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A NOTE ON CONNECTED SUBMETALINDELÖF SPACES

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In this paper, we shall show that if m is a natural number and for every $0 \leqq n \leqq m$, $2^{\omega_n} < 2^{\omega_{n+1}}$ and $2^{\omega} \leqq \omega_m$ are assumed, then connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^{\omega}$ are Lindelöf. Furthermore, we shall show that $2^{\omega} < 2^{\omega_1}$ if and only if connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^{\omega}$ and tightness $\leqq \omega$ are Lindelöf.

1. INTRODUCTION

It is known that $2^{\omega} < 2^{\omega_1}$ implies connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^{\omega}$ and tightness $\leqq \omega$ are Lindelöf (hence, connected, locally Lindelöf, normal Moore spaces are metrisable) see [**1**]. In this paper, we shall show that if m is a natural number and for every $0 \leqq n \leqq m$, $2^{\omega_n} < 2^{\omega_{n+1}}$ and $2^{\omega} \leqq \omega_m$ are assumed, then connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^{\omega}$ are Lindelöf. We shall also show that, in fact, the converse of Balogh's result above is also true (that is, if connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^{\omega}$ and tightness $\leqq \omega$ are Lindelöf, then $2^{\omega} < 2^{\omega_1}$).

First we present topological and set theoretical notation. All topological spaces are assumed to be regular T_1. A subset S of a topological space is said to be *normalised* if for any $S' \subseteq S$, S' and $S - S'$ can be separated by disjoint open sets. A subset S of a topological space is said to be *separated* if for any x of S there is a neighbourhood U_x of x such that $\{U_x : x \in S\}$ is disjoint. For a point x of space X, $\chi(x, X)$ denotes the least cardinality of a neighbourhood base at x.

A space is *Lindelöf* if every open cover has a countable subcover. But in this paper, in order to consider Lindelöf properties for limit cardinals, we define κ-Lindelöfness *which is different from κ-Lindelöfness in the usual sense,* as follows: for a cardinal κ, a space is *κ-Lindelöf* if every open cover has a subcover of cardinality $< \kappa$. Thus Lindelöfness is equivalent to ω_1-Lindelöfness, and compactness is equivelent to ω-Lindelöfness.

A space is *submetaLindelöf (submetacompact)* if every open cover has a countable family $\{\mathcal{U}_n \mid n \in \omega\}$ of open covers refining it such that for any x in X there is an n in ω such that $|(\mathcal{U}_n)_x| \leqq \omega$ ($< \omega$, respectively), where $(\mathcal{U}_n)_x = \{U \in \mathcal{U} \mid x \in U\}$.

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A space is *locally κ-Lindelöf* if every point has a κ-Lindelöf neighbourhood.

For an ordinal α and a set X, ${}^{\alpha}X$ denotes the set of all functions from α to X and X^{α} denotes the cardinality of ${}^{\alpha}X$. Furthermore ${}^{<\alpha}X$ denotes the set $\bigcup_{\beta<\alpha} {}^{\beta}X$ and similarly $X^{<\alpha}$ denotes the cardinality of ${}^{<\alpha}X$. A subset of an ordinal is said to be *club* if it is closed and unbounded in the order topology. For a function f, $f \mid A$ denotes the restriction of f to A. For other set theoretical or topological notions or notations, see [**4, 5**] and [**6**].

2. Results

To prove topological results, first we present a set theoretical assertion Φ and a topological assertion N.

Definition. Let κ be an uncountable regular cardinal, λ be a cardinal, and S be a subset of κ. $\Phi(\kappa, \lambda, S)$ denotes the following assertion:
For every $F: {}^{<\kappa}\lambda \to 2$, there exists a g in ${}^{\kappa}2$ such that for any f in ${}^{\kappa}\lambda$, $\{\alpha \in S: F(f \mid \alpha) = g(\alpha)\}$ is stationary in κ.

Note that if $\Phi(\kappa, \lambda, S)$ holds, then such an S must be stationary in κ.

$N(\kappa, \lambda, S)$ denotes the following assertion:
For every topological space X and every normalised sequence $\{x_\alpha: \alpha \in S\}$ of distinct points, if for every α in S, $\chi(x_\alpha, X) \leqq \lambda$, then there is a stationary subset S' of S such that $\{x_\alpha: \alpha \in S'\}$ is separated.

Using the techniques of [**3**] and [**7**], we can prove the next result.

Lemma 1. [**3, 7**]. *Let κ be an infinite cardinal. Then the following assertions are equivalent*

(1) $2^{\kappa} < 2^{\kappa^+}$;
(2) $\Phi(\kappa^+, 2, \kappa^+)$;
(3) $\Phi(\kappa^+, 2^{\kappa}, \kappa^+)$;
(4) $N(\kappa^+, 2^{\kappa}, \kappa^+)$.

(1) and (2) of the following result come directly from the definition. (3) of the following result is proved as in [**3**]. (4) is an easy consequence of (3).

Lemma 2. *Let κ be an uncountable regular cardinal. Then the following assertions hold:*

(1) If $S \subset S' \subset \kappa$ and $\Phi(\kappa, 2, S)$ holds, then so does $\Phi(\kappa, 2, S')$.

(2) Let S be a stationary subset of κ. Then the following are equivalent

(i) $\Phi(\kappa, 2, S)$ *holds;*
(ii) $\Phi(\kappa, 2, S \cap C)$ *holds for any club C of κ;*

(iii) $\Phi(\kappa, 2, S \cap C)$ *holds for some club* C *of* κ.

(3) Let $\{S_\alpha : \alpha < \kappa\}$ *be a family of subsets of* κ. *If* $\Phi(\kappa, 2, \nabla_{\alpha<\kappa} S_\alpha)$ *holds, then there is an* $\alpha < \kappa$ *such that* $\Phi(\kappa, 2, S_\alpha)$ *holds, where* $\nabla_{\alpha<\kappa} S_\alpha = \{\beta < \kappa : \exists \alpha < \beta(\beta \in S_\alpha)\}$

(4) Let $\lambda < \kappa$ *and* $\{S_\alpha : \alpha < \lambda\}$ *be a family of subsets of* κ. *If* $\Phi\left(\kappa, 2, \bigcup_{\alpha<\lambda} S_\alpha\right)$ *holds, then there is an* $\alpha < \lambda$ *such that* $\Phi(\kappa, 2, S_\alpha)$ *holds.*

The next lemma can be proved as in [**7**].

LEMMA 3. *Let* κ *be an infinite cardinal, and* S *be a subset of* κ^+. *If* $\Phi(\kappa^+, 2, S)$ *holds, then so does* $N(\kappa^+, 2^\kappa, S)$.

From now on we will prove our results.

LEMMA 4. *Let* κ *be an infinite cardinal,* S *be a stationary subset of* κ^+, X *be a submetaLindelöf normal space of character* $\leqq 2^\kappa$, *and* Y *be a subset* $\{x_\alpha : \alpha \in S\}$ *of distinct points of* X *such that for any point* x *of* X *there is a neighbourhood* U_x *of* x *with* $|U_x \cap Y| \leqq \kappa$. *Assume* $\Phi(\kappa^+, 2, S)$. *Then there is a stationary subset* S' *of* S *such that* $\Phi(\kappa^+, 2, S')$ *holds and* $\{x_\alpha : \alpha \in S'\}$ *is closed discrete in* X.

PROOF: Take U_x for each point x in X as above. Then $\mathcal{U} = \{U_x : x \in X\}$ is an open cover of X. By submetaLindelöfness, there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers refining $\mathcal{U}$ such that for any point x in X, there is an $n(x) \in \omega$ with $|(\mathcal{U}_{n(x)})_x| \leqq \omega$. Let Y_n be the subset $\{x_\alpha \in Y : n(x_\alpha) = n\}$ of Y, and S_n be the subset $\{\alpha \in S : x_\alpha \in Y_n\}$ of S. Then $Y = \bigcup_{n\in\omega} Y_n$ and $S = \bigcup_{n\in\omega} S_n$. By $\Phi(\kappa^+, 2, S)$ and (4) of Lemma 2, there is an $n \in \omega$ such that $\Phi(\kappa^+, 2, S_n)$. Define an equivalence relation $\simeq$ on S_n as follows: for α and α' in S_n, $\alpha \simeq \alpha'$ if and only if there is a finite sequence $U_0, \ldots, U_i$ of elements of $\mathcal{U}_n$ such that $x_\alpha \in U_0$, $x_{\alpha'} \in U_i$ and $U_j \bigcap U_{j+1} \bigcap Y_n \neq 0$ for $j \in i$.

Since $\mathcal{U}_n$ is a refinement of $\mathcal{U}$, each equivalence class of $\simeq$ is of cardinality $\leqq \kappa$. Let $\{S_{n\gamma} : \gamma \in \Gamma\}$ enumerate these equivalence classes. For γ in Γ, enumerate $S_{n\gamma} = \{\alpha_{\gamma\beta} : \beta < \kappa\}$ in type κ. For $\beta < \kappa$, let T_β be the set $\{\alpha_{\gamma\beta} : \gamma \in \Gamma\}$. Then $\{\{x_\alpha : \alpha \in T_\beta\} : \beta < \kappa\}$ is a partition of Y_n; and since $\mathcal{U}_n$ is an open cover, $\{x_\alpha : \alpha \in T_\beta\}$ is closed discrete in X for each $\beta < \kappa$. Then by (4) of Lemma 2, there is a $\beta < \kappa$ such that $\Phi(\kappa^+, 2, T_\beta)$. Hence this T_β is the desired S'. ∎

From now on, we assume that κ is an infinite cardinal.

LEMMA 5. $[2^\kappa < 2^{\kappa^+}]$. *Let* X *be a submetaLindelöf, normal space of character* $\leqq 2^\kappa$. *Then the closure of* κ^+*-Lindelöf subspaces of* X *are* κ^+*-Lindelöf.*

PROOF: Suppose, on the contrary, that there is a κ^+-Lindelöf subspace Z such that $\mathrm{cl}\, Z$ is not κ^+-Lindelöf.

FACT. *$\mathrm{cl}\, Z$ contains a closed discrete subspace A of cardinality κ^+.*

Proof of the FACT. Let $\mathcal{U}$ be an open cover of $\mathrm{cl}\, Z$ with no subcover of cardinality $\leqq \kappa$. Since X is submetaLindelöf, so is $\mathrm{cl}\, Z$. Hence there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $\mathrm{cl}\, Z$ each refining $\mathcal{U}$ such that for any point x in $\mathrm{cl}\, Z$ there is an $n(x) \in \omega$ such that $|(\mathcal{U}_{n(x)})_x| \leqq \omega$. For $n \in \omega$, let Z_n be the subspace $\{x \in \mathrm{cl}\, Z : n(x) = n\}$. By the axiom of choice, take a maximal subset A_n of Z_n such that no member of $\mathcal{U}_n$ contains two elements of A_n. Since the $\mathcal{U}_n$'s are open covers of the closed subspace $\mathrm{cl}\, Z$, the A_n's are closed discrete in X. Let $\mathcal{U}'_n$ be the subfamily $\{U \in \mathcal{U}_n : U \bigcap A_n \neq 0\}$ of $\mathcal{U}_n$. Then by maximality, $\mathcal{U}'_n$ covers Z_n for each $n \in \omega$. Thus $\mathcal{U}' = \bigcup_{n \in \omega} \mathcal{U}'_n$ is an open cover of $\mathrm{cl}\, Z$ which refines $\mathcal{U}$. Since $\mathcal{U}$ has no subcover of cardinality $\leqq \kappa$, there is an $n \in \omega$ such that the cardinality of $\mathcal{U}'_n$ is greater than κ. But since for any point x in A_n, $|(\mathcal{U}_n)_x| \leqq \omega$, it follows that $\kappa < |A_n|$. Take a subset A of A_n of cardinality κ^+. This A is as desired. The proof of the fact is complete.

By the fact, let $\{x_\alpha : \alpha < \kappa^+\}$ be a closed discrete subset of $\mathrm{cl}\, Z$. By $2^\kappa < 2^{\kappa^+}$ and Lemma 1, there is a stationary subset S of κ^+ such that $\{x_\alpha : \alpha \in S\}$ is separated. Then by normality, take a discrete family $\{U_\alpha : \alpha \in S\}$ of open sets such that $x_\alpha \in U_\alpha$ for each $\alpha \in S$. Then $\{U_\alpha \bigcap Z : \alpha \in S\}$ is a discrete family of open sets of Z of cardinality κ^+. This contradicts the κ^+-Lindelöfness of Z. $\square$

THEOREM 6. $[2^\kappa < 2^{\kappa^+}]$ *Let X be a connected, κ^{++}-Lindelöf, locally Lindelöf, submetaLindelöf, normal space of character $\leqq 2^\kappa$. Then X is κ^+-Lindelöf.*

PROOF: Assume that X is κ^{++}-Lindelöf but not κ^+-Lindelöf. Then there is an open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa^+\}$ of X such that for any $\alpha < \kappa^+$, $\mathrm{cl}\, U_\alpha$ is Lindelöf and $U_\alpha - \bigcup_{\beta<\alpha} U_\beta \neq 0$. By connectedness, take a point x_α of $\partial\left(\bigcup_{\beta<\alpha} U_\beta\right)$ for each $\alpha < \kappa^+$. For $\alpha < \kappa^+$, take $g(\alpha) \geq \alpha$ such that $x_\alpha \in U_{g(\alpha)}$. Then $C = \{\alpha < \kappa^+ : g''\alpha \subset \alpha\}$ is a club set in κ^+, and the points of $Y = \{x_\alpha : \alpha \in C\}$ are all distinct. Then by Lemma 1 and 2) of Lemma 2, $\Phi(\kappa^+, 2, C)$ holds. Since $\mathcal{U}$ is an open cover of X such that each member of $\mathcal{U}$ intersects Y in at most κ-many points, by Lemma 4 there is a stationary subset S of C such that $\{x_\alpha : \alpha \in S\}$ is closed discrete and $\Phi(\kappa^+, 2, S)$ holds. Hence by Lemma 3, there is a stationary subset S_0 of S such that $\{x_\alpha : \alpha \in S_0\}$ is separated. Using normality, take a discrete open family $\{V_\alpha : \alpha \in S_0\}$ such that $x_\alpha \in V_\alpha$ for each α in S_0. Since x_α is in $\partial\left(\bigcup_{\beta<\alpha} U_\beta\right)$ for any $\alpha \in S_0$, there is an $f(\alpha) < \alpha$ such that $U_{f(\alpha)} \cap V_\alpha$ is non-empty. By the pressing down lemma, there are a stationary subset S_1 of S_0 and a $\beta < \kappa^+$ such that $f(\alpha) = \beta$ for every α in S_1. Then $\{V_\alpha \cap \mathrm{cl}\, U_\beta : \alpha \in S_0\}$ is a discrete open family of cardinality κ^+. This contradicts the Lindelöfness of $\mathrm{cl}\, U_\beta$. ∎

THEOREM 7. *Let m be a natural number. Assume that for each n, $0 \leqq n \leqq m$, $2^{\omega_n} < 2^{\omega_{n+1}}$ holds. Let X be a connected, locally Lindelöf, submetaLindelöf, normal space of character $\leqq 2^\omega$ and tightness $\leqq \omega_m$. Then X is Lindelöf.*

PROOF: Let $\mathcal{U}$ be an open cover of X whose elements have non-empty Lindelöf closures. By Lemma 5 and $2^{\omega_m} < 2^{\omega_{m+1}}$, the closures of ω_{m+1}-Lindelöf subspaces are ω_{m+1}-Lindelöf. Hence by induction on ω_{m+1}, take a sequence $\{\mathcal{U}_\beta : \beta \in \omega_{m+1}\} \subset [\mathcal{U}]^{\leqq \omega_m}$. such that $\mathcal{U}_\beta$ is non-empty and $\mathrm{cl}(\bigcup \mathcal{U}_\beta) \subseteq \bigcup \mathcal{U}_{\beta+1}$ for each β in ω_{m+1}. Let $\mathcal{U}'$ be the family $\bigcup_{\beta < \omega_{m+1}} \mathcal{U}_\beta$. Then $\bigcup \mathcal{U}'$ is clopen in X. Indeed, take a point x in $\mathrm{cl}(\bigcup \mathcal{U}')$. Since the tightness is $\leqq \omega_m$, take an ω_m-sequence $A = \{x_\alpha : \alpha < \omega_m\}$ of points in $\bigcup \mathcal{U}'$ such that $x \in \mathrm{cl}\, A$. For $\alpha < \omega_m$, take a $\beta(\alpha) < \omega_{m+1}$ such that $x_\alpha \in \bigcup \mathcal{U}_{\beta(\alpha)}$. Let $\beta = \sup\{\beta(\alpha) : \alpha < \omega_m\} < \omega_{m+1}$. Then $A \subseteq \bigcup \mathcal{U}_\beta$. Hence, $x \in \mathrm{cl}(\bigcup \mathcal{U}_\beta) \subset \bigcup \mathcal{U}'$. This implies $\bigcup \mathcal{U}'$ is a closed supspace of X. It is evident that $\bigcup \mathcal{U}'$ is open. Hence it is clopen in X. Then by the connectedness of X, $\bigcup \mathcal{U}' = X$. Since $|\mathcal{U}'| \leqq \omega_{m+1}$, the above argument implies that X is ω_{m+2}-Lindelöf. Therefore by Theorem 6, X is ω_{m+1}-Lindelöf. Then by a finite number of applications of Theorem 6, it follows that X is ω_1-Lindelöf (that is, X is Lindelöf). The proof is complete ∎

COROLLARY 8. *Let m be a natural number. Assume that for each n, $0 \leqq n \leqq m$, $2^{\omega_n} < 2^{\omega_{n+1}}$ holds. Let X be a locally connected, locally Lindelöf, submetaLindelöf, normal space of character $\leqq 2^\omega$ and tightness $\leqq \omega_m$. Then X is a free union of Lindelöf subspaces (hence it is strongly paracompact).*

PROOF: Apply the above theorem to each connected component. ∎

COROLLARY 9. *Assume that for each n, $0 \leqq n \leqq m$, $2^{\omega_n} < 2^{\omega_{n+1}}$ holds and $2^\omega \leqq \omega_m$. Then connected (locally connected), locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^\omega$ are Lindelöf (a free union of Lindelöf subspaces, respectively).*

PROOF: Since the tightness is not greater than the character which is $2^\omega \leqq \omega_m$, we can apply Theorem 7 (Corollary 8, respectively). ∎

The next result was announced in [**1**].

COROLLARY 10. *[$CH + 2^{\omega_1} < 2^{\omega_2}$] Connected (locally connected) locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^\omega$ are Lindelöf (a free union of Lindelöf subspaces, respectively).*

The next result was proved in [**1**].

COROLLARY 11. *If $2^\omega < 2^{\omega_1}$ holds, then connected (locally connected), locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^\omega$ and tightness $\leqq \omega$ are Lindelöf (a free union of Lindelöf subspaces, respectively).*

Next we shall show that, in fact, the converse of Corollary 11 is also true.

THEOREM 12. *The following assertions are equivalent:*

(1) $2^\omega < 2^{\omega_1}$ *holds;*

(2) *connected, locally Lindelöf, submetaLindelöf, normal spaces of character $\leqq 2^\omega$ and tightness $\leqq \omega$ are Lindelöf;*

(3) *connected, locally Lindelöf, submetacompact, normal spaces of character (weight) $\leqq 2^\omega$ and tightness $\leqq \omega$ are Lindelöf.*

PROOF: (1) $\to$ (2): by Corollary 11.

(2) $\to$ (3): evident.

(3) $\to$ (1): assume $2^\omega = 2^{\omega_1}$. We shall show that the negation of (3) holds. Then there is a collection of ω_1-many free ultrafilters on ω, say $\{x_\alpha : \alpha < \omega_1\}$, such that for any subset D of ω_1 there is a subset U of ω such that $U \in x_\alpha$ for every $\alpha \in D$ and $\omega - U \in x_\alpha$ for every $\alpha \in \omega_1 - D$, by [2].

Let $\mathbb{R}$ be the real line. Since $\mathbb{R}$ is normal and ω is closed in $\mathbb{R}$ (hence ω is $C^\star$-embedded in $\mathbb{R}$), $\beta\omega = \mathrm{cl}_{\beta\mathbb{R}}\,\omega \subset \beta\mathbb{R}$ is valid. Here βY denotes the Stone-Čech compactification of a Tychonoff space Y. Let X be $\mathbb{R} \bigcup \{x_\alpha : \alpha < \omega_1\}$. Equip X with the subspace topology on $\beta\mathbb{R}$. We shall show that this X has the desired properties.

Since $\mathbb{R}$ is connected and dense in X, X is connected.

To show the normality of X, it is enough to show that the subspace $X - \mathbb{R}$ is normalised in X. Let D be a subset of ω_1, and U be a subset of ω as above. Let W be the set $\bigcup\{(n - 1/2, n + 1/2) : n \in U\}$, and W' be an open set of X such that $W' \bigcap \mathbb{R} = W$. Then it is not hard to show that $\{x_\alpha : \alpha \in D\} \subseteq W'$ and $\mathrm{cl}\, W' \bigcap \{x_\alpha : \alpha \in \omega_1 - D\} = 0$. Hence X is normal. This argument implies $\{x_\alpha : \alpha < \omega_1\}$ is closed discrete in X. Therefore X is not Lindelöf.

Since points of $\mathbb{R}$ have compact neighbourhoods in X, to show the local Lindelöfness of X we must show that points of $X - \mathbb{R}$ have closed Lindelöf neighbourhoods in X. Take an open neighbourhood U of x_α such that $\mathrm{cl}\, U \cap \{x_\alpha : \alpha < \omega_1\} = \{x_\alpha\}$. Since $\mathbb{R}$ is hereditarily Lindelöf, it follows that $\mathrm{cl}\, U$ is Lindelöf. Thus X is locally Lindelöf.

To show X is submetacompact, let $\mathcal{U}$ be an open cover of X. For $\alpha < \omega_1$, fix an $U_\alpha \in \mathcal{U}$ such that $x_\alpha \in U_\alpha$. For $n \in \omega$ and $\alpha \in \omega_1$, let $U_{\alpha n}$ be the open set $U_\alpha - ([-n, n] \cup \{x_\beta : \beta \neq \alpha, \quad \beta < \omega_1\})$ of X. By the paracompactness of $\mathbb{R}$, take a locally finite open family $\mathcal{V}$ refining $\mathcal{U}$ such that $\bigcup \mathcal{V} = \mathbb{R}$. Then $\{\mathcal{U}_n : n \in \omega\}$, where $\mathcal{U}_n = \mathcal{V} \cup \{U_{\alpha n} : \alpha < \omega_1\}$, shows the submetacompactness of X.

Since $\mathbb{R}$ has a countable basis, the weight of X is not greater than 2^{ω}.

Finally since $\mathbb{R}$ is hereditarily separable, the tightness of X is ω. The proof is complete. ∎

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