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CONSTRUCTION OF REGULAR SEMIGROUPS WITH INVERSE TRANSVERSALS

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1. Preliminaries

Let S be a regular semigroup. An inverse subsemigroup S° of S is an *inverse transversal* if $|V(x)\cap S^\circ|=1$ for each $x\in S$, where $V(x)$ denotes the set of inverses of x. In this case, the unique element of $V(x)\cap S^\circ$ is denoted by x°, and $x^{\circ\circ}$ denotes $(x^\circ)^{-1}$. Throughout this paper S denotes a regular semigroup with an inverse transversal S°, and $E(S^\circ)=E^\circ$ denotes the semilattice of idempotents of S°. The sets $\{e\in S: ee^\circ=e\}$ and $\{f\in S: f^\circ f=f\}$ are denoted by I_S and Λ_S, respectively, or simply I and Λ. Though each element of these sets is idempotent, they are not necessarily sub-bands of S. When both I and Λ are sub-bands of S, S° is called an *S-inverse transversal*. An inverse transversal S° is *multiplicative* if $x^\circ xyy^\circ\in E^\circ$, and S° is *weakly multiplicative* if $(x^\circ xyy^\circ)^\circ\in E^\circ$ for every $x,y\in S$. A band B is *left* [resp. *right*] *regular* if $efe=ef$ [resp. $efe=fe$], and B is *left* [resp. *right*] *normal* if $efg=egf$ [resp. $efg=feg$] for every $e,f,g\in B$. A subset Q of S is a *quasi-ideal* of S if $QSQ\subseteq S$.

We list already obtained results in [**3, 4, 5, 6**], which will be used in this paper:

(1.1) $(xy)^\circ=(x^\circ xy)^\circ x^\circ=y^\circ(xyy^\circ)^\circ$ for every $x,\ y\in S$.

(1.2) If S° is a quasi-ideal of S, then I [resp. Λ] is a left [resp. right] normal band with an inverse transversal E°

(1.3) If S° is an S-inverse transversal, then I [resp. Λ] is a left [resp. right] regular band with an inverse transversal E°.

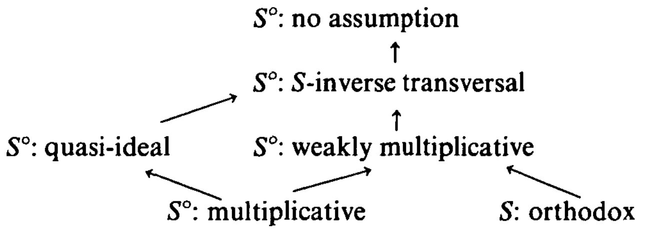
(1.4) S° is weakly multiplicative if and only if $I\Lambda=\{ef: e\in I, f\in\Lambda\}$ is the idempotent-generated subsemigroup of S with inverse transversal E°.

(1.5) S° is an S-inverse transversal of S if and only if $(x^\circ y)^\circ=y^\circ x^{\circ\circ}$ and $(xy^\circ)^\circ=y^{\circ\circ}x^\circ$ for every $x,y\in S$.

(1.6) S is isomorphic to the set $\{(e,x,f)\in I\times S^\circ\times\Lambda: e^\circ=xx^{-1}, f^\circ=x^{-1}x\}$ under the multiplication

$$(e,x,f)(g,y,h)=(exfgy(xfgy)^\circ, x(fg)^{\circ\circ}y, (xfgy)^\circ xfgyh).$$

(1.7) The following diagram is obtained:



Proposition 1.8. *S is orthodox if and only if $(xy)^\circ = y^\circ x^\circ$ for every $x, y \in S$.*

Proof. If S is orthodox, then $y^\circ x^\circ \in V(xy) \cap S^\circ$ for every $x, y \in S$, so that $(xy)^\circ = y^\circ x^\circ$. Conversely, if $(xy)^\circ = y^\circ x^\circ$ for every $x, y \in S$, then $e \in E(S)$ if and only if $e^\circ \in E^\circ$. Because, $e \in E(S)$ implies $e^\circ = (ee)^\circ = e^\circ e^\circ$ and $e^\circ \in E^\circ$ implies $e = ee^\circ e = e(e^\circ)^2 e = e(e^2)^\circ e = e(e^2)^\circ e^2 (e^2)^\circ e = ee^\circ eee^\circ e = ee$. Let $e, f \in E(S)$. Then $(ef)^\circ = f^\circ e^\circ \in E^\circ$, so that $ef \in E(S)$. Thus S is orthodox.

The above result has been obtained, when S° is multiplicative, by T.S. Blyth and R. McFadden (cf. [**1**]).

2. Main theorem

To achieve our aim, we need several lemmas.

Lemma 2.1. *For each $a \in E^\circ$, let $L_a = \{e \in I : e^\circ = a\}$ and $R_a = \{f \in \Lambda : f^\circ = a\}$. Then:*

(1) *L_a [resp. R_a] is a left [resp. right] zero-semigroup,*

(2) *if $e \in L_a$, $g \in L_b$ with $b \leqq a$, then $eg \in L_b$, and if $f \in R_a$, $h \in R_b$ with $b \leqq a$, then $hf \in R_b$, and*

(3) *$I = \Sigma\{L_a : a \in E^\circ\}$ and $\Lambda = \Sigma\{R_a : a \in E^\circ\}$, where Σ denotes disjoint union.*

Proof. (1) For $e, g \in L_a$, we have $eg = ee^\circ g = eg^\circ g = eg^\circ = ee^\circ = e$.

(2) Let $e \in L_a$ and $g \in L_b$. Then $egb = egg^\circ = eg$. If $b \leqq a$, then $beg = baeg = g^\circ e^\circ eg = g^\circ e^\circ g = g^\circ g = g^\circ = b$. Thus $eg \in L_b$.

(3) This is clear.

Let Y be a semilattice, and T_α a semigroup for each $\alpha \in Y$. Let $T = \Sigma\{T_\alpha : \alpha \in Y\}$. If a partial binary operation $\circ$ is defined in T such that

(1) for $x, y, z \in T$, $x \circ (y \circ z) = (x \circ y) \circ z$ if $x \circ y$, $(x \circ y) \circ z$, $y \circ z$ and $x \circ (y \circ z)$ are defined in T,

(2) $x \circ y = xy$ if $x, y \in T_\alpha$, where xy is the product of x and y in T_α, and

(3) for $x \in T_\alpha$ and $y \in T_\beta$ with $\beta \leqq \alpha$, $x \circ y$ [resp. $y \circ x$] is defined and $x \circ y$ [resp. $y \circ x$] $\in T_\beta$,

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then the resulting system $T(\circ)$ is called a *lower* [resp. *upper*] *partial chain* of $\{T_\alpha : \alpha \in Y\}$. In particular, if each T_α contains $\bar{\alpha}$, and $\{\bar{\alpha}: \alpha \in Y\}$ forms a semilattice isomorphic to Y under the binary operation $\circ$, then $\{\bar{\alpha}: \alpha \in Y\}$ is called a *semilattice transversal* of $T(\circ)$.

By Lemma 2.1, I [resp. Λ] is a lower [resp. upper] partial chain of left [resp. right] zero semigroups $\{L_a : a \in E^\circ\}$ [resp. $\{R_a : a \in E^\circ\}$], and I and Λ have a common semilattice transversal E°.

Lemma 2.2. *If S° is an S-inverse transversal of S, then I [resp. Λ] is a semilattice of left [resp. right] zero semigroups $\{L_a : a \in E^\circ\}$ [resp. $\{R_a : a \in E^\circ\}$].*

Proof. Let $e \in L_a$ and $g \in L_b$. Then, by (1.1) and (1.5), we have $(eg)^\circ = (e^\circ eg)^\circ e^\circ = (e^\circ g)^\circ e^\circ = g^\circ e^\circ e^\circ = g^\circ e^\circ = ab$. Since $eg \in I$, $eg \in L_{ab}$.

Lemma 2.3. *Let $e \in I$ and $f \in \Lambda$. Then:*

(1) $f^\circ (fe)^{\circ\circ} e^\circ = (fe)^{\circ\circ}$,

(2) $(f^\circ e^\circ)^{\circ\circ} = f^\circ e^\circ$,

(3) $(ff^\circ)^{\circ\circ} = f^\circ$ *and* $(e^\circ e)^{\circ\circ} = e^\circ$,

(4) *if* $f^\circ = (fe^\circ)^{\circ\circ}(fe^\circ)^\circ$ [*resp.* $e^\circ = (f^\circ e)^\circ (f^\circ e)^{\circ\circ}$], *then* $f^\circ = (fe^\circ)^{\circ\circ}$ [*resp.* $e^\circ = (f^\circ e)^{\circ\circ}$],

(5) *if S° is an S-inverse transversal of S, then* $(f^\circ e)^{\circ\circ} = (fe^\circ)^{\circ\circ} = f^\circ e^\circ$,

(6) *if S° is weakly multiplicative, then* $(fe)^{\circ\circ} \in E^\circ$.

(7) *if S° is a quasi-ideal of S, then* $(fe)^{\circ\circ} = fe$, *and*

(8) *if S is orthodox , then* $(fe)^{\circ\circ} = f^\circ e^\circ$.

Proof. (1) By (1.1) we have $(fe)^\circ = e^\circ (f^\circ fee^\circ)^\circ f^\circ = e^\circ (fe)^\circ f^\circ$, so that $(fe)^{\circ\circ} = f^\circ (fe)^{\circ\circ} e^\circ$. (2) and (3) are clear. (4) Let $f^\circ = (fe^\circ)^{\circ\circ}(fe^\circ)^\circ$. Then we have $fe^\circ = f^\circ fe^\circ = (fe^\circ)^{\circ\circ}(fe^\circ)^\circ fe^\circ$, so that $fe^\circ (fe^\circ)^\circ = (fe^\circ)^{\circ\circ}(fe^\circ)^\circ = f^\circ$. Thus we have $f(fe^\circ)^\circ f = fe^\circ (fe^\circ)^\circ f = f^\circ f = f$ and $(fe^\circ)^\circ f (fe^\circ)^\circ = (fe^\circ)^\circ fe^\circ (fe^\circ)^\circ = (fe^\circ)^\circ$, so that $f^\circ = (fe^\circ)^\circ$. Thus $f^\circ = (fe^\circ)^{\circ\circ}$. (5) By (1.5), this is clear. (6) Since $(fe)^\circ = (f^\circ fee^\circ)^\circ \in E^\circ$, $(fe)^{\circ\circ} \in E^\circ$. (7) Since $fe = f^\circ fee^\circ \in S^\circ S S^\circ \subseteq S^\circ$, $(fe)^{\circ\circ} = fe$. (8) By Proposition 1.8, this is clear.

Lemma 2.4. *For each $(x, y) \in S^\circ \times S^\circ$, let $\alpha_{(x,y)}: R_{x^{-1}x} \times L_{yy^{-1}} \to I$ and $\beta_{(x,y)}: R_{x^{-1}x} \times L_{yy^{-1}} \to \Lambda$ be mappings defined by $(f, e)\alpha_{(x,y)} = xfey(xfey)^\circ$ and $(f, e)\beta_{(x,y)} = (xfey)^\circ xfey$, respectively. Then:*

(1) $(f, e)\alpha_{(x,y)} \in L_{x(fe)^{\circ\circ}y(x(fe)^{\circ\circ}y)^{-1}}$ *and* $(f, e)\beta_{(x,y)} \in R_{(x(fe)^{\circ\circ}y)^{-1}x(fe)^{\circ\circ}y}$,

(2) *if* $f \in R_{x^{-1}x}, g \in L_{yy^{-1}}, h \in R_{y^{-1}y}$ *and* $k \in L_{zz^{-1}}$, *then*

$$(f, g)\alpha_{(x,y)}((f, g)\beta_{(x,y)}h, k)\alpha_{(x(fg)^{\circ\circ}y, z)} = (f, g(h, k)\alpha_{(y,z)})\alpha_{(x, y(hk)^{\circ\circ}z)},$$

$$(f, g(h, k)\alpha_{(y,z)})\beta_{(x, y(hk)^{\circ\circ}z)}(h, k)\beta_{(y,z)} = ((f, g)\beta_{(x,y)}h, k)\beta_{(x(fg)^{\circ\circ}y, z)}$$

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and

$$(fg)^{\circ\circ}y((f,g)\beta_{(x,y)}hk)^{\circ\circ}=(fg(h,k)\alpha_{(y,z)})^{\circ\circ}y(hk)^{\circ\circ},$$

(3) $(x^{-1}x, yy^{-1})\alpha_{(x,y)}=xy(xy)^{-1}$ *and* $(x^{-1}x, yy^{-1})\beta_{(x,y)}=(xy)^{-1}xy$, *and*
(4) *if* S° *is an S-inverse transversal of S, then* $(f^\circ, e)\alpha_{(f^\circ,e^\circ)}=f^\circ e$ *and* $(f,e^\circ)\beta_{(f^\circ,e^\circ)}=fe^\circ$.

Proof. (1) Let $(f,e)\in R_{x^{-1}x}\times L_{yy^{-1}}$. Then, by (1.1), we have

$$(xfey)^\circ=y^{-1}(x^{-1}xfeyy^{-1})^\circ x^{-1}=y^{-1}(f^\circ fee^\circ)x^{-1}=y^{-1}(fe)^\circ x^{-1}=(x(fe)^{\circ\circ}y)^{-1},$$

so that

$$((f,e)\alpha_{(x,y)})^\circ=(xfey(xfey)^\circ)^\circ=(efey)^{\circ\circ}(xfey)^\circ=x(fe)^{\circ\circ}y(x(fe)^{\circ\circ}y)^{-1}.$$

(2) By using (1.1), we can tediously but easily show that

$$(f,g)\alpha_{(x,y)}((f,g)\beta_{(x,y)}h,k)\alpha_{(x(fg)^{\circ\circ}y,z)}$$

$$=xfgyhkz(xfgyhkz)^\circ=(f,g(h,k)\alpha_{(y,z)})\alpha_{(x,y(hk)^{\circ\circ}z)},\ (f,g(h,k)\alpha_{(y,z)})\beta_{(x,y(hk)^{\circ\circ}z)}(h,k)\beta_{(y,z)}$$

$$=(xfgyhkz)^\circ xfgyhkz=((f,g)\beta_{(x,y)}h,k)\beta_{(x(fg)^{\circ\circ}y,z)}$$

and

$$(fg)^{\circ\circ}y((f,g)\beta_{(x,y)}hk)^{\circ\circ}=(fgyhk)^{\circ\circ}=(fg(h,k)\alpha_{(y,z)})^{\circ\circ}y(hk)^{\circ\circ}.$$

(3) By the definition, this can be easily proved.
(4) By using (1.5), this can be easily proved.

Let M and N be two sets. A *partial mapping* from M to N is a mapping from a subset C of M into N. The set of all partial mappings form M to N is denoted by $PT(M,N)$. Then, by Lemma 2.4, $\alpha_{(x,y)}\in PT(\Lambda\times I, I)$ and $\beta_{(x,y)}\in PT(\Lambda\times I,\Lambda)$ with $\operatorname{dom}(\alpha_{(x,y)})=\operatorname{dom}(\beta_{(x,y)})=R_{x^{-1}x}\times L_{yy^{-1}}$.

Theorem 2.5. *Let* S° *be an inverse semigroup with the semilattice* E° *of idempotents, and let* I *be a lower partial chain of left zero semigroups* $\{L_a: a\in E^\circ\}$ *and* Λ *an upper partial chain of right zero semigroups* $\{R_a; a\in E^\circ\}$. *Suppose that* I *and* Λ *have a common semilattice transversal* E°. *Let* $\Lambda\times I\to S^\circ$, $(f,e)\to f*e$ *be a mapping satisfying:*

(1*) $f^\circ(f*e)e^\circ=f*e$,
(2*) $f^\circ*e^\circ=f^\circ e^\circ$,
(3*) $f*f^\circ=f^\circ$ *and* $e^\circ*e=e^\circ$ *and*

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(4*) *if* $f^\circ=(f*e^\circ)(f*e^\circ)^{-1}$, *then* $f^\circ=f*e^\circ$, *and if* $e^\circ=(f^\circ*e)^{-1}(f^\circ*e)$, *then* $e^\circ=f^\circ*e$.

Suppose that, for each $(x,y)\in S^\circ\times S^\circ$, *there exist* $\alpha_{(x,y)}\in PT(\Lambda\times I,I)$ *and* $\beta_{(x,y)}\in PT(\Lambda\times I,\Lambda)$ *satisfying:*

(a*) $\mathrm{dom}(\alpha_{(x,y)})=\mathrm{dom}(\beta_{(x,y)})=R_{x^{-1}x}\times L_{yy^{-1}}$, $(f,e)\alpha_{(x,y)}\in L_{x(f*e)y(x(f*e)y)^{-1}}$ *and* $(f,e)\beta_{(x,y)}\in R_{(x(f*e)y)^{-1}x(f*e)y}$,

(b*) if $f\in R_{x^{-1}x}$, $g\in L_{yy^{-1}}$, $h\in R_{y^{-1}y}$ *and* $k\in L_{zz^{-1}}$, *then*

$$(f,g)\alpha_{(x,y)}((f,g)\beta_{(x,y)}h,k)\alpha_{(x(f*g)y,z)}=(f,g(h,k)\alpha_{(y,z)})\alpha_{(x,y(h*k)z)},$$

$$(f,g(h,k)\alpha_{(y,z)})\beta_{(x,y(h*k)z)}(h,k)\beta_{(y,z)}=((f,g)\beta_{(x,y)}h,k)\beta_{(x(f*g)y,z)}$$

and

$$(f*g)y((f,g)\beta_{(x,y)}h*k)=(f*g(h,k)\alpha_{(y,z)})y(h*k),\quad \textit{and}$$

(c*) $(x^{-1}x,yy^{-1})\alpha_{(x,y)}=xy(xy)^{-1}$ *and* $(x^{-1}x,yy^{-1})\beta_{(x,y)}=(xy)^{-1}xy$.

Define a multiplication on the set $W=\{(e,x,f)\in I\times S^\circ\times\Lambda : e\in L_{xx^{-1}}, f\in R_{x^{-1}x}\}$ *by* $(e,x,f)(g,y,h)=(e(f,g)\alpha_{(x,y)},\ x(f*g)y,\ (f,g)\beta_{(x,y)}h)$. *Then* W *is a regular semigroup with an inverse transversal isomorphic to* S°.

Conversely, every regular semigroup with an inverse transversal can be constructed in this way.

Proof. We can easily show, by using (a*) and (b*), that W is a semigroup. Let $W^\circ=\{(e,x,f)\in W : e,f\in E^\circ\}$. Then $(e,x,f)\in W^\circ$ if and only if $e=xx^{-1}$ and $f=x^{-1}x$. By (2*) and (c*), we obtain $(xx^{-1},x,x^{-1}x)(yy^{-1},y,y^{-1}y)=(xy(xy)^{-1},xy,(xy)^{-1}xy)$, which shows $W^\circ\simeq S^\circ$, so that W° is an inverse subsemigroup of W.

For $(e,x,f)\in W$, by (3*), we have $x(f*f^\circ)x^{-1}=xf^\circ x^{-1}=xx^{-1}=e^\circ$, so that $(f,f^\circ)\alpha_{(x,x^{-1})}\in L_{e^\circ}$ and $(f,f^\circ)\beta_{(x,x^{-1})}\in R_{e^\circ}$. Thus we have $(e,x,f)(f^\circ,x^{-1},e^\circ)=(e(f,f^\circ)\alpha_{(x,x^{-1})},e^\circ,(f,f^\circ)\beta_{(x,x^{-1})}e^\circ)=(e,e^\circ,e^\circ)$. Again by (3*), $e^\circ(e^\circ*e)x=x$, so that $(e^\circ,e)\alpha_{(e^\circ,x)}\in L_{e^\circ}$ and $(e^\circ,e)\beta_{(e^\circ,x)}\in R_{f^\circ}$. Thus we have $(e,x,f)(f^\circ,x^{-1},e^\circ)(e,x,f)=(e,e^\circ,e^\circ)(e,x,f)=(e(e^\circ,e)\alpha_{(e^\circ,x)},x,(e^\circ,e)\beta_{(e^\circ,x)}f)=(e,x,f)$ Similarly we obtain $(f^\circ,x^{-1},e^\circ)(e,x,f)(f^\circ,x^{-1},e^\circ)=(f^\circ,x^{-1},e^\circ)$. Consequently (f°,x^{-1},e°) is an inverse of (e,x,f) in W°.

Let $(g^\circ,y,h^\circ)\in W^\circ$ be an inverse of $(e,x,f)\in W$. Then

$$(e,x,f)=(e,x,f)(g^\circ,y,h^\circ)(e,x,f)=(\dots,x(f*g^\circ)y,(f,g^\circ)\beta_{(x,y)}h^\circ)(e,x,f)$$

$$=(\dots,x(f*g^\circ)y((f,g^\circ)\beta_{(x,y)}h^\circ*e)x,\dots).$$

Thus we have

$$xx^{-1}\geqq x(f*g^\circ)y(x(f*g^\circ)y)^{-1}$$

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$$\geqq x(f*g^\circ)y((f,g^\circ)\beta_{(x,y)}h^\circ*e)x(x(f*g^\circ)y((f,g^\circ)\beta_{(x,y)}h^\circ*e)x)^{-1}$$

$$=xx^{-1},$$

so that $xx^{-1}=x(f*g^\circ)y(x(f*g^\circ)y)^{-1}$. By (1*)

$$x^{-1}x=x^{-1}xx^{-1}x=x^{-1}x(f*g^\circ)yy^{-1}(f*g^\circ)^{-1}x^{-1}x$$

$$=f^\circ(f*g^\circ)g^\circ(f*g^\circ)^{-1}f^\circ=(f*g^\circ)(f*g^\circ)^{-1}.$$

By (4*), we obtain $f^\circ=f*g^\circ$, and similarly $e^\circ=h^\circ*e$. Since

$$e^\circ=xx^{-1}=x(f*g^\circ)y(x(f*g^\circ)y)^{-1},$$

$(f,g^\circ)\alpha_{(x,y)}\in L_{e^\circ}$. Thus we have

$$(g^\circ,y,h^\circ)=(g^\circ,y,h^\circ)(e,x,f)(g^\circ,y,h^\circ)=(g^\circ,y,h^\circ)(e(f,g^\circ)\alpha_{(x,y)},\ x(f*g^\circ)y,\dots)$$

$$=(g^\circ,y,h^\circ)(e,xf^\circ y,\dots)=(g^\circ,y,h^\circ)(e,xy,\dots)=(\dots,y(h^\circ*e)xy,\dots)$$

$$=(\dots,ye^\circ xy,\dots)=(\dots,yxy,\dots),$$

so that $y=yxy$. Since $y^{-1}y\geqq(xy)^{-1}xy\geqq(yxy)^{-1}yxy=y^{-1}y, y^{-1}y=(xy)^{-1}xy$ and $(f,g^\circ)\beta_{(x,y)}\in R_{h^\circ}$. Thus we have $(e,x,f)=(e,x,f)(g^\circ,y,h^\circ)(e,x,f)=(e,xy,h^\circ)(e,x,f)=(\dots,xyx,\dots)$, so that $x=xyx$. Consequently $y=x^{-1}$. Thus each element $(e,x,f)\in W$ has the unique inverse (f°,x^{-1},e°) in W°.

Conversely, let S be a regular semigroup with an inverse transversal S°. By Lemma 2.1, I_S and Λ_S are a lower partial chain of left zero semigroups $\{L_a: a\in E^\circ\}$ and an upper partial chain of right zero semigroups $\{R_a: a\in E^\circ\}$, respectively. For $(f,e)\in\Lambda_S\times I_S$, put $f*e=(fe)^{\circ\circ}$. Then, by Lemma 2.3, $*$ is a mapping from $\Lambda_S\times I_S$ into S° satisfying the conditions (1*)–(4*). For each $(x,y)\in S^\circ\times S^\circ$ and for every $(f,e)\in R_{x^{-1}x}\times L_{yy^{-1}}$, put $(f,e)\alpha_{(x,y)}=xfey(xfey)^\circ$ and $(f,e)\beta_{(x,y)}=(xfey)^\circ xfey$. Then, $\alpha_{(x,y)}\in PT(\Lambda\times I,I)$ and $\beta_{(x,y)}\in PT(\Lambda\times I,\Lambda)$, and by Lemma 2.4, they satisfy the conditions (a*)–(c*). Thus we can construct a semigroup $W=\{(e,x,f)\in I_S\times S^\circ\times\Lambda_S: e\in L_{xx^{-1}}, f\in R_{x^{-1}x}\}$ under the multiplication $(e,x,f)(g,y,h)=(exfgy(xfgy)^\circ, x(fg)^{\circ\circ}y, (xfgy)^\circ xfgyh)$. Then, by (1.6), $W\simeq S$. The proof is complete.

3. Application to special cases

(1) S-inverse transversals

Lemma 3.1. *In Theorem 2.5, if the mapping $*$ satisfies the condition (1*), and (2°) $f*e^\circ=f^\circ*e=f^\circ e^\circ$ instead of the conditions (2*), (3*) and (4*), then W is a regular*

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semigroup with an S-inverse transversal isomorphic to S°. Conversely, every such semigroup can also be so constructed.

Furthermore, if a binary operation $\circ$ is defined on I [resp. Λ] by $e\circ g=e(e^\circ,g)\alpha_{(e^\circ,g^\circ)}$ for $e,g\in I$ [resp. $h\circ g=(f,h^\circ)\beta_{(f^\circ,h^\circ)}h$ for $f,h\in\Lambda$], then $I(\circ)\simeq I_W$ [resp. $\Lambda(\circ)\simeq\Lambda_W$].

Proof. It is clear that (2°) implies (2*), (3*) and (4*). Thus, it is enough to show that W° is an S-inverse transversal of W. Let $(e,x,f),(g,y,h)\in W$, Then, by (2°), we have

$$\begin{aligned}((e,x,f)^\circ(g,y,h))^\circ &= ((f^\circ,x^{-1},e^\circ)(g,y,h))^\circ\\ &= (\ldots,x^{-1}(e^\circ * g)y,\ldots)^\circ\\ &= (\ldots,x^{-1}e^\circ g^\circ y,\ldots)^\circ=(\ldots,x^{-1}y,\ldots)^\circ\\ &= ((x^{-1}y)^{-1}x^{-1}y,(x^{-1}y)^{-1},x^{-1}y(x^{-1}y)^{-1})\\ &= (y^{-1}x(y^{-1}x)^{-1},y^{-1}x,(y^{-1}x)^{-1}y^{-1}x)\\ &= (y^{-1}y,y^{-1},yy^{-1})(xx^{-1},x,x^{-1}x)\\ &= (g,y,h)^\circ(e,x,f)^{\circ\circ},\end{aligned}$$

and similarly $((e,x,f)(g,y,h)^\circ)^\circ=(g,y,h)^{\circ\circ}(e,x,h)^\circ$, so that, by (1.5), W° is an S-inverse transversal. By (5) of Lemma 2.3, the converse assertion is clear.

For the last assertion, by a part of the proof of Theorem 2.5, $(e,x,f)(e,x,f)^\circ=(e,e^\circ,e^\circ)$, which shows that $(e,x,f)\in I_W$ if and only if $x=f=e^\circ$. Let (e,e°,e°), $(g,g^\circ,g^\circ)\in I_W$. Then, by (1*) and (2°), $e^\circ(e^\circ * g)g^\circ=e^\circ * g=e^\circ g^\circ$, so that $(e,e^\circ,e^\circ)(g,g^\circ,g^\circ)=(e(e^\circ,g)\alpha_{(e^\circ,g^\circ)},e^\circ g^\circ,(e^\circ,g)\beta_{(e^\circ,g^\circ)}g^\circ)$. Since I_W is a sub-band of W, $(e(e^\circ,g)\alpha_{(e^\circ,g^\circ)},e^\circ g^\circ,(e^\circ,g)\beta_{(e^\circ,g^\circ)}g^\circ)\in I_W$, so that $(e^\circ,g)\beta_{(e^\circ,g^\circ)}g^\circ=e^\circ g^\circ$. Consequently $(e,e^\circ,e^\circ)(g,g^\circ,g^\circ)=(e\circ g,e^\circ g^\circ,e^\circ g^\circ)$, which shows that $I_W\simeq I(\circ)$.

By Lemmas 2.2 and 3.1, we obtain:

Theorem 3.2. *Let S° be an inverse semigroup with the semilattice E° of idempotents, and let I be a semilattice of left zero semigroups $\{L_a\colon a\in E^\circ\}$ and Λ a semilattice of right zero semigroups $\{R_a\colon a\in E^\circ\}$. Suppose that I and Λ have a common semilattice transversal E°. Let $\Lambda\times I\to S^\circ,(f,e)\to f*e$ be a mapping satisfying:*

(1*) *$f^\circ(f*e)e^\circ=f*e$ and* (2°) *$f^\circ*e=f*e^\circ=f^\circ e^\circ$.*

Suppose that, for each $(x,y)\in S^\circ\times S^\circ$, there exist $\alpha_{(x,y)}\in PT(\Lambda\times I,I)$ and $\beta_{(x,y)}\in PT(\Lambda\times I,\Lambda)$ satisfying the conditions (a*), (b*), *and* (c*) *in Theorem 2.5, and* (d*) *$(f^\circ,e)\alpha_{(f^\circ,e^\circ)}=f^\circ e$ and $(f,e^\circ)\beta_{(f^\circ,e^\circ)}=fe^\circ$. Define a multiplication on the set $W=\{(e,x,f)\in I\times S^\circ\times\Lambda\colon e\in L_{xx^{-1}},\ f\in R_{x^{-1}x}\}$ as in Theorem 2.5. Then W is a regular semigroup with an S-inverse transversal isomorphic to S°, and $I_W\simeq I$ and $\Lambda_W\simeq\Lambda$. Conversely, every such semigroup can be so constructed.*

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Proof. It is enough to show that $I_W \simeq I$ and $\Lambda_W \simeq \Lambda$. Let $e, g \in I$. Then, by (d*), $e(e^\circ, g)\alpha_{(e^\circ, g^\circ)} = ee^\circ g = eg$, so that the binary operation $\circ$ in Lemma 3.1 coincides with the product in I. Thus $I_W \simeq I$, and similarly $\Lambda_W \simeq \Lambda$.

(2) *Weakly multiplicative inverse transversals*

Lemma 3.3. *In Theorem 2.5, if the mapping $*$ is $\Lambda \times I \to E^\circ$ instead of $\Lambda \times I \to S^\circ$, then W is a regular semigroup with a weakly multiplicative inverse transversal isomorphic to S°. Conversely, every such semigroup can be so constructed.*

Proof. It is enough to show that W° is weakly multiplicative. Let $(e, x, f), (g, y, h) \in W$. Then, since $f * g \in E^\circ$, we have

$$((e,x,f)^\circ(e,x,f)(g,y,h)(g,y,h)^\circ)^\circ = ((f^\circ, f^\circ, f)(g, g^\circ, g^\circ))^\circ = (\ldots, f^\circ(f*g)g^\circ, \ldots)^\circ$$

$$= (\ldots, f*g, \ldots)^\circ = (f*g, f*g, f*g) \in E(W^\circ),$$

so that W° is weakly multiplicative. By (6) of Lemma 2.3, the converse assertion is clear.

By Theorem 3.2 and Lemma 3.3, we obtain:

Theorem 3.4. *Let S°, E°, I and Λ be as in Theorem 3.2, Let $\Lambda \times I \to E^\circ, (f, e) \to f * g$ be a mapping satisfying the condition (1*) and (2°). Suppose that, for each $(x, y) \in S^\circ \times S^\circ$, there exist $\alpha_{(x,y)} \in PT(\Lambda \times I, I)$ and $\beta_{(x,y)} \in PT(\Lambda \times I, \Lambda)$ satisfying the conditions (a*)–(d*). Define a multiplication on the set $W = \{(e, x, f)) \in I \times S^\circ \times \Lambda : e \in L_{xx^{-1}},\ f \in R_{x^{-1}x}\}$ as in Theorem 2.5. Then W is a regular semigroup with a weakly multiplicative inverse transversal isomorphic to S°, and $I_W \simeq I$ and $\Lambda_W \simeq \Lambda$. Conversely, every such semigroup can be so constructed.*

(3) *Orthodox semigroups*

Lemma 3.5. *In Theorem 2.5, if the mapping $*$ satisfies (1°) $f * e = f^\circ e^\circ$ instead of (1*)–(4*), then W is an orthodox semigroup with an inverse transversal isomorphic to S°. Conversely, every such semigroup can be so constructed.*

Proof. It is clear that (1°) implies (1*)–(4*). Thus, it is enough to show that W is orthodox. Let $(e, x, f), (g, y, h) \in W$. Then we have

$$((e,x,f)(g,y,h))^\circ = (\ldots, x(f*g)y, \ldots)^\circ = (\ldots, xf^\circ g^\circ y, \ldots) = (\ldots, xy, \ldots)^\circ$$

$$= ((xy)^{-1}xy, (xy)^{-1}, xy(xy)^{-1})$$

$$= (y^{-1}x^{-1}(y^{-1}x^{-1})^{-1}, y^{-1}x^{-1}, (y^{-1}x^{-1})^{-1}y^{-1}x^{-1})$$

$$=(y^{-1}y,y^{-1},yy^{-1})(x^{-1}x,x^{-1},xx^{-1})=(g,y,h)^{\circ}(e,x,f)^{\circ}.$$

By Proposition 1.8, W is orthodox. By (8) of Lemma 2.3, the converse assertion is clear.

If the mapping $*$ in Theorem 2.5 satisfies the condition (1°), then $x(f*e)y=xy$ for $(f,e)\in R_{x^{-1}x}\times L_{yy^{-1}}$, so that we can omit the mapping $*$. Thus, by Theorem 3.2 and Lemma 3.5, we obtain:

Theorem 3.6. *Let S°, E°, I and Λ be as in Theorem 3.2. Suppose that, for each $(x,y)\in S^{\circ}\times S^{\circ}$, there exist $\alpha_{(x,y)}\in PT(\Lambda\times I,I)$ and $\beta_{(x,y)}\in PT(\Lambda\times I,\Lambda)$ satisfying:*

(a°) $\operatorname{dom}(\alpha_{(x,y)})=\operatorname{dom}(\beta_{(x,y)})=R_{x^{-1}x}\times L_{yy^{-1}}$, $\operatorname{ran}(\alpha_{(x,y)})\subseteq L_{xy(xy)^{-1}}$ and $\operatorname{ran}(\beta_{(x,y)})\subseteq R_{(xy)^{-1}xy}$,

(b°) if $f\in R_{x^{-1}x}, g\in L_{yy^{-1}}$, $h\in R_{y^{-1}y}$ and $k\in L_{zz^{-1}}$, then

$$(f,g)\alpha_{(x,y)}((f,g)\beta_{(x,y)}h,k)\alpha_{(xy,z)}=(f,g(h,k)\alpha_{(y,z)})\alpha_{(x,yz)},$$

$$(f,g(h,k)\alpha_{(y,z)})\beta_{(x,yz)}(h,k)\beta_{(y,z)}=((f,g)\beta_{(x,y)}h,k)\beta_{(xy,z)},$$

and (c*) *and* (d*) *in Theorem* 3.2. *Define a multiplication on the set*

$$W=\{(e,x,f))\in I\times S^{\circ}\times\Lambda: e\in L_{xx^{-1}}, f\in R_{x^{-1}x}\}$$

by $(e,x,f)(g,y,h)=(e(f,g)\alpha_{(x,y)},xy,(f,g)\beta_{(x,y)}h)$. Then W is an orthodox semigroup with an inverse transversal isomorphic to S°, and $I_W\simeq I$ and $\Lambda_W\simeq\Lambda$. Conversely, every such semigroup can be so constructed.

A *left* [resp. *right*] *inverse semigroup* is an orthodox semigroup whose band of idempotents is left [resp. right] regular.

Lemma 3.7. *In Theorem 3.2, if each right zero semigroup R_a, $a\in E^{\circ}$, is trivial, that is, $\Lambda=E^{\circ}$, then W is a left inverse semigroup with an inverse transversal isomorphic to S°, and $I_W\simeq I$. Conversely, every such semigroup can be so constructed.*

Proof. Let $(e,x,f)\in W$. Since $f\in E^{\circ}$, $f=x^{-1}x$. Let $(e,x,x^{-1}x)\in W$ be an idempotent. Then $(e,x,x^{-1}x)=(e,x,x^{-1}x)(e,x,x^{-1}x)=(\dots,x(x^{-1}x*e)x,\dots)=(\dots,xx^{-1}xe^{\circ}x,\dots)=(\dots x^2,\dots)$, so that $x=x^2\in E^{\circ}$. Thus, $(e,x,x^{-1}x)=(e,e^{\circ},e^{\circ})$, which shows $E(W)=I_W$. Consequently, the set $E(W)$ of idempotents of W is left regular, so that W is a left inverse semigroup.

In Lemma 3.7, for $g\in L_{yy^{-1}}$, we have $x(x^{-1}x*g)y=xx^{-1}xg^{\circ}y=xy$, so that the mapping $*$ can be omitted.

Corollary 3.8 ([8, Theorem 1]). *Let S° and I be as in Theorem 3.2. Let σ be an anti-homomorphism S° into* End(I), *$x\to\sigma(x)$ satisfying* (1) *$L_{yy^{-1}}\sigma(x)\subseteq L_{xy(xy)^{-1}}$*, (2) *$(yy^{-1})\sigma(x)=xy(xy)^{-1}$ and* (3) *$e\sigma(f^{\circ})=ef^{\circ}$.*

Define a multiplication on the set $W=\{(e,x)\in I\times S^{\circ}: e\in L_{xx^{-1}}\}$ *by* $(e,x)(g,y)=(e(g\sigma(x)),xy)$. *Then W is a left inverse semigroup with an inverse transversal isomorphic to* S°, *and* $I_W\simeq I$. *Conversely, every such semigroup can be so constructed.*

Proof. For each $(x,y)\in S^{\circ}\times S^{\circ}$ and for every $e\in L_{yy^{-1}}$, we take $(x^{-1}x,e)\alpha_{(x,y)}=e\sigma(x)$ and $(x^{-1}x,e)\beta_{(x,y)}=(xy)^{-1}xy$. Then we can show that $\alpha_{(x,y)}$ and $\beta_{(x,y)}$ satisfy the condition (a*)–(d*) in Theroem 3.2. Thus, by Lemma 3.7, we can construct a left inverse semigroup $W'=\{(e,x,x^{-1}x)\in I\times S^{\circ}\times E^{\circ}: e\in L_{xx^{-1}}\}$ under the multiplication

$$(e,x,x^{-1}x)(g,y,y^{-1}y)=(e(x^{-1}x,g)\alpha_{(x,y)},xy,(x^{-1}x,g)\beta_{(x,y)}y^{-1}y)=(e(g\sigma(x)),xy,(xy)^{-1}xy).$$

Let $W'\to W$ be a mapping given by $(e,x,x^{-1}x)\to(e,x)$. Then the mapping is clearly an isomorphism. Thus W is a left inverse semigroup with an inverse transversal isomorphic to S°, and $I_W\simeq I$. The proof of the converse is the same as in [**8**].

(4) *Quasi-ideal inverse transversals*

Corollary 3.9. ([3, Theorem 4.2]). *Let* S°, I *and* Λ *be as in Theorem* 3.2., *and let* $\Lambda\times I\to S^{\circ},(f,e)\to f*e$ *be a mapping satisfying the conditions* (1*) *and* (2°) *in Theorem* 3.2. *Suppose that, for each* $(x,y)\in S^{\circ}\times S^{\circ}$, *there exist* $\alpha_{(x,y)}$ *and* $\beta_{(x,y)}$ *in* $PT(\Lambda\times I,E^{\circ})$ *satisfying the condition* (a*) *in Theorem* 3.2. *Define a multiplication on the set* $W=\{(e,x,f))\in I\times S^{\circ}\times\Lambda: e\in L_{xx^{-1}}, f\in R_{x^{-1}x}\}$ *as in Theorem* 3.2. *Then W is a regular semigroup with a quasi-ideal inverse transversal isomorphic to* S°, *and* $I_W\simeq I$ *and* $\Lambda_W\simeq\Lambda$. *Conversely, every such semigroup can be so constructed.*

Proof. For $(f,e)\in R_{x^{-1}x}\times L_{yy^{-1}}$, since $(f,e)\alpha_{(x,y)}\in E^{\circ},(f,e)\alpha_{(x,y)}=x(f*e)y(x(f*e)y)^{-1}$ and similarly $(f,e)\beta_{(x,y)}=(x(f*e)y)^{-1}x(f*e)y$. Then we can easily show that $\alpha_{(x,y)}$ and $\beta_{(x,y)}$ satisfy the conditions (b*)–(d*) in Theorem 3.2. Thus W is a regular semigroup with an S-inverse transversal isomorphic to S°, and $I_W\simeq I$ and $\Lambda_W\simeq\Lambda$. For $(e,x,f)\in W$ and for $(g^{\circ},y,h^{\circ}),(k^{\circ},z,m^{\circ})\in W^{\circ}$, we can show that $(g^{\circ},y,h^{\circ})(e,x,f)(k^{\circ},z,m^{\circ})=(yxz(yxz)^{-1},yxz,(yxz)^{-1}yxz)\in W^{\circ}$, so that W° is a quasi-ideal of W. By (7) of Lemma 2.3, the converse assertion is clear.

Corolloray 3.10 ([**9**, Theorem 2]). *In Corollary* 3.9, *if the mapping* $*$ *is* $\Lambda\times I\to E^{\circ}$ *instead of* $\Lambda\times I\to S^{\circ}$, *then W is a regular semigroup with a multiplicative inverse transversal isomorphic to* S°, *and* $I_W\simeq I$ *and* $\Lambda_W\simeq\Lambda$. *Conversely, every such semigroup can be so constructed.*

Proof. For $(f,e)\in R_{x^{-1}x}\times L_{yy^{-1}}$, since $f*e\in E^{\circ}$, we have $(f,e)\alpha_{(x,y)}=x(f*e)x^{-1}$ and $(f,e)\beta_{(x,y)}=y^{-1}(f*e)y$. From this fact, we have $(e,x,f)^{\circ}(e,x,f)(g,y,h)(g,y,h)^{\circ}=(f^{\circ}f^{\circ},f)(g,g^{\circ},g^{\circ})=(f*g,f*g,f*g)\in E(W^{\circ})$, so that W° is multiplicative.

By (1.2), I_W [resp. Λ_W] in Corollaries 3.9 and 3.10 is a left [resp. right] normal band. Since $I_W\simeq I$ [resp. $\Lambda_W\simeq\Lambda$], I [resp. Λ] is necessarily a left [resp. right] normal band.

Though the condition (1) $g^\circ(f*e)=g^\circ f*e$ and $(f*e)h^\circ=f*eh^\circ$ has been used instead of (1*) $f^\circ(f*e)e^\circ=f*e$ in [3] and [9], Corollaries 3.9 and 3.10 can be obtained under the condition (1*) which is weaker than (1).

Moreover, we can obtain construction theorems on idempotent-generated regular semigroups with inverse transversals and bands with inverse transversals, by taking $S^\circ=E^\circ$ in Theorem 3.4 and Theorem 3.6, respectively.

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