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Gabber rigidity in hermitian K-theory

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We note that Gabber's rigidity theorem for the algebraic K-theory of henselian pairs also holds true for hermitian K-theory with respect to arbitrary form parameters.

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Let R be a commutative ring and $\mathfrak{m} \subseteq R$ an ideal such that $(R, \mathfrak{m})$ is a henselian pair. Standard examples include henselian local rings like valuation rings of complete nonarchimedean fields as well as pairs where R is $\mathfrak{m}$-adically complete or where $\mathfrak{m}$ is locally nilpotent. We write $F = R/\mathfrak{m}$ and let n be a natural number which is invertible in R. Then Gabber's rigidity theorem [5] says that the canonical map

$$K(R)/n \longrightarrow K(F)/n$$

is an equivalence; this result was preceded by work of Suslin [13] who showed this conclusion for henselian valuation rings. See also [4] for an extension of this result, involving topological cyclic homology, to the case where n need not be invertible in R and a general discussion of henselian pairs. The purpose of this short note is to use the results of [2, 3] as well as [6] to show that Gabber's rigidity property also holds true for hermitian K-theory, a.k.a. Grothendieck–Witt theory.

To state the main result, let λ be a form parameter over R in the sense of [12, §3], see also [1, definition 4.2.26]. In loc. cit. it is explained that such a form parameter λ is equivalently described by a Poincaré structure $Ϙ_R^{g\lambda}$ in the sense of [1] on $\mathcal{D}^p(R)$ which sends projective R-modules to discrete spectra. Here, $\mathcal{D}^p(R)$ denotes the stable ∞-category of perfect complexes over R. We will assume that the $\mathbb{Z}$-module with involution over R underlying the form parameter λ is given by $\pm R$, that is, given by the R-module R with C_2-action either the identity or multiplication by -1, viewed as an $R \otimes R$-module via the multiplication map. There is then an induced form parameter on F whose associated Poincaré structure on $\mathcal{D}^p(F)$ we will denote by $Ϙ_F^{g\lambda}$, see remark 4 below for details. The construction is made so that the extension of scalars functor canonically refines to a Poincaré functor $(\mathcal{D}^p(R), Ϙ_R^{g\lambda}) \to (\mathcal{D}^p(F), Ϙ_F^{g\lambda})$ and therefore a map on Grothendieck–Witt theory. Standard examples of form parameters capture the notion of quadratic, even and symmetric forms (as well as their skew-quadratic, skew-even and skew-symmetric cousins) with associated Poincaré structures $\Omega^{\pm \mathrm{gq}}$, $\Omega^{\pm \mathrm{ge}}$ and $\Omega^{\pm \mathrm{gs}}$. A further example is provided by the Burnside Poincaré structure Ω^{b} whose L-theory was calculated explicitly for $\mathbb{Z}$ in [**3**, example 1.3.18] and whose 0'th Grothendieck–Witt group was studied for commutative rings with 2 invertible in the PhD thesis of Dylan Madden [**10**]. With this notation fixed, we have the following result.

THEOREM 1. *Let* $(R, \mathfrak{m})$ *be a henselian pair,* $F = R/\mathfrak{m}$ *and let* n *be a natural number invertible in* R*. Then the canonical map*

$$\mathrm{GW}(R; \Omega^{\mathrm{g}\lambda}_R)/n \longrightarrow \mathrm{GW}(F; \Omega^{\mathrm{g}\lambda}_F)/n$$

is an equivalence.

Proof. The main result of [**2**] gives a diagram of horizontal fibre sequences

$$\begin{array}{ccccc} \mathrm{K}(R)_{hC_2} & \longrightarrow & \mathrm{GW}(R; \Omega^{\mathrm{g}\lambda}_R) & \longrightarrow & \mathrm{L}(R; \Omega^{\mathrm{g}\lambda}_R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}(F)_{hC_2} & \longrightarrow & \mathrm{GW}(F; \Omega^{\mathrm{g}\lambda}_F) & \longrightarrow & \mathrm{L}(F; \Omega^{\mathrm{g}\lambda}_F) \end{array}$$

and by Gabber rigidity, the left vertical map becomes an equivalence after tensoring with $\mathbb{S}/n$. Therefore, the statement of the theorem is equivalent to the statement that the map

$$\mathrm{L}(R; \Omega^{\mathrm{g}\lambda}_R)/n \longrightarrow \mathrm{L}(F; \Omega^{\mathrm{g}\lambda}_R)/n$$

is an equivalence. We then consider the diagram

$$\begin{array}{ccc} \mathrm{L}(R; \Omega^{\mathrm{q}}_{\pm R}) & \longrightarrow & \mathrm{L}(R; \Omega^{\mathrm{g}\lambda}_R) \\ \downarrow & & \downarrow \\ \mathrm{L}(F; \Omega^{\mathrm{q}}_{\pm F}) & \longrightarrow & \mathrm{L}(F; \Omega^{\mathrm{g}\lambda}_F) \end{array}$$

where $\Omega^{\mathrm{q}}_{\pm R}$ denotes the homotopy quadratic Poincaré structure associated with the invertible module with involution $\pm R$ which is part of the form parameter λ, and likewise for $\Omega^{\mathrm{q}}_{\pm F}$. We now observe that the formula for relative L-theory obtained in [**6**] shows that the top and bottom horizontal cofibres are $\mathbb{S}[\frac{1}{n}]$-modules.

Indeed, [**6**] shows that the cofibre of the top horizontal arrow is a filtered colimit of objects of the form

$$\mathrm{Eq}\left(\mathrm{map}_R(T \otimes_R T, R) \rightrightarrows (\Sigma^{1-\sigma}\mathrm{map}_R(T \otimes_R T, R))_{hC_2}\right)$$

for $T \in \mathcal{D}^{\mathrm{p}}(R)$, the bottom horizontal cofibre is described similarly[1]. Since $\mathrm{map}_R(T \otimes_R T, R)$ is canonically an R-module and n is invertible in R, it is also an $\mathbb{S}\left[\frac{1}{n}\right]$-module. Moreover, since $\mathrm{Mod}\left(\mathbb{S}\left[\frac{1}{n}\right]\right) \subseteq \mathrm{Sp}$ is a full subcategory closed

[1] In [**6**], the authors in fact show that the relative L-theory in question, also known as normal L-theory, is given by the C_2-geometric fixed points of the real topological cyclic homology of $(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{g}\lambda}_R)$. The equalizer formula is then reminiscent of the Nikolaus–Scholze formula for TC [**11**].

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under colimits and limits, both terms in the equalizer, and therefore also the equalizer itself belong to $\mathrm{Mod}\left(\mathbb{S}\left[\frac{1}{n}\right]\right)$. Consequently, the horizontal maps in the above diagram become equivalences upon tensoring with $\mathbb{S}/n$. The statement of the main theorem is therefore equivalent to the statement that the left vertical map in the above commutative square is an equivalence. This is a consequence of the work of Wall's [14] as explained in [3, prop. 2.3.7 and remark 2.3.8]. □

Remark 2. Restricting the situation above to form parameters rather than general Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$ was merely a cosmetic choice to obtain a result about classical Grothendieck–Witt theory: Indeed, it is again a consequence of the main theorem of [2] that the diagram

$$\begin{array}{ccc} \mathrm{GW}(R; Ϙ^{\mathrm{q}}_{\pm R}) & \longrightarrow & \mathrm{GW}(R; Ϙ) \\ \downarrow & & \downarrow \\ \mathrm{L}(R; Ϙ^{\mathrm{q}}_{\pm R}) & \longrightarrow & \mathrm{L}(R; Ϙ) \end{array}$$

is a pullback diagram for any Poincaré structure $Ϙ$ on $\mathcal{D}^{\mathrm{p}}(R)$ whose $\mathbb{Z}$-module with involution over R is given by $\pm R$. The proof presented above therefore shows that for any ring R in which n is invertible, the canonical map

$$\mathrm{GW}(R; Ϙ^{\mathrm{q}}_{\pm R})/n \longrightarrow \mathrm{GW}(R; Ϙ)/n$$

is an equivalence so that Gabber rigidity also holds for the Poincaré structure $Ϙ$.

In particular, Gabber rigidity also applies to the homotopy symmetric Poincaré structure $Ϙ^{\pm \mathrm{s}}$ as well as the Tate Poincaré structure $Ϙ^{\mathrm{t}}_R$, see [1, example 3.2.12].

Remark 3. Rigidity in hermitian K-theory has of course been studied in several works before, see for instance [7–9, 15] for the case of rings with involution. The main purpose here is to show how to use the formalism of Poincaré categories and the main result of [2] to reduce rigidity in hermitian K-theory to rigidity in algebraic K-theory and L-theory in a way that allows to treat general form parameters.

Remark 4. In this remark, we describe how extension of scalars can be used to prolong a form parameter over R along a map $R \to R'$ of rings. It is here that the assumption on the underlying module with involution is used. Indeed, we will describe a general construction on Hermitian structures, and the assumption is used to ensure that the given Poincaré structure is sent to a Poincaré structure rather than merely a Hermitian structure.

Namely, in [1, §3.3], we have shown that the category of Hermitian structures on $\mathcal{D}^{\mathrm{p}}(R)$ is equivalent to the category $\mathrm{Mod}_{\mathrm{N}(R)}(\mathrm{Sp}^{C_2}) = \mathrm{Mod}(\mathrm{N}(R))$, that is, the category of modules over the multiplicative norm[2] $\mathrm{N}(R)$ in the category Sp^{C_2} of genuine C_2-spectra. Moreover, the category $\mathrm{Mod}(\mathrm{N}(R))$ is equipped with a canonical t-structure whose heart is equivalent to the category of (possibly degenerate) form parameters over R, see [1, remark 4.2.27]. Objects in $\mathrm{Mod}(\mathrm{N}(R))$ are described

[2] Also known as the Hill–Hopkins–Ravenel norm.

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by triples (M, N, α) where

- M is an object of $\mathrm{Mod}_{R\otimes R}(\mathrm{Sp}^{BC_2})$, where $R \otimes R$ is an algebra in spectra with C_2-action where the action flips the two tensor factors,
- N is an object of $\mathrm{Mod}(R)$ and
- α is a map $N \to M^{tC_2}$ of R-modules,

see [1]; the Poincaré structures then consist of the above triples where M is *invertible* in the sense of [1, def. 3.1.4]. We warn the reader that caution has to be taken in regards to how M^{tC_2} is to be viewed as an R-module, see e.g. [3, p. 7] for the details. An object (M, N, α) is connective in the canonical t-structure on $\mathrm{Mod}(\mathrm{N}(R))$ if and only if M and N are connective.

The Poincaré structure associated with the triple (M, N, α) is denoted by Ω^{α}_{M}. Assuming that M is in the image of the canonical functor $\mathrm{Fun}(BC_2, \mathrm{Mod}(R)) \to \mathrm{Mod}_{R\otimes R}(\mathrm{Fun}(BC_2, \mathrm{Sp}))$, the triple

$$(M', N', \alpha') = (R' \otimes_R M, R' \otimes_R N, R' \otimes_R N \to R' \otimes_R M^{tC_2} \to (R' \otimes_R M)^{tC_2})$$

gives rise to a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R')$ for which the extension of scalar functor canonically refines to a Poincaré functor $(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\alpha}_{M}) \to (\mathcal{D}^{\mathrm{p}}(R'), \Omega^{\alpha'}_{M'})$, see [1, lemma 3.4.3]. Now, if (M, N, α) was associated with a form parameter, then the same need not be true for the triple (M', N', α'): Indeed, this is the case if and only if $\Omega^{\alpha'}_{M'}(R')$ is a discrete spectrum which in general need not be the case (but by construction it is always a connective spectrum). However, we may consider the composite

$$M'_{hC_2} \longrightarrow \Omega^{\alpha'}_{M'}(R') \longrightarrow \tau_{\leqslant 0}\Omega^{\alpha'}_{M'}(R')$$

and denote its cofibre by N''. The pushout diagram of spectra

$$\begin{array}{ccc} \Omega^{\alpha'}_{M'}(R') & \longrightarrow & N' \\ \downarrow & & \downarrow \\ (M')^{hC_2} & \longrightarrow & (M')^{tC_2} \end{array}$$

and the fact that $(M')^{hC_2}$ is coconnective shows that there is a canonical map $\alpha''\colon N'' \to (M')^{tC_2}$. By construction, the triple (M', N'', α'') is an object of $\mathrm{Mod}(\mathrm{N}(R'))^{\heartsuit}$ and in fact identifies with $\tau_{\leqslant 0}(M', N', \alpha')$. This object determines a Poincaré structure $\Omega^{\mathrm{g}\lambda'}$ associated with a form parameter λ' over R' for which the extension of scalars functor refines to a Poincaré functor

$$(\mathcal{D}^{\mathrm{p}}(R), \Omega^{\mathrm{g}\lambda}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(R'), \Omega^{\mathrm{g}\lambda'}).$$

To give an example of this construction, we recall the genuine Poincaré structures $\Omega^{\geqslant m}_{\pm R}$ which, for $m = 0, 1, 2$ are the Poincaré structures $\Omega^{\mathrm{gq}}_{\pm R}$, $\Omega^{\mathrm{ge}}_{\pm R}$ and $\Omega^{\mathrm{gs}}_{\pm R}$ associated with the classical (skew-) quadratic, even and symmetric form parameter over R, respectively, see [3, remark R.3 and R.5]. In this case, the extension of scalars functor associated with a ring map $R \to R'$ indeed sends $\Omega^{\geqslant m}_{\pm R}$ to $\Omega^{\geqslant m}_{\pm R'}$.

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Remark 5. For the Poincaré structures $\Qoppa^{\geqslant m}_{\pm R}$ one can give the following argument that the map

$$\mathrm{GW}(R; \Qoppa^{\geqslant m}_{\pm R})/n \longrightarrow \mathrm{GW}(F; \Qoppa^{\geqslant m}_{\pm R})/n$$

is an equivalence without appealing to the general formula for relative L-theory of [**6**]. Namely, in [**3**, prop. 3.1.14] we have shown that the map

$$\mathrm{L}(R; \Qoppa^{\mathrm{q}}_{\pm R})\left[\tfrac{1}{2}\right] \longrightarrow \mathrm{L}(R; \Qoppa^{\geqslant m}_{\pm R})\left[\tfrac{1}{2}\right]$$

is an equivalence for all $m \in \mathbb{Z}$. Therefore, the proof of the theorem applies in the case where 2 does not divide n. In the case where 2 divides n, we deduce that 2 is invertible in R in which case already the map $\Qoppa^{\mathrm{q}}_{\pm R} \to \Qoppa^{\geqslant m}_{\pm R}$ is an equivalence of Poincaré structures, see [**3**, remark R.4].

Remark 6. Suppose that R is an associative ring which is $\mathfrak{m}$-adically complete for an ideal $\mathfrak{m} \subset R$. Then the result of Wall, see again [**3**, prop. 2.3.7], says that the map $\mathrm{L}^{\pm \mathrm{q}}(R) \to \mathrm{L}^{\pm \mathrm{q}}(R/\mathfrak{m})$ is an equivalence. To the best of our knowledge, it is not known whether also the map $K(R)/n \to K(R/\mathfrak{m})/n$ is an equivalence. However, if it is, this argument shows that the same is true for Grothendieck–Witt theory and vice versa.

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