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# THE PROJECTIVE GEOMETRY ARISING FROM A HOLLOW MODULE

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**Abstract**

We discuss the projective geometry defined in terms of the hollow factor modules of a given module. In particular, we derive an explicit expression for the division ring obtained in coordinatizing such a projective geometry.

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In [2] an independence structure was defined on the set of uniform submodules of a module, and was shown to be modular. Thus, if it is connected and of rank at least 3, it corresponds naturally to a projective geometry, which is Desarguesian. The division rings obtainable by coordinatizing such projective geometries were discussed there in detail. Dually, in [3], an independence space, also modular, was defined on the set of hollow factor modules of a module. In this paper we discuss the division rings obtained by coordinatizing the associated projective geometries.

An *independence structure* $\mathscr{E}$ on a set $E$ is a collection of subsets (the *independent sets*), satisfying certain axioms, not unlike the properties of linear independence when $E$ is a subset of a vector space (see [7] for full details). The *rank* of $A \subseteq E$ is the cardinality of any maximal independent subset of $A$, and for $r$ finite, an *r-flat* is a maximal set of rank $r$. If $\mathrm{rk}(A) = r$, then $[A]$ denotes the unique $r$-flat containing $A$, and we may write $[a, b]$ for $[\{a, b\}]$, for example. The 1-flats partition $E$; by collapsing each to a single element we get the *simple* independence space naturally associated with $\mathscr{E}$. A pair of elements $e, f \in E$ is *connected* if they are both contained in some *circuit* (minimal dependent set); connectedness is an

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equivalence relation, the classes being called *connected components*. An independence structure is *modular* if $\mathrm{rk}(A) + \mathrm{rk}(B) = \mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B)$ for any flats $A, B \subseteq E$; some equivalent definitions are quoted in [2]. Further details are in [2], [7] etc.

Let $R$ be a ring with 1; all modules will be unitary left $R$-modules. A submodule $K$ of a module $M$ is *small* ($K \leqslant_s M$) if $K + L = M \Rightarrow L = M$. A *hollow* module is not the sum of two proper submodules; let $Hf(M) = \{N \leqslant M;\ M/N \text{ is hollow}\}$. We define $\mathcal{G}d(M) \subseteq \mathcal{P}(Hf(M))$ (the set of subsets of $Hf(M)$) by

(a) For $\{K_1,\dots,K_r\} \subseteq Hf(M)$, $\{K_1,\dots,K_r\} \in \mathcal{G}d(M)$ if, for each $l = 1,\dots,r$, $K_l + \bigcap_{j \neq l} K_j = M$. (In this case, for $\phi \subset J \subset I = \{1,\dots,r\}$, $\bigcap_{i \in I \setminus J} K_i + \bigcap_{j \in J} K_j = M$.)

(b) For $\{K_i\colon i \in I\} \subseteq Hf(M)$, $\{K_i\colon i \in I\}$ is in $\mathcal{G}d(M)$ if every finite subset of it is, according to (a).

The next theorem outlines the background to the present work.

THEOREM 1. (i) *$\mathcal{G}d(M)$ is a modular independence structure on $Hf(M)$.*

(ii) *If a connected component has rank at least* 3 *then its* 1*-flats and* 2*-flats form the points and lines of a projective geometry, which, if Desarguesian, is coordinatizable over a unique division ring $D$.*

PROOF. (i) is [3], Theorems 2.3 and 2.6. For (ii) combine standard results, as is done in [2], Theorem 9.

In examining when $\mathcal{G}d(M)$ is connected, we obtain the following result; its dual follows easily from [2], Lemma 10.

LEMMA 2. *Let $N_1, N_2 \in Hf(M)$. Then $N_1$ and $N_2$ are connected if and only if $M/N_1$ and $M/N_2$ have isomorphic non-trivial factor modules.*

PROOF. Suppose $N_1$ and $N_2$ are connected. If $\{N_1, N_2\} \notin \mathcal{G}d(M)$, then $M/(N_1 + N_2)$ is a common non-trivial factor module. Otherwise, since $\mathcal{G}d(M)$ is modular, there is a circuit $\{N_1, N_2, N'\}$ for some $N' \in Hf(M)$. If $N = N' + (N_1 \cap N_2)$, then by [3], Lemma 2.2, $N < M$. Also, $N + N_1 = N + N_2 = N_1 + N_2 = M$. We define a map $\theta$: $M/N_1 \to M/N$ by $(m + N_1)\theta = n_2 + N$, where $m = n_1 + n_2$, $n_i \in N_i$. To show $\theta$ is well-defined, let $m \in N_1$, $m = n_1 + n_2$, $n_i \in N_i$. Then $n_2 \in N_1 \cap N_2 \leqslant N$. As $\operatorname{im}\theta = (N_2 + N)/N = M/N$, we have $(M/N_1)/\ker\theta \cong M/N$. Similarly $M/N_2$ has a factor module isomorphic to $M/N$.

Conversely, let $\phi_i$: $(M/N_i) \to L \neq 0$ ($i = 1,2$) be surjections. Define a map $\theta$: $M \to L$ by $m\theta = (m + N_1)\phi_1 + (m + N_2)\phi_2$. As $N_2\theta = ((N_2 + N_1)/N_1)\phi_1 = L$,

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$\operatorname{im}\theta = L$. Let $N = \ker\theta$, so $M/N \cong L$ and $N \in Hf(M)$. As $N_1 \cap N_2 \leqslant N$, $(N_1 \cap N_2) + N < M$ and $\{N_1, N_2, N\} \notin \mathcal{G}d(M)$. If $\{N_1, N_2\} \notin \mathcal{G}d(M)$ then clearly $N_1$ and $N_2$ are connected; otherwise, $N_1 + N_2 = M$ and it remains to show that $N_1 + N = N_2 + N = M$, whence $\{N_1, N_2, N\}$ is a circuit. Let $n_1 \in N_1$; as $M = N_1 + N_2$ and $\phi_2$ is onto, we can choose $n_2 \in N_2$ such that $(n_2 + N_1)\phi_1 = (n_1 + N_2)\phi_2$. Then $n_1 = (n_1 - n_2) + n_2 \in \ker\theta + N_2$, that is, $N_1 \leqslant N + N_2$. Thus $M = N + N_2$, similarly $M = N + N_1$, and the result is shown.

We may make assumptions about the structure of $M$ while leaving the projective geometry, or at least one of its planes, unchanged.

LEMMA 3. (i) *Let* $K \leqslant_s M$. *Then* $\mathcal{G}d(M/K)$ *and* $\mathcal{G}d(M)$ *have isomorphic associated simple independence spaces.*

(ii) *Let* $\{N_1, N_2, N_3\} \in \mathcal{G}d(M)$, *let* $K = N_1 \cap N_2 \cap N_3$. *Then the associated simple independence space of* $\mathcal{G}d(M/K)$ *is isomorphic to that of the subspace* $[N_1, N_2, N_3]$ *of* $\mathcal{G}d(M)$.

PROOF. Define a map $\theta$: $Hf(M) \to Hf(M/K) \cup \{M/K\}$ by $\theta(N) = (N + K)/K$. For (i), as $K \leqslant_s M$, $N + K < M$; for (ii), $N + K < M$ if and only if $N \in [N_1, N_2, N_3]$, as follows from [3], Lemma 2.2. In this case, $N + K \in Hf(M)$ and, equivalently, $(N + K)/K \in Hf(M/K)$; also $[N] = [N + K]$ in $\mathcal{G}d(M)$. Thus, if $\{L_i\colon i \in I\} \subseteq Hf(M)$ in (i), or $\{L_i\colon i \in I\} \subseteq [N_1, N_2, N_3]$ in (ii), then

$$\begin{aligned}\{L_i\colon i \in I\} \in \mathcal{G}d(M) &\Leftrightarrow \{L_i + K\colon i \in I\} \in \mathcal{G}d(M)\\ &\Leftrightarrow \{(L_i + K)/K\colon i \in I\} \in \mathcal{G}d(M/K).\end{aligned}$$

THEOREM 4. *The projective planes in the projective geometries of Theorem* 1(ii) *are precisely those arising from* $M = H^3$, *H a hollow module; they are Desarguesian.*

PROOF. Let $\{N_1, N_2, N_3\}$ be independent, in a connected component of $\mathcal{G}d(M)$. By Lemma 2, let $K_i \geqslant N_i$ such that $M/K_1 \cong M/K_2 \cong M/K_3 \cong H$ say, $H \neq 0$. Let $K = K_1 \cap K_2 \cap K_3$, $M' = M/K$ and $K_i' = K_i/K$ $(i = 1,2,3)$. Then, by Lemma 3(ii), the projective plane determined by $[N_1, N_2, N_3]$ $(= [K_1, K_2, K_3])$ is that of $\mathcal{G}d(M')$. As $(K_1' \cap K_2') + K_3' = M'$, $(K_1' \cap K_2') + (K_1' \cap K_3') = K_1'$; as $K_1' \cap K_2' \cap K_3' = 0$, this sum is direct, as is $K_1' + (K_2' \cap K_3') = M'$. This last also implies $K_2' \cap K_3' \cong M'/K_1' \cong H$; similar results give $M' = K_1' \cap K_2' + K_1' \cap K_3' + K_2' \cap K_3' \cong H^3$. Now $\mathcal{G}d(H^4)$ gives a projective geometry of rank 4 (dimension 3), necessarily Desarguesian; therefore its planes, which are isomorphic to $\mathcal{G}d(H^3)$ by Lemma 3(ii), are Desarguesian. Conversely, for $H$ hollow, $\mathcal{G}d(H^3)$

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(with basis $\{(H, H, 0), (H, 0, H), (0, H, H)\}$) is connected by Lemma 2, and therefore gives a projective plane, which is again Desarguesian.

We now describe the results of coordinatizing $\mathscr{G}d(H^3)$. Let us define the natural projections $p$: $H^2 \to (H, 0)$ $(= H \oplus 0)$, and $q$: $H^2 \to (0, H)$.

THEOREM 5. *The division ring which coordinatizes* $\mathscr{G}d(H^3)$ is anti-isomorphic to the following.

$$D = \{[M]: M \leqslant H^2, p(M) = H, (0, H) \nleqslant M\}, \quad \textit{where}$$
$$[M] = [N] \Leftrightarrow M + N < H^2 \Leftrightarrow (0, H) \nleqslant M + N.$$
$$0_D = [(H, 0)], 0_D = [M] \Leftrightarrow q(M) < H, \quad \textit{and} \quad 1_D = [\{(h, h): h \in H\}].$$
$$[M] + [N] = [(M, N, +)] \quad \textit{and} \quad [M] \times [N] = [(M, N, \times)], \textit{where}$$
$$(M, N, +) = \{(m_1, m_2 + n_2): (m_1, m_2) \in M, (n_1, n_2) \in N, m_1 = n_1\} \quad \textit{and}$$
$$(M, N, \times) = \{(m_1, n_2): (m_1, m_2) \in M, (n_1, n_2) \in N, m_2 = n_1\}.$$

*Also,* $-[M] = [\{(m_1, -m_2): (m_1, m_2) \in M\}]$ *and, for* $[M] \neq 0_D$, $[M]^{-1} = [\{m_2, m_1\}: (m_1, m_2) \in M]$.

PROOF. We follow the coordinatization rule of [5], p. 209. Let the coordinate line $D \cup \{\infty_D\}$ be $[(H, 0, H), (0, H, H)]$. If $N \in [(H, 0, H), (0, H, H)]$ then $N + (0, 0, H) < H^3$ and so $[N] = [N + (0, 0, H)]$. We will therefore consider $D \cup \{\infty_D\}$ as the set of 1-flats of $\mathscr{G}d(H^2)$, under the well-defined 1-1 correspondence $[N] \leftrightarrow [N \oplus H]$ $(N \in hf(H^2), N \oplus H \in Hf(H^3))$. Choose $0_D$ and $1_D$ as stated, and $\infty_D = [(0, H)]/$ Since, for $M, N \in Hf(H^2)$, $[M] = [N]$ when $M + N < H^2$, we have $[M] = 0_D$ when $q(M) < H$ and $[M] = \infty_D$ when $p(M) < H$. Let $M \leqslant H^2$ such that $p(M) = H$. Then $M < H^2 \Leftrightarrow (0, H) \nleqslant M$, and in this case $M \in Hf(H^2)$, by [3], Lemma 3.5(i), since $M + (0, H) = H^2$. Likewise $M + N < H^2 \Leftrightarrow (0, H) \nleqslant M + N$. Thus $D$ is as stated. The coordinatization procedure then gives the operations. We omit the details, but the following Lemma is used in the construction.

LEMMA 6. *Let* $[A, B]$ *and* $[C, D]$ *be two distinct lines (with* $A, B, C, D \in Hf(H^3)$*). Then* $[A, B] \cap [C, D] = [N]$*, where* $N = A \cap B + C \cap D$.

PROOF. As $\mathrm{rk}(\mathscr{G}d(H^3)) = 3$, $\{A, B, C, D\}$ contains a circuit, which is not contained in either $\{A, B\}$ or $\{C, D\}$. Therefore, by [3], Lemma 2.2, $N < H^3$. We show $H^3/N$ is hollow. Suppose $N \leqslant K', L' < H^3$. Let $K \geqslant K'$, $L \geqslant L'$ such that $K, L \in Hf(H^3)$, by [3], Theorem 2.5. As $K \geqslant N \geqslant A \cap B$, and similarly, we

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have $\{K, L\} \subseteq [A, B] \cap [C, D]$; as

$$\operatorname{rk}([A, B] \cap [C, D]) = \operatorname{rk}([A, B]) + \operatorname{rk}([C, D]) - \operatorname{rk}([A, B, C, D])$$
$$= 2 + 2 - 3 = 1,$$

$[K] = [L]$; that is, $K + L < H^3$. Thus $N \in Hf(H^3)$, and clearly $N \in [A, B] \cap [C, D]$.

Naturally, it can be verified directly that $D$ is a division ring. Clearly, $(M, N, +)$ and $(M, N, \times)$ are submodules of $H^2$ which project onto $(H, 0)$. As $(0, H)$ is hollow, $(M, N, +) \cap (0, H) = M \cap (0, H) + N \cap (0, H) < (0, H)$, so $(M, N, +) < H^2$. To check that $(M, N, \times) < H^2$ requires the following interesting lemma.

LEMMA 7. *Let $N' \leqslant N < H^2$, such that $p(N) = q(N) = H$. Then $p(N') = H \Leftrightarrow q(N') = H$, and in this case $[N'] = [N]$.*

PROOF. Suppose $p(N') = H$. Then $N' \in Hf(H^2)$, and since $N' + N = N < H$, $[N'] = [N]$. As $q(N) = H$, $[N'] = [N] \neq 0_D$, so $q(N') = H$. The converse is by symmetry.

Consider $(M, N, \times)$ where $[M], [N] \in D$, $[N] \neq 0_D$. Let $N' = \{(n_1, n_2) \in N: (0, n_1) \in M\}$. By the lemma, we get

$$(0, H) \leqslant (M, N, \times) \Rightarrow q(N') = H \Rightarrow p(N') = H \Rightarrow (0, H) \leqslant M,$$

which is not so. Thus $(M, N, \times) < H^2$. It is easy to show that calculating $[M] - [N]$ gives $0_D$ if and only if $[M] = [N]$, and this leads to a proof that the operations are well-defined. The remaining details are easy to verify (noting that to show, say, $[A] = [B]$, it is enough to show that, for example, $A \geqslant B$).

We turn now to some special cases. Since a hollow module is either cyclic or not finitely generated, we consider $H$ cyclic, $H = Rh$. Let $H \cong R/I$, $I$ a left ideal of $R$. For $[M] \in D$, $p(M) = H$ and so we may choose $(h, m) \in M$. Then by Lemma 7, $[R(h, m)] = [M]$, and it also follows that $Rm = H$ if and only if $q(M) = H$. If we denote $[R(h, m)]$ by $\langle m \rangle$, we get

$$D = \{\langle m \rangle : m \in H, R(h, m) < H^2\}, \quad \text{where}$$
$$\langle m \rangle = \langle n \rangle \Leftrightarrow R(h, m) + R(h, n) < H^2,$$
$$0_D = \langle 0 \rangle;\ 0_D = \langle n \rangle \Leftrightarrow Rn < H;\ 1_D = \langle h \rangle,$$
$$\langle m \rangle \pm \langle n \rangle = \langle m \pm n \rangle,$$
$$\langle m \rangle \times \langle n \rangle = \langle rn \rangle \text{ and } \langle n \rangle^{-1} = \langle sh \rangle, \quad \text{where } m = rh \text{ and } h = sn.$$

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Note that, if $I = \mathrm{Ann}(h)$, then $R(h, m) < H^2 \Leftrightarrow Im < H$.

However, the case where $H$ is cyclic is always covered by the following Theorem (see [4], Corollary 2.2). Let the division ring $D$ described in Theorem 5 be called $Dd(H^3)$.

THEOREM 8. *If $H$ is hollow and $K < H$, then $Dd(H^3) \cong Dd((H/K)^3)$. If $H$ also has a maximal submodule (that is, $J(H) < H$), then $Dd(H^3) \cong \mathrm{En}(H/J(H))$.*

PROOF. As $H$ is hollow, $K \leqslant_s H$, $K^3 \leqslant_s H^3$ (by [1], 5.20(1)), and, from Lemma 3(i), $Dd(H^3) \cong Dd(H^3/K^3) \cong Dd((H/K)^3)$. If $H$ has a maximal submodule, then it is unique, since $H$ is hollow, and so $J(H)$ is maximal. Let $N = H/J(H)$ and, as $N$ is simple, let $N = Rh \cong R/I$.

Define $f$: $\mathrm{En}(N) \to Dd(N^3)$ by $f(\psi) = \langle h\psi \rangle$. Now $\psi = 0 \Leftrightarrow h\psi = 0 \Leftrightarrow \langle h\psi \rangle = 0_D$ as $N$ is simple. To show $f$ is onto, let $\langle m \rangle \in Dd(N^3)$. Thus $Im < N$, so $Im = 0$, and we may define $\psi \in \mathrm{En}(N)$ by $(rh)\psi = rm$. Also, for $\psi \in \mathrm{En}(N)$, $I(h\psi) = (Ih)\psi = 0$. Clearly $f$ preserves the operations, and so is an isomorphism.

It can be verified that in the case where $H$ is cyclic, $H = Rh$, then $\langle h\psi \rangle \leftrightarrow \psi$ is an isomorphism from $Dd(H^3)$ to $\mathrm{En}(H/J(H))$. This last theorem is the dual of part of [2], Theorem 15. The proof is not similar because projective covers need not exist.

Suppose that in fact $H$ does have a projective cover $P$, that is, $H \cong P/K$, $K \leqslant_s P$. Then $P$ is also hollow. By [1], 17.14, $P$ has a maximal submodule $M$; as $K \leqslant_s P$, $K \leqslant M$, and so $M/K$ is maximal in $H$. Thus Theorem 8 applies. Also, $M/K$ and hence $M$ are unique maximal submodules, of $H$ and $P$ respectively, so $P/J(P) \cong H/J(H)$. From [1], 17.12 and 17.10 we have $\mathrm{En}(P/J(P)) \cong \mathrm{En}(P)/J(\mathrm{En}(P))$. Thus $Dd(H^3) \cong \mathrm{En}(P)/J(\mathrm{En}(P))$, corresponding to [2], Theorem 14. It follows from this that $\mathrm{En}(P)$ is (quasi-)local, for $P$ hollow projective; this is also shown in [6], Proposition 4.1 and Theorem 4.2, which characterize hollow projective modules (see also [1], 17.19).

There remain the hollow modules with no maximal submodule (and therefore no projective cover). We look at the example $H = \mathbf{Z}_{p^\infty}$ (a $\mathbf{Z}$-module). This is hollow, and has no maximal submodule, since all proper submodules are finitely generated (indeed finite and cyclic), see [4], Section 5.

Let $N \in Hf(H^2)$, with $p(N) = H$. As $(0, H) \nleqslant N$, $N \cap (0, H) = \mathbf{Z}(0, 1/p^e)$ for some $e \geqslant 0$. Thus, if $(1/p^i, m)$ and $(1/p^j, n)$ $(i > j > 0)$ are in $N$, then $p^{i-j}(1/p^i, m) - (1/p^j, n) \in \mathbf{Z}(0, 1/p^3)$. So, if

$$m \in \frac{a_k}{p^{i-k}} + \frac{a_{k+1}}{p^{i-k-1}} + \cdots + \frac{a_{i-e-1}}{p^{e+1}} + \mathbf{Z}\left(\frac{1}{p^e}\right),$$

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then

$$n \in \frac{a_k}{p^{j-k}} + \frac{a_{k+1}}{p^{j-k-1}} + \cdots + \frac{a_{j-e-1}}{p^{e+1}} + \mathbf{Z}\left(\frac{1}{p^e}\right).$$

Let us therefore describe $N$ by the power series expression $a_k p^k + a_{k+1} p^{k+1} + \cdots$ $(0 \leqslant a_i < p,\ a_k \neq 0)$. Any coefficient $a_l$ is determined by choosing $j > l + e$ and $(1/p^j, n) \in N$; then $a_l$ appears in the expression for $n$. $Dd(H^3)$ is the set of such power series expressions, addition and multiplication being natural; it is the $p$-adic completion of the rationals.

Another example would be $H = \mathbf{Z}[1/p]$. However, since $\mathbf{Z}_{p^\infty} = \mathbf{Z}[1/p]/\mathbf{Z}$, the same division ring arises, by Theorem 8.

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