MINIMAL INTERCHANGES OF (0, 1)-MATRICES AND DISJOINT CIRCUITS IN A GRAPH

DAVID W. WALKUP

1. Introduction. In this paper we obtain a partial answer in graph-theoretic form to a question raised by Ryser (**2**, p. 68) concerning the minimal number of interchanges required to transform equivalent (0, 1)-matrices into each other.

For given positive integers m and n we consider the collection of $m \times n$ (0, 1)-*matrices* $A = \{a_{ij}\}$, i.e. $a_{ij} = 0$ or 1 for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. We say the (0, 1)-matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are *equivalent* and write $A \sim B$ if and only if they have the same row and column sums, that is, if and only if

$$r_i = \textstyle\sum_j a_{ij} = \sum_j b_{ij}, \qquad s_j = \sum_i a_{ij} = \sum_i b_{ij}.$$

We note immediately that $A \sim B$ if and only if $B - A \sim O$, where O designates the $m \times n$ matrix of zeros.

Given a (0, 1)-matrix A, we can obtain an equivalent one, A', by finding a 2×2 minor of A of the form

$$\begin{matrix} 0 & .\,. & 1 \\ . & & . \\ . & & . \\ 1 & .\,. & 0 \end{matrix}$$

and replacing it by one of form

$$\begin{matrix} 1 & .\,. & 0 \\ . & & . \\ . & & . \\ 0 & .\,. & 1 \end{matrix}$$

or vice versa. Ryser calls this transformation from A to A' an *interchange* and shows (**1**; **2**, p. 68) that any matrix equivalent to A may be obtained from it by a suitable sequence of interchanges. We shall show the following:

THEOREM 1. *If A and B are equivalent* (0, 1)-*matrices, then B can be obtained from A by a sequence of*

$$\tfrac{1}{2}\alpha(A, B) - \beta(G) \tag{1.1}$$

and no fewer interchanges, where $\alpha(A, B)$ is the number of positions at which

Received June 1, 1964.

A and B disagree, G is the directed, bipartite graph derived from $B - A \sim O$, *and* $\beta(G)$ *is the maximum number of edge disjoint circuits in G.*

Experimentation with a number of reasonably small examples has shown that determination of the maximum number of interchanges by evaluating $\beta(G)$ is considerably easier than by a direct examination of the matrices A and B. However, no simple algorithm for computing $\beta(G)$ has been found.

In §2, we develop some convenient methods and notations concerning graphs and matrices. In §3, we reprove Ryser's result that a sequence of interchanges exists, showing, in fact, that a sequence of length (1.1) exists. In §4, we prove a general result on graphs and show it implies that (1.1) is a lower bound for the number of interchanges.

2. Preliminaries.

Definition 1. By a *graph G with multiplicities*, or *graph* for short, we mean a set $V = \{v_1, v_2, \ldots, v_t\}$ of *vertices*, and an integer-valued function F on the ordered pairs of $V \times V$ satisfying $F(v_i, v_j) = -F(v_j, v_i)$, so that in particular $F(v_i, v_i) = 0$. We designate by $\mathfrak{E}$ the collection of ordered pairs (v_i, v_j) of $V \times V$ for which $F(v_i, v_j) > 0$. We choose to write the elements of $\mathfrak{E}$ in the form $E(v_i, v_j)$ and say that $E(v_i, v_j)$ is an *arc of G directed from* v_i *to* v_j *of multiplicity* $F(v_i, v_j)$.

A graph with multiplicities may be thought of, if desired, as an undirected loopless graph where $F(v_i, v_j) \neq 0$ is a flux from v_i to v_j through the only edge connecting v_i and v_j.

The class of all graphs with given vertex set V is designated by $\mathfrak{G} = \mathfrak{G}(V)$. Throughout we shall suppose that V is arbitrary but fixed. Of special interest is the subclass $\mathfrak{G}^* \subset \mathfrak{G}(V)$ consisting of *basic graphs*—graphs with arcs of multiplicity 1 only. Basic graphs may be thought of as directed graphs with at most one arc, regardless of direction, connecting any distinct vertices. Given any basic graph G^* from $\mathfrak{G}^*$, we define a subset $\mathfrak{G}(G^*)$ of $\mathfrak{G}(V)$ as follows: G is in $\mathfrak{G}(G^*)$ if and only if for each arc $E(v_i, v_j)$ of G either $E(v_i, v_j)$ or $E(v_j, v_i)$ is an arc of G^*.

PROPOSITION 1. *If* G_1 *and* G_2 *are graphs in* $\mathfrak{G}$ *with functions* F_1 *and* F_2, *then the function F given by*

$$F(v_i, v_j) = F_1(v_i, v_j) + F_2(v_i, v_j)$$

is the function of a graph G which we may call the sum $G_1 + G_2$. $\mathfrak{G}(V)$ *is an additive group under this composition and each* $\mathfrak{G}(G^*)$ *is a subgroup.*

We shall say that a sum $\sum G_i$ of graphs in a class $\mathfrak{G}(G^*)$ is *conjoint* if for each arc E of G^* the non-zero integers $F_i(E)$ have the same sign, that is, if there is no cancellation in forming the sum $F = \sum F_i$ for G, or if, in the undirected graph interpretation, all fluxes reinforce. If, in fact, for each E in G^* at most one $F_i(E)$ is non-zero, we shall say that the sum $\sum G_i$ is disjoint.

It will be seen that conjointness in a sum of graphs is thus a generalization of the usual concept of edge disjointness. By a *circuit of length* r (an r-*circuit*) we mean a graph in $\mathfrak{G}^*$ having exactly $r \geqslant 3$ distinct arcs

$$E(p_1, p_2), E(p_2, p_3), \ldots, E(p_r, p_1),$$

joining r distinct vertices $p_1, p_2, \ldots, p_r$ of V.

We say a graph is *conservative* if the sum of multiplicities of arcs leaving each vertex equals the sum of multiplicities of entering arcs. Any circuit is conservative, but also:

PROPOSITION 2. *If the graph G of $\mathfrak{G}(G^*)$ is conservative, it can be written as a conjoint sum of circuits in $\mathfrak{G}(G^*)$.*

If we wish to consider *bipartite* graphs, we can suppose the vertex set V is the disjoint union of sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, $m + n = t$, and restrict attention to the subclass $\mathfrak{G}^0 \subset \mathfrak{G}(V)$ containing those graphs which have no arcs connecting two points in X or two points in Y. We shall suppose the integers m and n and the sets X and Y understood when considering a class $\mathfrak{G}^0$. The definitions and results on circuits, conservative graphs, and subgroups $\mathfrak{G}(G^*)$ will carry over to the bipartite case.

If $\mathfrak{G}^0$ is the class of bipartite graphs on vertex sets X and Y of m and n elements respectively, we can define for each $m \times n$ matrix of integers $A = \{a_{ij}\}$ the graph $G(A)$ in $\mathfrak{G}^0$ whose function F is given by $F(x_i, y_j) = -F(y_j, x_i) = a_{ij}$. The correspondence $A \leftrightarrow G(A)$ is an isomorphism between the additive group of $m \times n$ matrices and the group $\mathfrak{G}^0$. Accordingly, we shall speak of these matrices and graphs interchangeably when convenient.

PROPOSITION 3. *An $m \times n$ matrix A is equivalent to zero if and only if $G(A)$ is conservative.*

We note that if the graph $G(C)$ in $\mathfrak{G}^0$ corresponding to the matrix C is an r-circuit, we may permute the rows and columns of C to obtain an $m \times n$ matrix

$$\begin{array}{|c|c|} \hline T & O \\ \hline O & O \\ \hline \end{array},$$

where T is an $r \times r$ matrix with r 1's on the diagonal, $r - 1$ -1's on the superdiagonal, and a -1 in the lower left. Propositions 2 and 3 combine to give:

PROPOSITION 4. *Every $m \times n$ matrix A equivalent to zero is the conjoint sum of bipartite circuits. If the only entries of A are* 0, 1, *and* -1, *the sum is disjoint.*

3. Proof that (1.1) can be attained. For any conservative graph G in some class $\mathfrak{G}(G^*)$, let $\alpha = \alpha(G)$ be the sum of multiplicities of G, and let $\beta = \beta(G)$ be the largest integer for which G can be written as a conjoint sum of β circuits.

It is easily seen that $\alpha(G)$ and $\beta(G)$ are independent of the particular choice of G^*. In the remainder of this section only, we direct our attention exclusively to a class $\mathfrak{G}^0$ of bipartite graphs. We note that if A and B are equivalent (0, 1)-matrices, then $\alpha(A, B)$, the number of disagreements between A and B, equals $\alpha(G(B - A))$.

LEMMA 1. *If A and B are equivalent* (0, 1)-*matrices, then there exists a sequence*

$$A = A_0, A_1, A_2, \ldots, A_\beta = B, \qquad \beta = \beta(G(B - A)),$$

of equivalent (0, 1)-*matrices such that each difference*

$$C_i = A_i - A_{i-1}$$

is a circuit (*of length* r_i) *and*

$$B - A = \sum_{j=1}^{\beta} C_j \tag{3.1}$$

is a disjoint sum. Moreover,

$$\alpha(A, B) = \sum_{i=1}^{\beta} r_i.$$

Proof. $B - A$ satisfies the stronger conditions of Proposition 4; hence the disjoint sum (3.1) exists. The partial sums

$$A_i = A + \sum_{j=1}^{i} C_j$$

are all equivalent to A, since the C_i are equivalent to zero. The disjointness in (3.1) and the fact that A and B are (0, 1)-matrices imply that the A_i are also (0, 1)-matrices.

LEMMA 2. *If A and B are equivalent* (0, 1)-*matrices and $C = B - A$ is an r-circuit, then $r = 2s$, and there exists a sequence*

$$A = A_0, A_1, A_2, \ldots, A_{s-1} = B \tag{3.2}$$

of equivalent (0, 1)-*matrices for which the differences*

$$D_i = A_i - A_{i-1}$$

are circuits of length 4; *i.e.* A_i *and* A_{i-1} *differ only by an interchange.*

Proof. All graphs in $\mathfrak{G}^0$ are bipartite; hence the circuit $B - A$ has even length $r = 2s$. A weak result

$$B = A + \sum_{j=1}^{s-1} D_i'$$

for certain 4-circuits D_i' follows easily upon examination of Figure 1. Note

https://doi.org/10.4153/CJM-1965-081-0 Published online by Cambridge University Press

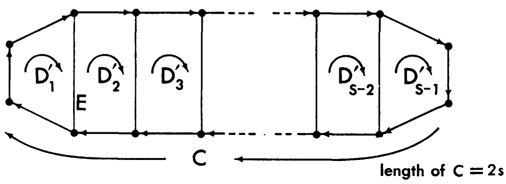

FIGURE 1

that the D_i' will necessarily visit vertices in X and Y alternately and hence are indeed elements of $\mathfrak{G}^0$. We seek a reordering D_i of D_i' so that

$$A_i = A + \sum_{j=1}^{i} D_j$$

are (0, 1)-matrices. Clearly the A_i will be equivalent. The sum

$$H = \sum_{j=2}^{s-1} D_j'$$

is a $(2s - 2)$-circuit in $\mathfrak{G}^0$, and a matrix of zeros and ones. We assert that either

(i) $A, A + D_1', A + D_1' + H = B$ or

(ii) $A, A + H, A + H + D_1' = B$

is a sequence of equivalent (0, 1)-matrices. The only possible difficulty is the value of the middle terms for the ordered pair (x_i, y_j) corresponding to E in Figure 1. But D_1' and H take opposite values for this pair; hence exactly one of $A + D_1'$ and $A + H$ is a (0, 1)-matrix. By applying the same argument to the circuit H instead of C, we may place additional terms between $A + D_1'$ and B if (i) holds or between A and $A + H$ if (ii) holds. Repeating this process a sufficient number of times, we shall reach simultaneously the sequence (3.2) and the proper reordering of the D_i'.

LEMMA 3. *If A and B are equivalent* (0, 1)-*matrices, there exists a sequence*

$$A = A_0, A_1, A_2, \ldots, A_k = B \tag{3.3}$$

of equivalent (0, 1)-*matrices for which the differences $A_i - A_{i-1}$ are* 4-*circuits and*

$$k = \tfrac{1}{2}\alpha(A, B) - \beta(G(B - A)).$$

Proof. The existence of the sequence (3.3) follows from Lemmas 1 and 2. The value of k derives from the computation

$$\sum_{i=1}^{\beta} (\tfrac{1}{2} r_i - 1) = \frac{1}{2} \sum_{i=1}^{\beta} r_i - \sum_{i=1}^{\beta} 1 = \frac{1}{2}\alpha(A, B) - \beta.$$

4. Proof that (1.1) is a lower bound. Let G be any conservative graph in a subgroup $\mathfrak{G}(G^*) \subset \mathfrak{G}$. We have defined $\alpha(G)$ and $\beta(G)$. For any positive integer $\delta \geqslant 3$ let $\gamma = \gamma(G, \delta)$ be the smallest integer for which G can be written as the sum of γ circuits from $\mathfrak{G}(G^*)$ of length δ or less. If G cannot be so written, set $\gamma = \infty$.

THEOREM 2. *If G is a finite, conservative graph in $\mathfrak{G}(G^*)$, then*

$$\gamma(G, \delta) \geqslant \frac{\alpha(G) - 2\beta(G)}{\delta - 2}.$$

Proof. We need consider the case $\gamma < \infty$ only. We fix δ and define the function $\phi(G)$ so that

$$\phi(G) = \alpha(G) - 2\beta(G) - (\delta - 2)\cdot\gamma(G).$$

We must show that

$$\phi(G) \leqslant 0 \qquad \text{for all conservative } G \text{ in } \mathfrak{G}(G^*). \tag{4.1}$$

Suppose (4.1) is false. Choose a conservative graph G_0 from $\mathfrak{G}(G^*)$ for which $\alpha(G_0)$ is as small as possible subject to

$$\phi(G_0) > 0. \tag{4.2}$$

Since the empty graph satisfies (4.1), we have

$$\alpha(G_0) > 0, \qquad \beta(G_0) > 0, \qquad \gamma(G_0) > 0.$$

Let

$$G_0 = \sum_{i=1}^{\gamma} D_i \tag{4.3}$$

be some expression for G_0 as a sum of a minimum number of circuits of $\mathfrak{G}(G^*)$ of length δ or less. For each D_i let $q(D_i)$ be the number of arcs of D_i which coincide (with proper orientation) with an arc of G_0. There must exist a D_k for which $q(D_k) \geqslant \delta - 1$ for otherwise we would have

$$\alpha(G_0) \leqslant \sum_{i=1}^{\gamma} q(D_i) \leqslant (\delta - 2)\cdot\gamma(G_0),$$

in violation of (4.2).

We suppose first that $q(D_k) = \delta$. Consider the conservative graph

$$G' = \sum_{\substack{i=1 \\ (i \neq k)}}^{\gamma(G_0)} D_i. \tag{4.4}$$

By exhibiting a specific sum for G', (4.4) shows that

$$\gamma(G_0) \geqslant \gamma(G') + 1. \tag{4.5}$$

Further, let

$$G' = \sum_{i=1}^{\beta(G')} C_i \tag{4.6}$$

be a representation of G' as a conjoint sum of a maximal number of circuits.

https://doi.org/10.4153/CJM-1965-081-0 Published online by Cambridge University Press

Then, because $q(D_k) = \delta$,

$$G_0 = \sum_{i=1}^{\beta(G')} C_i + D_k$$

is a conjoint sum for G_0, implying

$$\beta(G_0) \geqslant \beta(G') + 1. \tag{4.7}$$

Combining (4.5), (4.7), and $\alpha(G_0) = \alpha(G') + \delta$, we conclude that

$$\phi(G_0) \leqslant \phi(G'), \tag{4.8}$$

which contradicts the choice of G_0 as a smallest conservative graph satisfying (4.2). In the same way, the assumption $q(D_k) = \delta - 1$ for a circuit D_k of length $\delta - 1$ leads to (4.8) with strict inequality.

As a third and last alternative, we assume there exists a circuit D_k in (4.3) of length δ for which $q(D_k) = \delta - 1$. Let E be the only arc of D_k which does not coincide with an arc of G_0. Again we form G' as in (4.4) and find an expansion (4.6). We see that G' is again in $\mathfrak{G}(G^*)$. In G', $\delta - 1$ multiplicities of G_0 have been decreased, and one corresponding to E^-, the arc reverse to E, has been increased (possibly from zero to one). Thus

$$\alpha(G_0) = \alpha(G') + \delta - 2. \tag{4.9}$$

As before, (4.5) must hold. Let C_h be any circuit in (4.6) which has an arc coinciding with E^-. Now $C_h + D_k$ may not be a circuit, but it is a non-vacuous, conservative graph which, by Proposition 2, is the conjoint sum

$$C_h + D_k = \sum_{i=1}^{\epsilon} \tilde{C}_i$$

of at least one circuit. Therefore, we have

$$G_0 = \sum_{\substack{i=1\\ i\neq k}}^{\beta(G')} C_i + \sum_{i=1}^{\epsilon} \tilde{C}_i,$$

and this sum is easily seen to be conjoint. Accordingly,

$$\beta(G_0) \geqslant \beta(G'). \tag{4.10}$$

But (4.5), (4.9), and (4.10) imply (4.8) again, and we are forced to conclude that (4.1) always holds. This concludes the proof of Theorem 2.

Theorem 1 now follows directly from Lemma 3 and Theorem 2 for $\delta = 4$, $\mathfrak{G}(G^*) = \mathfrak{G}^0$.

A theorem similar to Theorem 1 can be proved for the case $\delta = 3$.

THEOREM 3. *If G is a conservative graph in $\mathfrak{G}(V)$, $\alpha(G)$ is the sum of multiplicities of arcs of G, and $\beta(G)$ is the largest integer for which G can be written as a conjoint sum of circuits, then G can be written as the sum of*

$$(4.11) \qquad \alpha(G) - 2\beta(G)$$

and no fewer 3-circuits from $\mathfrak{G}(V)$.

Proof. Theorem 2 states that (4.11) is a lower bound. The proof that (4.11) can be realized follows from Figure 2 in the same way that Theorem 1 and Lemmas 1, 2, and 3 follow from Figure 1.

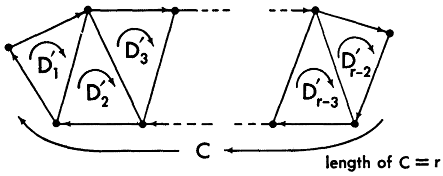

FIGURE 2

REFERENCES

1. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math., *9* (1957), 371–377.
2. ——— *Combinatorial mathematics*, The Carus Mathematical Monographs no. 14 (Mathematical Association of America, 1963).

Boeing Scientific Research Laboratories, Seattle, Washington