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CORRIGENDUM

LINEAR RELATIONS ON HEREDITARILY INDECOMPOSABLE NORMED SPACES – CORRIGENDUM

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The purpose of this note is to correct the proof of [1, Proposition 12] concerning the existence of nonempty essential spectra of a linear relation on a complex normed space.

Our notation is the same as in [1]. Let $T \in LR(X)$ where X is a complex normed space. The essential resolvents, $\rho_{e+}(T)$ and $\rho_e(T)$, are defined by

$$\rho_{e+}(T) := \{\lambda \in \mathbb{C} : \lambda - T \in F_+\}$$

and

$$\rho_e(T) := \{\lambda \in \mathbb{C} : \lambda - T \in F_+ \cap F_- \text{ and } \overline{k}(\lambda - T) = 0\}.$$

Applying properties of Fredholm relations, we may equivalently define the above essential resolvents as follows:

$$\rho_{e+}(T) := \{\lambda \in \mathbb{C} : \lambda - \widetilde{T} \in \phi_+\}$$

and

$$\rho_e(T) := \{\lambda \in \mathbb{C} : \lambda - \widetilde{T} \in \phi_+ \cap \phi_- \text{ and } k(\lambda - \widetilde{T}) = 0\}.$$

The essential spectra, $\sigma_{e+}(T)$ and $\sigma_e(T)$, are the respective complements of the essential resolvents: $\sigma_{e+}(T) := \mathbb{C} \setminus \rho_{e+}(T)$ and $\sigma_e(T) := \mathbb{C} \setminus \rho_e(T)$.

Recall that the spectrum of T is defined in [2, Section VI.1(4)] as $\sigma(T) := \mathbb{C} \setminus \rho(T)$ where

$$\rho(T) := \{\lambda \in \mathbb{C} : T_\lambda := (\lambda - \widetilde{T})^{-1} \text{ is everywhere defined and single-valued}\}.$$

It is clear from the closed graph theorem that $\rho(T)$ consists of all $\lambda \in \mathbb{C}$ for which $\lambda - T$ is injective, open and has dense range.

We now consider the essential spectrum, $\sigma_{e-}(T)$, defined by $\sigma_{e-}(T) := \mathbb{C} \setminus \rho_{e-}(T)$ where $\rho_{e-}(T) := \{\lambda \in \mathbb{C} : \lambda - T \in F_-\}$ which coincides with the set $\{\lambda \in \mathbb{C} : \lambda - \widetilde{T} \in \phi_-\}$.

The proof of [1, Proposition 12] relies on the following statement:

If T is a partially continuous linear relation with dense domain then T'' is bounded single-valued.

This, however, need not be satisfied.

REMARK 1. If the linear relation T is partially continuous and densely defined, but not single-valued, then $\widetilde{T}$ and T'' are bounded but both linear relations are not single-valued.

PROOF. We first show that:

(1) *if $S \in LR(E, F)$ where E and F are normed spaces, then S'' single-valued implies $\widetilde{S}$ single-valued, which implies S single-valued.*

We note that from [2, Proposition III.1.4(d)] we have $D(\widetilde{S}') \subset (D(\widetilde{S}')^{\top})^{\perp} = \widetilde{S}(0)^{\perp}$ and then from [2, Proposition III.1.4(b)] we have $\widetilde{S}(0)^{\perp\perp} \subset D(\widetilde{S}')^{\perp} = D(S')^{\perp} = S''(0)$. Hence $\widetilde{S}(0)^{\perp\perp} \subset S''(0)$. This last property together with the fact that S is single-valued if and only if $S(0) = \{0\}$ (see [2, Corollary I.2.9]) implies that $\widetilde{S}$ (and hence S) is single-valued whenever S'' is single-valued.

We now prove that:

(2) *$T \in LR(X)$ partially continuous implies that $\widetilde{T}$, T' and T'' are continuous.*

This is immediate from the following chain of implications:

T partially continuous $\Rightarrow T'$ continuous (by [2, Corollary V.9.6]) $\Leftrightarrow \widetilde{T}$ continuous (by [2, Proposition III.4.9]) and T' continuous $\Rightarrow T''$ continuous (by [2, Corollary III.1.13]).

(3) *T partially continuous with dense domain implies that $\widetilde{T}$ and T'' are bounded.*

By virtue of (2), to establish assertion (3) it only remains to prove that $\widetilde{T}$ and T'' are everywhere defined.

Since $\widetilde{T}$ is continuous if and only if $D(\widetilde{T})$ is closed [2, Theorem III.4.2], we have that $D(\widetilde{T}) = (D(\widetilde{T})^{\perp})^{\top} = \widetilde{T}'(0)^{\top} = T'(0)^{\top} = (D(T)^{\perp})^{\top}$ [2, Proposition III.1.4(b)] $= \widetilde{X}$ (as T has dense domain). Hence $D(\widetilde{T}) = \widetilde{X}$.

That $D(T'') = X''$ follows from the observation that T' is continuous if and only if $T'(0)^{\perp} = D(T'') = D(T)^{\perp\perp} = X''$. The first equality follows from [2, Proposition III.4.6(a)], the second from [2, Proposition III.1.4(b)] and the third from $\overline{D(T)} = X$. □

The following lemma is elementary but is used several times in the proof of Proposition 3. Note that $R(T) := T(D(T))$ denotes the range of T and $N(T) := T^{-1}(0)$ denotes the null space of T.

LEMMA 2. *Let $T \in LR(X)$ such that $\widetilde{T}$ is everywhere defined and let $\eta \in \rho(T)$. Then for all $\lambda \neq \eta$ we have:*

(i) $\lambda - \widetilde{T} = S(\eta - \widetilde{T})$;
(ii) $R(\lambda - \widetilde{T}) = R(S)$;
(iii) $N(\lambda - \widetilde{T}) = N(S)$,

where $S := (\eta - \lambda)((\eta - \lambda)^{-1} - T_{\eta})$.

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PROOF. (i) Since $\eta - \widetilde{T}$ is injective and $D(\eta - \widetilde{T}) = D(\widetilde{T}) = \widetilde{X}$ it follows from [2, Section I.1.3(9)] that

$$\begin{aligned} S(\eta - \widetilde{T}) &= (I - (\eta - \lambda)T_\eta)(\eta - \widetilde{T}) \\ &= ((\eta - \widetilde{T})^{-1}(\eta - \widetilde{T}) - (\eta - \lambda)(\eta - \widetilde{T})^{-1})(\eta - \widetilde{T}). \end{aligned}$$

Then, since $D((\eta - \widetilde{T})^{-1}) = R(\eta - \widetilde{T}) = \widetilde{X}$, it follows from [2, Proposition I.4.2(e)] that

$$S(\eta - \widetilde{T}) = ((\eta - \widetilde{T})^{-1}(\eta - \widetilde{T} - (\eta - \lambda)))(\eta - \widetilde{T})$$

and thus, by [2, Proposition VI.5.1],

$$S(\eta - \widetilde{T}) = (\eta - \widetilde{T})^{-1}(\lambda - \widetilde{T})(\eta - \widetilde{T}) = (\eta - \widetilde{T})^{-1}(\eta - \widetilde{T})(\lambda - \widetilde{T}) = \lambda - \widetilde{T}.$$

(ii) Follows immediately from (i) upon noting that $R(\eta - \widetilde{T}) = \widetilde{X}$.
(iii) Let $x \in \widetilde{X} \setminus \{0\}$. Then, using the fact that $\eta \in \rho(\widetilde{T}) = \rho(T)$,

$$\begin{aligned} x \in N(\lambda - \widetilde{T}) &\Leftrightarrow (\lambda - \widetilde{T})x = (\lambda - \widetilde{T})(0) \\ &\Leftrightarrow (\eta - \widetilde{T})x = (\lambda - \widetilde{T})x + (\eta - \lambda)x \\ &\qquad = (\eta - \lambda)x + \widetilde{T}(0) = (\eta - \lambda)x + (\eta - \widetilde{T})(0) \\ &\Leftrightarrow x = (\eta - \lambda)T_\eta x + T_\eta(\eta - \widetilde{T})(0) = (\eta - \lambda)T_\eta x \\ &\Leftrightarrow 0 = (I - (\eta - \lambda)T_\eta)x \\ &\Leftrightarrow x \in N(S). \end{aligned}$$

□

We proceed now to give a correct proof of [1, Proposition 12].

PROPOSITION 3. *Let X be a complex normed space and let $T \in LR(X)$ be partially continuous such that $\overline{D(T)} = X$ and $\rho(T) \neq \emptyset$. Then the sets $\sigma(T)$, $\sigma_{e+}(T)$, $\sigma_{e-}(T)$ and $\sigma_e(T)$ are nonempty.*

PROOF. $\sigma(T) \neq \emptyset$. By (3) of Remark 1 we have that $\widetilde{T}$ is bounded and thus it follows from [2, Theorem VI.3.3] that the spectrum of $\widetilde{T}$ is nonempty. The assertion now follows upon noting that $\sigma(T) = \sigma(\widetilde{T})$.

In order to prove that the essential spectra of T are nonempty subsets of $\mathbb{C}$ let us consider two cases for T.

Case 1. T is bounded single-valued and X is complete.

It is well known that in this case, $\sigma_e(T)$ coincides with the spectrum of the image of T in the Calkin algebra $B(X)/K(X)$ where $B(X)$ and $K(X)$ denote the class of all bounded operators and bounded compact operators on X respectively (see, for example, [3]), so that $\sigma_e(T)$ is nonempty. This property, together with the fact that $\sigma_e(T)$ is closed [1, Proposition 11], implies that the boundary of $\sigma_e(T)$, denoted $\sigma_e(T)^b$, is also a nonempty set.

Furthermore, by the stability of the index of a bounded semi-Fredholm operator under small perturbation (see, for example [4, 2.c.9]) we deduce that the boundary

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of $\sigma_e(T)$ is contained both in $\sigma_{e+}(T)$ and in $\sigma_{e-}(T)$. Therefore $\sigma_{e+}(T)$ and $\sigma_{e-}(T)$ are nonempty sets.

Case 2. T is partially continuous with dense domain and has a nonempty resolvent set.

Let $\eta \in \rho(T)$. Then by the open mapping theorem for linear relations [2, Theorem III.4.2], T_η is a bounded single-valued linear relation. Moreover, it is clear that $N(T_\eta) = (\eta - \widetilde{T})(0) = \widetilde{T}(0)$ and $R(T_\eta) := R((\eta - \widetilde{T})^{-1}) = D(\eta - \widetilde{T}) = D(\widetilde{T}) = \widetilde{X}$ (by (3)). Consequently $0 \in \rho_{e-}(T_\eta)$ and since $\sigma_{e-}(T_\eta)$ is nonempty by Case 1, we conclude from Lemma 2 that there exists $\lambda \neq \eta$ such that $\lambda \in \sigma_{e-}(T)$. Hence $\sigma_{e-}(T) \neq \emptyset$ and since $\sigma_{e-}(T) \subset \sigma_e(T)$ we have that $\sigma_e(T)$ is also a nonempty set.

It only remains to prove that $\sigma_{e+}(T) \neq \emptyset$. To this end, let us consider two possibilities for $\widetilde{T}(0)$:

(a) $\dim \widetilde{T}(0) < \infty$. In this case $0 \in \rho_{e+}(T_\eta)$ and then it follows from Case 1 and Lemma 2 that $\sigma_{e+}(T) \neq \emptyset$;

(b) $\dim \widetilde{T}(0) = \infty$. Then $0 \in \sigma_{e+}(T_\eta)$ since clearly $N(T_\eta) = \widetilde{T}(0)$. Now, as $\emptyset \neq \sigma_e(T_\eta)^b \subset \sigma_{e+}(T_\eta) \cap \sigma_{e-}(T_\eta)$ (see the proof of Case 1) and $0 \in \rho_{e-}(T_\eta)$ we obtain that $\sigma_{e+}(T_\eta)$ contains nonzero elements, and thus from Lemma 2 we deduce that $\sigma_{e+}(T) \neq \emptyset$, as desired. □

References

[1] T. Álvarez, 'Linear relations on hereditarily indecomposable normed spaces', *Bull. Aust. Math. Soc.* **74** (2006), 289–300.

[2] R. W. Cross, *Multivalued Linear Operators* (Marcel Dekker, New York, 1998).

[3] H. G. Heuser, *Functional Analysis* (Wiley, Chichester, 1982).

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces* (Springer, Berlin, 1977).