f) < \infty$, then $|B|(\phi_\rho, f) < \infty$ and*

$$(5.5)\quad \lim_{\rho\to 0} |B|(\phi_\rho, f) = 0.$$

Proof. $|B|(\phi_\rho, f) \leqq |B|(\phi, f)$ by the way we have defined ϕ_ρ. From a Lebesgue dominated convergence theorem argument, the result follows.

Remark. We are now in a position to clarify the situation. Take $f \sim T$, $\phi \in A(\mathbf{T})$, supp $T \subseteq Z\phi$, and assume (5.4). From Proposition 5.1 and Proposition 5.2 we have (5.3) and (5.5). Now, $\phi_\rho = \phi$ on a neighborhood of supp T. Consequently it is not unreasonable to expect (in light of (5.3)) that with a strengthening of (5.4) we could further conclude that

$$(5.6) \quad B(\phi, f) = B(\phi_\rho, f)$$

(even though generally $\phi_\rho \in C(\mathbf{T}) \backslash A(\mathbf{T})$). Obviously, (5.3), (5.5), and (5.6) yield the yearned for annihilation, $\langle T, \phi \rangle = 0$.

THEOREM 5.1. *Given* $f \sim T$, $\phi \in A(\mathbf{T})$, supp $T \subseteq Z\phi$, *and assume that*

$$(5.7) \quad |B_2|(\phi, f) \equiv \int_0^\infty \frac{1}{\tau^2} ||\phi, \tau|| \, ||f, \tau|| d\tau < \infty.$$

Then $\langle T, \phi \rangle = 0$.

Proof. $|B|(\phi, f) \leqq |B_2|(\phi, f)$. In light of the previous remark we need only show that (5.7) implies (5.6). Set $\psi = \phi_\rho - \phi$ so that $\psi = 0$ on a neighborhood of $Z\phi$. Choose a C^∞-approximate identity δ_n so that $\psi_n = \psi * \delta_n$ is 0 on a neighborhood of $Z\phi$. Since $|\hat{\delta}_n(m)| \leqq 1$ we have

$$||\psi_n, \tau||^2 = 4 \sum_m |\hat{\psi}(m)\hat{\delta}_n(m)|^2 \sin^2 \frac{m\tau}{2} \leqslant ||\psi, \tau||^2.$$

Consequently from (5.7) we can use the dominated convergence theorem and have

$$\lim B(\psi_n - \psi, f) = 0.$$

Now $B(\psi_n, f) = \langle T, \psi_n \rangle$ from Proposition 5.1 so that

$$B(\psi_n, f) = 0.$$

Hence, $B(\phi, f) = B(\phi_\rho, f)$.

Obviously, in order to apply Theorem 5.1, we need only check the finiteness of the integral in (5.7) over the range of integration (0, 1).

The Beurling-Pollard result then follows:

PROPOSITION 5.3. *Let* $\phi \in \Lambda_\alpha$, $\alpha > 1/2$. *For all* $T \in A'(Z\phi)$, $\langle T, \phi \rangle = 0$.

Proof. This follows as $|B_2|(\phi, f) < \infty$ by Proposition 2.1(c).

As an obvious generalization we have

PROPOSITION 5.4. *If* $\phi = 0$ *on* supp T, $\phi \in A(\mathbf{T})$, $T \in A'(\mathbf{T})$, *and*

$$\int_0^{2\pi} \frac{\omega_2(\phi, \tau)}{\tau^{3/2}} d\tau < \infty,$$

then $\langle T, \phi \rangle = 0$.

Proposition 5.4 remains true if the integral condition is replaced by

$$\int_0^{2\pi} \frac{\omega_\infty(\phi, \tau)^{1/2} d\tau}{\tau} < \infty.$$

In this form every bounded variation function $\phi \in \Lambda_\alpha$, $\alpha > 0$, is synthesizable (ϕ is automatically in $A(\mathbf{T})$ with these hypotheses); of course, this does not yield the complete Katznelson result that each bounded variation function in $A(\mathbf{T})$ is synthesizable.

It is easy to check that if $\phi \in C(\mathbf{T}), f \sim T, T \in M(\mathbf{T})$, and $|B_2|(\phi, f) < \infty$, then

$$B(\phi, f) = \int \phi df = -\int f d\phi.$$

If $f \in C(\mathbf{T})$ then

$$\int \chi_{(a,b)} df = f(b) - f(a) = B(\chi_{(a,b)}, f).$$

Thus, we have

PROPOSITION 5.5. *If* $f \sim T, f \in C(\mathbf{T})$, *and* $\{x\} = \bigcap I_n$, I_n *an interval, then* $\lim_n B(\chi_{I_n}, f) = 0$.

Consequently, if $f \sim T, f \in C(\mathbf{T})$, we have $T(\{x\}) = 0$.

Remark. The Beurling integral with its Lebesgue dominated convergence theorem can obviously only give point mass 0 if such is to exist. This suggests a closer adherence to Stieltjes integral representation for the solution of synthesis problems not dealt with in this section.

Recalling that

$$\lim_N \frac{1}{\pi N} \sum_{|n| \leqslant N} \hat{\mu}(n) e^{inx} = f(x+) - f(x-),$$

for $f \sim \mu$, $\mu \in M(\mathbf{T})$, we note that

$$\lim_N \frac{1}{2N+1} \sum_{|n| \leqslant N} \hat{T}(n) e^{inx} = 0$$

for $T \in A_c'(\mathbf{T})$. In this regard we know that for $T \in A'(\mathbf{T})$, T is almost

https://doi.org/10.4153/CJM-1974-092-3 Published online by Cambridge University Press

periodic if supp T is countable whereas supp T is perfect if $T \in A_c'(\mathbf{T})$. Also if $f(x\pm)$ exist for $f \sim T$ we can well-define $T(\{x\}) = f(x+) - f(x-)$ by a de la Vallée-Poussin kernel calculation.

Epilogue. Because of the limited application of Beurling's integral to synthesis problems, and the relation between synthesizable pseudo-measures and an adequate Stieltjes integral representation for the operation of $T \in A'(\mathbf{T})$ on $\phi \in A(\mathbf{T})$, we have developed a theory of integral for synthesis. These results involve an inversion theory of convolution transforms on certain Bohr compactifications. A special case of this is the Denjoy-Kempisty integral [**9**] generalized and combined with a Beurling technique [**6**, pp. 1984-5]. This work is forthcoming.

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