THE BESSEL POLYNOMIALS

J. L. BURCHNALL

1. Krall and Frink [2] have recently considered in connection with certain solutions of the wave equation a system of polynomials $y_n(x)$, $(n = 0, 1, 2, \ldots)$, where y_n is defined as that polynomial solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + (2x + 2) \frac{dy}{dx} = n(n + 1)y \tag{1}$$

which is equal to unity when $x = 0$.

They note the relationship of these polynomials to Hankel's functions of imaginary argument and establish among other results:

$$\begin{aligned} y_n &= 2^{-n} e^{2/x} D^n (x^{2n} e^{-2/x}) \qquad \left(D = \frac{d}{dx}\right) \\ &= \sum_{r=0}^{w} \frac{(n + r)!}{(n - r)!\,r!} \left(\frac{x}{2}\right)^r, \end{aligned} \tag{2}$$

$$y_{n+1} = (2n + 1)xy_n + y_{n-1}, \tag{3}$$

$$\frac{1}{2\pi i} \int_C y_m y_n e^{-2/x} dx = \frac{(-)^{n+1} 2\epsilon_{mn}}{2n + 1}, \tag{4}$$

where C is the unit circle or any contour surrounding $x = 0$ and $\epsilon_{mn} = 0,1$ according as $m \neq n$, $m = n$.

It seems worthwhile to point out that the polynomials y_n are effectively the same as those encountered by T. W. Chaundy and the author in the course of a wider investigation [1]. Recognition of this fact leads to a more economical determination of the principal formulae of [2] as well as to other properties not mentioned by the authors of that paper. I therefore develop in more detail than was previously possible the properties of the polynomials in question.

2. It was shown in [1, pp. 478, 485] that the differential equation

$$\delta(\delta - 2n - 1)y = x^2 y \qquad \left(\delta = x\frac{d}{dx}\right), \tag{5}$$

where n is zero or a positive integer, has the solutions

$$y = \theta_n(x)e^{-x},\ \theta_n(-x)e^x,$$

where θ_n is a polynomial of degree n in x defined by

$$\theta_0 = 1,\ \theta_n = (-)^n e^x (\delta - 1)(\delta - 3) \ldots (\delta - 2n + 1)e^{-x}. \tag{6}$$

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We note that in θ_n the coefficient of x^n is unity and that more explicitly

(7) $$\theta_n = \sum_{r=0}^{n} \frac{(n+r)!\,x^{n-r}}{2^r(n-r)!\,r!}.$$

A comparison of (7) with (2) shows that

(8) $$y_n(x) = x^n\theta_n(1/x),$$

an identification which may be made without a knowledge of the explicit forms of θ_n, y_n by observing that, on setting $y = \theta_n e^{-x}$ in (5), we obtain

$$\delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n,$$

whence, without difficulty, $x^n\theta_n(1/x)$ is a solution of

(9) $$\delta z + \tfrac{1}{2}x(\delta - n)(\delta + n + 1)z$$

which is the "δ" form of (1).

3. The zeros of θ_n, and so of y_n, have properties not mentioned in [2]. It was for instance shown in [1], as a corollary to a more general argument, that zeros a_r $(r = 1, 2, \ldots, n)$ of θ_n satisfy the relations

(10) $$\sum_{r=1}^{n} a_r^{-1} = -1, \quad \sum_{r=1}^{n} a_r^{1-2s} = 0 \qquad (s = 2, 3, \ldots, n).$$

Hence the zeros b_r of y_n obey the relations

(11) $$\sum_{r=1}^{n} b_r = -1, \quad \sum_{r=1}^{n} b_r^{2s-1} = 0 \qquad (s = 2, 3, \ldots, n).$$

An *ad hoc* proof of (10) is immediate: for let σ_k denote the sum of the kth powers of the zeros of θ_n and let $\theta_n = e^x\phi_n$, then

(12) $$\frac{\phi'_n}{\phi_n} + 1 = \frac{\theta'_n}{\theta_n} = -\sum_{k=1}^{\infty} x^{k-1}\sigma_{-k}.$$

Reference to (6) shows that the expansion of ϕ_n in ascending powers contains no odd powers of x of index less than $2n + 1$. In consequence the expansion of ϕ'_n/ϕ_n contains no even powers of x with index less than $2n$ and so, on equating coefficients in (12), we have (10).

We may also establish the following results:

(13) *The polynomials θ_n, θ_{n+1} have no zero in common and no θ_n has a repeated zero.*

(14) *The polynomial θ_n has at most one real zero.*

Proofs of the statements in (13) are obtained by a familiar argument from the identities[1]

[1]It is worthy of notice that, if we define $\theta_{-n} = x^{1-2n}\theta_{n-1}$ the relations (15), (16) as well as the differential equation

$$\delta(\delta - 2n - 1)\theta_n = 2x(\delta - n)\theta_n$$

are also satisfied for negative integral n.

(15) $$\theta'_n - \theta_n = -x\theta_{n-1},$$

(16) $$\theta_{n+1} - x^2\theta_{n-1} = (2n+1)\theta_n,$$

the latter of which is equivalent to (3).

To establish (15) we observe that

$$\begin{aligned} e^{-x}(\delta - x)\theta_n &= \delta\theta_n e^{-x} \\ &= (-)^n(\delta-3)(\delta-5)\ldots(\delta-2n+1)\,\delta(\delta-1)e^{-x} \\ &= (-)^n(\delta-3)\ldots(\delta-2n+1)x^2e^{-x} \\ &= (-)^n x^2(\delta-1)\ldots(\delta-2n+3)e^{-x} \\ &= -x^2e^{-x}\theta_{n-1}. \end{aligned}$$

Again from (6)

$$(\delta - 2n - 1)\theta_n e^{-x} = -\theta_{n+1}e^{-x},$$

i.e.,

$$x(\theta'_n - \theta_n) = (2n+1)\theta_n - \theta_{n+1}$$

and, substituting on the left from (15), we have (16).

To prove (14) let

$$z_1 = e^{-x}\theta_n(x), \quad z_2 = e^x\theta_n(-x),$$

then from (5),

$$z_1z'_2 - z_2z'_1 = Cx^{2n}$$

or,

(17) $$\theta_n(-x)\theta'_n(x) + \theta_n(x)\theta'_n(-x) = 2\theta_n(x)\theta_n(-x) + Cx^{2n},$$

where C is a constant shown by a simple calculation to be $2(-1)^{n+1}$. Now in $\theta_n(x)$ all coefficients are positive and so all real zeros are negative. If possible let $-\alpha$, $-\beta$ be two consecutive real zeros. Then, from (17),

$$\theta_n(\alpha)\theta'_n(-\alpha) = C\alpha^{2n}, \quad \theta_n(\beta)\theta'_n(-\beta) = C\beta^{2n}.$$

But $\theta_n(\alpha)$, $\theta_n(\beta)$ are both positive and so $\theta'_n(-\alpha)$, $\theta'_n(-\beta)$ have the same sign which is impossible. Hence θ_n has at most one real zero.

It is natural to enquire whether the zeros (b_r) of $y_n(x)$, in some order or other furnish the only solutions of the system of equations

$$\sum_{r=1}^{n} x_r = -1, \quad \sum_{r=1}^{n} x_r^{2s-1} = 0 \qquad (s = 2, 3, \ldots, n).$$

This is in fact the case for, if (x_r) is any solution, let $-x_r = y_r$. Then

$$\sum_{r=1}^{n} b_r^{2s-1} + \sum_{r=1}^{n} y_r^{2s-1} = 0 \qquad (s = 1, 2, \ldots, n).$$

The elementary symmetric functions A_{2s-1} of the numbers (b_r), (y_r) *taken together* therefore vanish when $s = 1, 2, \ldots, n$ and the (b_r), (y_r) are the roots of an equation

$$t^{2n} + A_2t^{2n-2} + \ldots + A_{2n} = 0,$$

containing only even powers of t. To every root t there corresponds a root $-t$. Now reference to (17) shows that no two zeros of θ_n can differ in sign only and the same is therefore true of the (b_r). Hence the (y_r) are the (b_r) in some order or other.

4. A pseudo-generating function for $\theta_n(x)$. I establish the formula

$$\frac{e^{2xu}}{1-2u} = \sum_{n=0}^{\infty} \frac{\theta_n(x)}{n!} [2u(1-u)]^n \tag{18}$$

for sufficiently small values[2] of u, from which the generating function for y_n and other variants in [2] may be derived. For the right-hand side of (18) is

$$e^x \sum_{n=0}^{\infty} (-)^n \frac{[2u(1-u)]^n}{n!} (\delta - 1)(\delta - 3) \ldots (\delta - 2n + 1)\, e^{-x}$$

$$= e^x \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^n[2u(1-u)]^n}{n!} (m-1) \ldots (m - 2n + 1) \frac{(-x)^m}{m!}$$

$$= e^x \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-x)^m}{m!} \frac{(\frac{1}{2} - \frac{1}{2}m)_n}{n!} [4u(1-u)]^n$$

$$= e^x \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} (1 - 4u + 4u^2)^{\frac{1}{2}m - \frac{1}{2}}$$

$$= \frac{e^x}{1-2u} \sum_{m=1}^{\infty} \frac{(-x)^m(1-2u)^m}{m!}$$

$$= \frac{e^{2xu}}{1-2u}.$$

5. The authors of [2] have also considered the equation

$$x^2 \frac{d^2y}{dx^2} + (ax + b) \frac{dy}{dx} = n(n + a - 1)y,$$

or

$$b\delta y + x(\delta + n + a - 1)(\delta - n)y = 0,$$

with polynomial solutions $y_n(x,a,b)$ made definite by the supplementary condition[3] $y_n(0) = 1$. Defining $\phi_n(x,a,b) \equiv \phi_n$ by

$$\phi_n = x^n y_n(x^{-1},a,b),$$

we find that ϕ_n is a solution of

$$\delta(\delta + 1 - a - 2n)z = bx(\delta - n)z, \tag{19}$$

[2]We may take conveniently $0 < u < \frac{1}{2}$.

[3]It is evident from the second form of the differential equation that the constant b is a mere scale-factor, and nothing would be lost by considering only $b = \pm 1$. I retain b for the sake of comparison with the formulae of [2].

and that $e^{-bx}\phi_n$ is a solution of

$$\delta(\delta + 1 - a - 2n)w = - bx(\delta - n - a + 2)w. \tag{20}$$

We note that, if $a = 2$, (19) and (20) differ only in the sign of b and that (20) will then have the solutions

$$e^{-bx}\phi_n(x,2,b), \quad \phi_n(x,2, - b)$$

leading us back effectively to the theory of §2.

We observe also that, if $n + a - 2$ is zero or a positive integer, (20) will have a polynomial solution of that degree. I now show that *equation* (20) *has the solution*

$$w = (\delta - n - a + 1)(\delta - n - a) \dots (\delta - 2n - a + 2)e^{-bx}. \tag{21}$$

For

$$\begin{aligned}\delta(\delta - 2n - a + 1)w &= (\delta - n - a + 1) \dots (\delta - 2n - a + 1)\delta e^{-bx}\\ &= (\delta - n - a + 1) \dots (\delta - 2n - a + 1)(- bx)e^{-bx}\\ &= - bx\,(\delta - n - a + 2) \dots (\delta - 2n - a + 2)e^{-bx}\\ &= -bx(\delta - n - a + 2)w.\end{aligned}$$

Recalling that the coefficient of x^n in ϕ_n is unity we have

$$\begin{aligned}\phi_n(x,a,b) &= (- b)^{-n}e^{bx}(\delta - n - a + 1) \dots (\delta - 2n - a + 2)e^{-bx}\\ &= (- b)^{-n}e^{bx}x^{a+2n-1}D^n(x^{-a-n+1}e^{-bx}),\end{aligned} \tag{22}$$

the latter form being equivalent to (47) of [2] and to (2) of the present note on setting $a = b = 2$. Thus, in addition to the formula (6) for the θ_n of §2, we have

$$\begin{aligned}e^{-2x}\theta_n(x) &= (- \tfrac{1}{2})^n(\delta - n - 1) \dots (\delta - 2n)e^{-2x}\\ &= (- \tfrac{1}{2})^n x^{2n+1}D^n(x^{-n-1}e^{-2x}).\end{aligned} \tag{23}$$

6. The operational methods of the present note enable me to repair one omission in [2], namely the failure to supply a generating function for the polynomials $y_n(x,a,b)$. I establish in the first place the result

$$\frac{(1 - u)^{2-a}e^{bxu}}{1 - 2u} = \sum_{n=0}^{\infty} \frac{[bu(1 - u)]^n\phi_n}{n!}, \tag{24}$$

which reduces to (18) when $a = b = 2$. We require the auxiliary formula

$$\sum_{n=0}^{\infty} \frac{(n - k)_n[u(1 - u)]^n}{n!} = (1 - 2u)^{-1}(1 - u)^{k+1}, \tag{25}$$

which may be proved as follows.

The coefficient of u^s in

$$\sum_{n=0}^{\infty} \frac{(n - k)_n u^n(1 - u)^{n-1-k}}{n!} \quad \text{is} \quad \sum_{n=0}^{s} \frac{(-)^{s-n}(2n - k - s)_s}{n!(s - n)!}$$

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and this is the coefficient of t^s in the expansion of

$$\sum_{n=0}^{s} \frac{(-)^n s!}{n!(s-n)!}(1+t)^{k+s-2n} = (1+t)^{k+s}[1-(1+t)^{-2}]^s = (1+t)^{k-s}t^s(2+t)^s$$

in which the coefficient of t^s is 2^s.

Returning now to (24), on the right we have, by (22) and (25),

$$e^{bx}\sum_{n=0}^{\infty}\frac{(-)^n[u(1-u)]^n}{n!}\sum_{m=0}^{\infty}\frac{(m-2n-a+2)_n(-bx)^m}{m!}$$

$$= e^{bx}\sum_{m=0}^{\infty}\frac{(-bx)^m}{m!}\sum_{n=0}^{\infty}\frac{(n+a-1-m)_n[u(1-u)]^n}{n!}$$

$$= e^{bx}(1-2u)^{-1}\sum_{m=0}^{\infty}\frac{(-bx)^m(1-u)^{m+2-a}}{m!}$$

$$= (1-2u)^{-1}(1-u)^{2-a}e^{bxu},$$

as required. Writing x^{-1} for x and xu for u in (24), we have

$$\frac{(1-xu)^{2-a}e^{bu}}{1-2xu} = \sum_{n=0}^{\infty}\frac{b^n y_n(x,a,b)[u(1-xu)]^n}{n!} \tag{26}$$

and, setting

$$2u(1-xu) = t, \text{ or } 2xu = 1-(1-2xt)^{\frac{1}{2}}$$

in (26), we find

$$[\tfrac{1}{2}-\tfrac{1}{2}(1-2xt)^{\frac{1}{2}}]^{2-a}(1-2xt)^{-\frac{1}{2}}\exp\left[\frac{b}{2x}\left\{1-(1-2xt)^{\frac{1}{2}}\right\}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(b/2)^n y_n(x,a,b)t^n}{n!},$$

which may serve as a generating function for the polynomials $y_n(x,a,b)$.

7. In this section I assume that a is a positive integer. The results obtained may in certain circumstances be extended to negative integral and zero values of a but at the cost of their ceasing to hold for all n.

In the first place we observe that, when a is a positive integer (20) has a polynomial solution of degree $n + a - 2$ and a series solution in ascending powers of x led by x^{2n+a-1}. On the other hand we see that the expansion of the w defined by (21) lacks all terms with indices between $n + a - 1$ and $2n + a - 2$ inclusive. The expansion may in fact be divided by this gap into two parts furnishing respectively a polynomial and a series solution. A more important consequence of this gap in the expansion of w is the following. From (21) we have

$$e^{-bx}\phi_n = \sum_{r=0}^{a+n-2} c_r x^r + \sum_{r=-1}^{\infty} c_{2n+a+r}x^{2n+a+r}$$

where

$$c_0 = b^{-n}(n + a - 1)_n, \; c_{2n+a-1} = (-b)^{n+a-1}n!/(2n + a - 1)!.$$

Suppose now $m < n$; then in the expansion of $e^{-bx}\phi_m\phi_n$ the term in $x^{m+n+a-1}$ is missing, while in the expansion of $e^{-bx}\phi_n{}^2$ the coefficient of x^{2n+a-1} is c_0c_{2n+a-1}. Hence, if C is any contour surrounding $x = 0$,

$$\frac{1}{2\pi i}\int_C \frac{\phi_m\phi_n e^{-bx}}{x^{m+n+a}}\,dx = \frac{\epsilon_{mn}(-)^{n+a-1}b^{a-1}n!}{(2n + a - 1)(n+a-2)!} \tag{28}$$

and, on changing the variable to $1/x$,

$$\frac{1}{2\pi i}\int_C \frac{y_m(x,a,b)y_n(x,a,b)e^{-b/x}}{x^{2-a}}\,dx = \frac{\epsilon_{mn}(-)^{n+a-1}b^{a-1}n!}{(2n + a - 1)(n + a - 2)!} \tag{29}$$

which reduces to (4) when $a = b = 2$. The problem of an appropriate weight function when a is not integral has been considered in [2], and to that discussion I have nothing to add.

References

[1] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators.* II. *The identity* $P^n = Q^m$, Proc. Roy. Soc. A, vol. 134 (1931), 471-485.

[2] H. L. Krall and Orrin Frink, *A new class of orthogonal polynomials: The Bessel polynomials,* Trans. Amer. Math. Soc., vol. 65 (1949), 100-115.

Durham, England