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A flower theorem in dimension two

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Abstract. We prove a two-dimensional analog of the Leau–Fatou flower theorem for non-degenerate reduced biholomorphisms tangent to the identity.

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1. *Introduction*

Let $F \in \mathrm{Diff}(\mathbb{C}^n, 0)$ be a germ of a biholomorphism tangent to the identity. In dimension one, the dynamics of F is completely described by the Leau–Fatou flower theorem [6, 9], which guarantees the existence of simply connected domains with zero in their boundary, covering a punctured neighborhood of the origin, which are stable either for F or for F^{-1}; moreover, in each of these domains F is conjugated to the unit translation.

In dimension two, no complete description of the dynamics of F is known. Some partial analogs of the Leau–Fatou flower theorem have been obtained, guaranteeing the existence of either one-dimensional [1, 5, 8, 12] or two-dimensional stable manifolds [7, 15]. With no extra assumptions on F, the most general result is due to Abate [1], who showed that F always supports some stable dynamics: either F has a curve of fixed points or there exist one-dimensional stable manifolds of F with the origin in their boundary.

The proof of the above-mentioned results is crucially based on a resolution theorem for F, which reduces the study of the dynamics of F to some combinatorial data of the resolution and the study of the local dynamics of some reduced models of the transform of F. This resolution theorem was introduced by Abate in [1] and is based on the corresponding result for vector fields due mainly to Seidenberg [4, 14]. Before stating the resolution theorem, we establish a precise definition of the reduced models suited to our purposes (see also Remark A.4).

Definition 1.1. The germ of a biholomorphism in dimension two at a fixed point p is called reduced if it is analytically conjugate to one of the following models.

(i) Regular fixed points:

$$\tilde{F}(x, y) = (x + x^M y^N [1 + A(x, y)], y + x^M y^N B(x, y)),$$

where $M, N \geq 0$, $(M, N) \notin \{(0, 0), (1, 0)\}$, ord $A \geq 1$ and $B \in (y)$ if $N \geq 1$.

(ii) Non-degenerate fixed points:

$$\tilde{F}(x, y) = (x + x^{M+1} y^N [a + A(x, y)], y + x^M y^N [by + B(x, y)]),$$

where $M \geq 1$, $N \geq 0$, $ab \neq 0$, $a/b \notin \mathbb{Q}_{>0}$, ord $A \geq 1$, ord $B \geq 2$ and $B \in (y)$ if $N \geq 1$.

(iii) Saddle-node fixed points:

$$\tilde{F}(x, y) = (x + x^M y^N [x + A(x, y)], y + x^M y^N B(x, y)),$$

where $M, N \geq 0$, $M + N \geq 1$, ord A, ord $B \geq 2$, $A \in (x)$ if $M \geq 1$, $B \in (y)$ if $N \geq 1$ and $x + A$ and B have no common factors.

The resolution theorem, as we explain in Appendix A, guarantees the existence of a finite composition of blow-ups $\pi : (M, E) \to (\mathbb{C}^2, 0)$, with $E = \pi^{-1}(0)$, which transforms F into a map $\tilde{F} : (M, E) \to (M, E)$ that fixes E pointwise such that for every $p \in E$ the germ of $\tilde{F}$ at p is reduced according to the previous definition.

Model (i) corresponds to the points $p \in E$ which are not singular, in Abate's terminology [1] (that is, the points which are non-singular for the saturation of the associated vector field), and models (ii) and (iii) correspond to the points $p \in E$ which are singular; for the latter, we use the names 'non-degenerate' and 'saddle-node' by analogy with the standard terminology for vector fields, according to whether the linear part of the saturation of the associated vector field has two or one non-zero eigenvalues, respectively. The dynamics of biholomorphisms of the form (i) is described in [1, Proposition 2.1] when $M = 0$ and in [3, Theorem 5.3] when $N = 0$.

In this paper we study the dynamics of non-degenerate fixed points. These are the only models that appear at singular points in the resolution of a generic biholomorphism.

Actually, we consider in our study a slightly more general class of biholomorphisms, since we do not impose the non-resonance condition $a/b \notin \mathbb{Q}_{>0}$. We distinguish two cases, according to whether the fixed point set of F has one or two components. As a first case, we consider biholomorphisms of the form

$$F(x, y) = (x + x^{M+1}[a + A(x, y)], y + x^M [cx + by + B(x, y)]), \tag{A}$$

where $M \geq 1$, $a, b, c \in \mathbb{C}$, $ab \neq 0$, ord $A \geq 1$ and ord $B \geq 2$. Non-degenerate fixed points with $N = 0$ are included here. Biholomorphisms of this type appear, for instance, after one blow-up at the point corresponding to a so-called *non-degenerate characteristic direction* (see [8]). Écalle [5] and Hakim [8] showed that in this case there exist one-dimensional stable manifolds for F with 0 in their boundary, called parabolic curves. Moreover, if a and b satisfy the condition

$$\mathrm{Re}(b/a) > 0, \tag{A*}$$

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Hakim proved in [7] (see also [2]) the existence of two-dimensional stable manifolds with 0 in their boundary, called parabolic domains, where F is analytically conjugate to the map $(z, w) \mapsto (z+1, w)$.

As a second case, we consider biholomorphisms of the form

$$F(x, y) = (x + x^{M+1} y^N [a + A(x, y)], y + x^M y^{N+1} [b + B(x, y)]), \tag{B}$$

where $M, N \geq 1$, $ab \neq 0$ and ord A, ord $B \geq 1$. Non-degenerate fixed points with $N \geq 1$ are included here. If a and b are such that $aM + bN \neq 0$ and satisfy the condition

$$\operatorname{Re}\left(\frac{a}{aM+bN}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{b}{aM+bN}\right) > 0, \tag{B*}$$

Vivas proved in [15] the existence of parabolic domains for F.

The main result of this paper is an analog of the Leau–Fatou flower theorem for biholomorphisms of this type, providing a complete description of the dynamics in a whole neighborhood of the origin.

THEOREM 1.2. *Let F be a local biholomorphism of the form (A) or (B). In the first case, assume that F satisfies condition (A*) and set $d = M$ and $N = 0$; in the second case, assume that F satisfies condition (B*) and set $d = \gcd(M, N)$. Then, in any neighborhood of the origin there exist d pairwise disjoint connected open sets $\Omega_0^+, \Omega_1^+, \ldots, \Omega_{d-1}^+$, with $0 \in \partial\Omega_k^+$ for all k, and d pairwise disjoint connected open sets $\Omega_0^-, \Omega_1^-, \ldots, \Omega_{d-1}^-$, with $0 \in \partial\Omega_k^-$ for all k, such that the following assertions hold.*

(1) *The sets Ω_k^+ are invariant for F and $F^j \to 0$ as $j \to +\infty$ compactly on Ω_k^+ for all k, and the sets Ω_k^- are invariant for F^{-1} and $F^{-j} \to 0$ as $j \to +\infty$ compactly on Ω_k^- for all k.*

(2) *The sets $\Omega_0^+, \ldots, \Omega_{d-1}^+, \Omega_0^-, \ldots, \Omega_{d-1}^-$ together with the fixed set $\{xy^N = 0\}$ cover a neighborhood of the origin.*

(3) *For each k, there exist biholomorphisms $\varphi_k^+ : \Omega_k^+ \to W_k^+ \subset \mathbb{C}^2$ and $\varphi_k^- : \Omega_k^- \to W_k^- \subset \mathbb{C}^2$, with $W_k^+, W_k^- \subset \mathbb{C} \times \mathbb{C}^*$ if $N \geq 1$, with the following properties:*
   (a) *φ_k^+ and φ_k^- conjugate F with the map $(z, w) \mapsto (z+1, w)$.*
   (b) *The sets W_k^+ and W_k^- satisfy*

$$\bigcup_{\pm j \in \mathbb{N}} [W_k^\pm - (j, 0)] = \mathbb{C}^2 \text{ if } N = 0; \quad \bigcup_{\pm j \in \mathbb{N}} [W_k^\pm - (j, 0)] = \mathbb{C} \times \mathbb{C}^* \text{ if } N \geq 1.$$

Our second result shows that if conditions (A*) or (B*) are strictly not satisfied, then F has generic finite orbits in some neighborhood of the origin and so no two-dimensional stable sets. A version of this result has already been proved by Lisboa [10] for analytic vector fields.

THEOREM 1.3. *Let F be a local biholomorphism of the form (A) or (B). In the first case, assume that $\operatorname{Re}(b/a) < 0$; in the second case, assume that either*

$$\operatorname{Re}\left(\frac{a}{aM+bN}\right) < 0 \quad \textit{or} \quad \operatorname{Re}\left(\frac{b}{aM+bN}\right) < 0.$$

Then there exists a neighborhood $\mathcal{U}$ of the origin and there exist sets $\mathcal{P}^+, \mathcal{P}^- \subset \mathcal{U}$, which are one-dimensional submanifolds of $\mathcal{U}$ if F is of the form (A) and are empty otherwise, such that the following properties hold: given $p \in \mathcal{U} \setminus \mathcal{P}^+$ outside the fixed set, there exists $j \in \mathbb{N}$ such that $F^j(p) \notin \mathcal{U}$; given $p \in \mathcal{U} \setminus \mathcal{P}^-$ outside the fixed set, there exists $j \in \mathbb{N}$ such that $F^{-j}(p) \notin \mathcal{U}$. If F is of the form (A), then $\mathcal{P}^+$ is the set of points in $\mathcal{U}$ that are attracted under F to the parabolic curves of F, and $\mathcal{P}^-$ is the set of points in $\mathcal{U}$ that are attracted under F^{-1} to the parabolic curves of F^{-1}.

The following example shows the necessity of the hypotheses in Theorems 1.2 and 1.3.

Example 1.4. For $M \geq 1$, $N \geq 0$ and $a, b \in \mathbb{C}^*$ that do not satisfy the hypotheses in Theorems 1.2 and 1.3, consider the biholomorphism F given by the time-one flow of the vector field

$$X = x^M y^N \left[ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \right]$$

and let us show that for any neighborhood $\mathcal{U}$ of the origin there exists $p \in \mathcal{U}$ outside the fixed set such that the orbit $\{F^j(p) \colon j \in \mathbb{Z}\}$ is contained in $\mathcal{U}$ and bounded away from the origin.

Assume first that $N = 0$, so $\mathrm{Re}(b/a) = 0$. If $(x(t), y(t))$ is a solution of X and we set $(x_0, y_0) = (x(0), y(0))$, we get by integration that

$$x(t) = x_0[1 - aMx_0^M t]^{-1/M} \quad \text{and} \quad y(t) = y_0[1 - aMx_0^M t]^{-b/(aM)},$$

defined for all $t \in \mathbb{R}$ provided that $aMx_0^M \notin \mathbb{R}$. We have that $|x(t)| \leq C|x_0|$ for some $C > 0$ and for all $t \in \mathbb{R}$, and if we set $b/(aM) = i\beta$ we have

$$|y(t)| = |y_0| e^{\beta \arg(1 - aMx_0^M t)},$$

so $e^{-|\beta|\pi}|y_0| \leq |y(t)| \leq e^{|\beta|\pi}|y_0|$ for all $t \in \mathbb{R}$. Therefore, given a neighborhood $\mathcal{U}$ of the origin, if we choose (x_0, y_0) as above and sufficiently small with $y_0 \neq 0$ then its orbit is contained in $\mathcal{U}$ and bounded away from the origin.

Assume now that $N \geq 1$, so either $aM + bN = 0$ or $aM + bN \neq 0$, $\mathrm{Re}(a/(aM + bN)) \geq 0$ and $\mathrm{Re}(b/(aM + bN)) = 0$. If $(x(t), y(t))$ is a solution of X with $(x(0), y(0)) = (x_0, y_0)$ and we set $P(t) = x(t)^M y(t)^N$, we have $P' = (aM + bN)P^2$, $x' = aPx$ and $y' = bPy$. Suppose first that $aM + bN = 0$. Then $P(t) = P(0)$, $x(t) = x_0 e^{aP(0)t}$ and $y(t) = y_0 e^{bP(0)t}$, so

$$(x(t), y(t)) = (x_0 e^{a x_0^M y_0^N t}, y_0 e^{b x_0^M y_0^N t})$$

for all $t \in \mathbb{R}$. Note that, since $a/b = -N/M \in \mathbb{R}$, in any neighborhood $\mathcal{U}$ of the origin we can take (x_0, y_0) arbitrarily small with $x_0 y_0 \neq 0$ such that $\mathrm{Re}(a x_0^M y_0^N) = \mathrm{Re}(b x_0^M y_0^N) = 0$. In this case, the expression above shows that $|x(t)| = |x_0|$ and $|y(t)| = |y_0|$ for all $t \in \mathbb{R}$, so the orbit of (x_0, y_0) is bounded away from the origin and contained in $\mathcal{U}$ provided (x_0, y_0) is small enough. Suppose now that $aM + bN \neq 0$, $\mathrm{Re}(a/(aM + bN)) \geq 0$ and $\mathrm{Re}(b/(aM + bN)) = 0$. By integration we have that $P(t) = x_0^M y_0^N [1 - (aM + bN)x_0^M y_0^N t]^{-1}$ and

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$$x(t) = x_0[1-(aM+bN)x_0^M y_0^N t]^{-a/(aM+bN)},$$
$$y(t) = y_0[1-(aM+bN)x_0^M y_0^N t]^{-b/(aM+bN)},$$

defined for all $t \in \mathbb{R}$ provided $(aM+bN)x_0^M y_0^N \notin \mathbb{R}$. If we set $a/(aM+bN) = \alpha_1 + i\alpha_2$, with $\alpha_1 \geq 0$, and $b/(aM+bN) = i\beta$, we have

$$|x(t)| = |x_0||1-(aM+bN)x_0^M y_0^N t|^{-\alpha_1} e^{\alpha_2 \arg(1-(aM+bN)x_0^M y_0^N t)}$$

and

$$|y(t)| = |y_0| e^{\beta \arg(1-(aM+bN)x_0^M y_0^N t)},$$

so

$$|x(t)| \leq C e^{|\alpha_2|\pi}|x_0|, \quad e^{-|\beta|\pi}|y_0| \leq |y(t)| \leq e^{|\beta|\pi}|y_0|$$

for some $C > 0$ and for all $t \in \mathbb{R}$. Therefore, given a neighborhood $\mathcal{U}$ of the origin, if we choose (x_0, y_0) as above and sufficiently small then its orbit is contained in $\mathcal{U}$ and bounded away from the origin.

To conclude the introduction, let us briefly describe the structure of the paper. In §2 we prove some basic dynamic facts in the spirit of the Leau–Fatou flower theorem that we will use throughout the paper. Sections 3–7 are devoted to the proof of Theorem 1.2. In §§3 and 4 we show the existence of invariant domains and invariant functions for maps of the form (B), with $N \geq 0$, satisfying the hypotheses of Theorem 1.2; as we will explain in §7, this will also allow us to obtain Theorem 1.2 for maps of the form (A). In §5 we construct an approximation of Fatou coordinates (that is, conjugations with $(z, w) \mapsto (z+1, w)$), which we modify to actual Fatou coordinates in §6. The final details of the proof of Theorem 1.2 are provided in §7. Finally, in §8 we prove Theorem 1.3.

2. *Stable subdynamics in two variables*

In this section we prove Proposition 2.2, which provides some basic dynamic facts in the spirit of the Leau–Fatou flower theorem, adapted to our two-dimensional context.

Definition 2.1. Given $d \in \mathbb{N}$, $\varepsilon > 0$ and $\theta \in (0, \pi/2)$, we define the sets

$$S(d, \varepsilon, \theta) = \{z \in \mathbb{C} : |z^d| < \varepsilon, |\arg(z^d)| < \theta\},$$

whose connected components are the d sectors

$$S_k(d, \varepsilon, \theta) = \left\{ z \in \mathbb{C} : |z| < \varepsilon^{1/d}, \left| \arg z - \frac{2k\pi}{d} \right| < \frac{\theta}{d} \right\}$$

for $k = 0, \ldots, d-1$, and

$$\widetilde{S}(d, \varepsilon, \theta) = S(d, \varepsilon, \theta) \cup \left\{ z \in \mathbb{C} : \left| z^d - \frac{\varepsilon}{2}e^{-i\theta} \right| < \frac{\varepsilon}{2} \right\} \cup \left\{ z \in \mathbb{C} : \left| z^d - \frac{\varepsilon}{2}e^{i\theta} \right| < \frac{\varepsilon}{2} \right\},$$

whose connected components are d sectorial domains $\widetilde{S}_k(d, \varepsilon, \theta)$ of opening $(\pi + 2\theta)/d$ bisected by the rays $e^{2k\pi i/d}\mathbb{R}^+$ for $k = 0, \ldots, d-1$.

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PROPOSITION 2.2. *Let $G : \Omega \to \mathbb{C}^2$ be a holomorphic function, where Ω is a neighborhood of $0 \in \mathbb{C}^2$. Denote $(x_j, y_j) = G^j(x, y)$ for all $j \geq 0$. Let $m, n, d \in \mathbb{Z}_{\geq 0}$ be such that $m + n \geq 1$ and $d \geq 1$ and suppose that*

$$x_1^m y_1^n = x^m y^n - \frac{1}{d}(x^m y^n)^{d+1} + (x^m y^n)^{d+1} O(x, y).$$

There exist functions $\varepsilon_G, \delta_G, \tilde{\delta}_G : (0, \pi/2) \to \mathbb{R}^+$ with the following properties.

(1) *If $\mathcal{S}$ is a connected component of $S(d, \varepsilon, \theta)$ or $\widetilde{S}(d, \varepsilon, \theta)$ with $\theta \in (0, \pi/2)$ and $\varepsilon \leq \varepsilon_G(\theta)$, then for all (x, y) with $x^m y^n \in \mathcal{S}$, $|x| < \delta_G(\theta)$ and $|y| < \delta_G(\theta)$ we have that $x_1^m y_1^n \in \mathcal{S}$. Hence, if $x^m y^n \in \mathcal{S}$ and $|x_j| < \delta_G(\theta)$, $|y_j| < \delta_G(\theta)$ for all $j \geq 0$ then $x_j^m y_j^n \in \mathcal{S}$ for all $j \geq 0$. In this case, if $\mathcal{S}$ is a component of $S(d, \varepsilon, \theta)$ we have that*

$$\frac{1}{2}\frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \leq |x_j^m y_j^n|^d \leq \frac{2}{\cos\theta}\frac{|x^m y^n|^d}{1 + j|x^m y^n|^d}$$

for all $j \geq 0$, and if $\mathcal{S}$ is a component of $\widetilde{S}(d, \varepsilon, \theta)$ we have that

$$|x_j^m y_j^n|^d \geq \frac{1}{2}\frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \quad \text{for all } j \geq 0,$$

$$|x_j^m y_j^n|^d \leq 2\frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \quad \text{for all } j \geq \frac{6}{|x^m y^n|^d}.$$

In particular, $x_j^m y_j^n \to 0$ as $j \to +\infty$.

(2) *Consider a component $\widetilde{S}_k(d, \varepsilon, \theta)$ of $\widetilde{S}(d, \varepsilon, \theta)$ with $\varepsilon \leq \varepsilon_G(\theta)$ and assume the following additional hypothesis on G: there exist $\nu > 0$ and $\delta_\nu > 0$ such that*

$$|x_1| \leq |x|(1 + \nu|x^m y^n|^d) \quad \text{and} \quad |y_1| \leq |y|(1 + \nu|x^m y^n|^d)$$

whenever $x^m y^n \in \widetilde{S}_k(d, \varepsilon, \theta)$, $|x| < \delta_\nu$ and $|y| < \delta_\nu$. Then, given $\mu > 0$, we find $\tilde{\mu} > 0$ with the following property: if $x^m y^n \in \widetilde{S}_k(d, \varepsilon, \theta)$, $|x| < \tilde{\mu}$ and $|y| < \tilde{\mu}$, there exists $j_0 \geq 0$ such that $x_{j_0}^m y_{j_0}^n \in S_k(d, \varepsilon, \theta)$ and, for every $j \leq j_0$, $x_j^m y_j^n \in \widetilde{S}_k(d, \varepsilon, \theta)$, $|x_j| < \mu$ and $|y_j| < \mu$.

(3) *Take $\theta \in (0, \pi/2)$ and consider a point (x, y) with $x^m y^n \neq 0$ such that $|x_j| < \tilde{\delta}_G(\theta)$ and $|y_j| < \tilde{\delta}_G(\theta)$ for all $j \geq 0$. Then $x_j^m y_j^n \to 0$ as $j \to +\infty$ and there exists $k \in \{0, \ldots, d-1\}$ such that $x_j^m y_j^n \in S_k(d, \varepsilon, \theta)$ for any $\varepsilon > 0$ and for $j \geq 0$ big enough. Moreover, the two last inequalities of assertion 1 hold.*

We will need the following result.

LEMMA 2.3. *There exists $\kappa : (0, \pi/2) \to \mathbb{R}^+$ with the following property. If $f(z) = z - z^2$ for $z \in \mathbb{C}$ and either $S = S(1, \varepsilon, \theta)$ or $S = \widetilde{S}(1, \varepsilon, \theta)$ with $\theta \in (0, \pi/2)$ and $\varepsilon \leq \kappa(\theta)$, then $f(S) \subset S$ and for all $z \in S$,*

$$\operatorname{dist}(f(z), \partial S) \geq \kappa(\theta)|z|^2.$$

Proof. Suppose first that $S = S(1, \varepsilon, \theta)$. Set $\kappa(\theta) = \frac{1}{8}\sin(\theta/2)$ and assume that $\varepsilon \leq \kappa(\theta)$. An easy computation shows that the map $z \mapsto w = 1/z$ conjugates f to

$$g(w) = w + 1 + \frac{1}{w - 1}$$

and transforms S into

$$W = \{w \in \mathbb{C}\colon |w| > 1/\varepsilon,\ |\arg w| < \theta\}.$$

Fix $z \in S(1, \varepsilon, \theta)$ and set $z_1 = f(z)$, $w = 1/z$ and $w_1 = 1/z_1 = g(w)$. Consider the sectors $S_w(\theta)$ and $S_w(\theta/2)$, where

$$S_w(\alpha) = w + \{\zeta \in \mathbb{C}\colon |\zeta| > 0,\ |\arg \zeta| < \alpha\}.$$

Clearly, $S_w(\theta/2) \subset S_w(\theta) \subset W$ and

$$\operatorname{dist}(w + 1, \partial S_w(\theta/2)) = \sin(\theta/2).$$

Then, since

$$\operatorname{dist}(w + 1, w_1) = \frac{1}{|w - 1|} < \frac{1}{1/\kappa(\theta) - 1} = \frac{1}{8/\sin(\theta/2) - 1} < \frac{1}{4}\sin(\theta/2) < \sin(\theta/2),$$

we see that $w_1 \in S_w(\theta/2) \subset W$, which proves that $f(S) \subset S$. Note that

$$\operatorname{dist}(w_1, \partial S_w(\theta)) = |w_1 - w| \sin\beta,$$

where β is one of the angles determined by the ray $\overrightarrow{ww_1}$ in the sector $S_w(\theta)$. Since $w_1 \in S_w(\theta/2)$, we have that $\beta \geq \theta/2$ and therefore

$$\begin{aligned}\operatorname{dist}(w_1, \partial S_w(\theta)) > |w_1 - w| \sin(\theta/2) &\geq \left(1 - \frac{1}{|w - 1|}\right)\sin(\theta/2)\\ &> \left(1 - \tfrac{1}{4}\sin(\theta/2)\right)\sin(\theta/2) > \tfrac{3}{4}\sin(\theta/2).\end{aligned}$$

Then, since $S_w(\theta) \subset W$,

$$\operatorname{dist}(w_1, \partial W) \geq \operatorname{dist}(w_1, \partial S_w(\theta)) > \frac{3}{4}\sin(\theta/2),$$

so the disk $\mathcal{D} = \{\zeta \in \mathbb{C}\colon |\zeta - w_1| < \frac{3}{4}\sin(\theta/2)\}$ is contained in W. If we set $h(\zeta) = 1/\zeta$, we have that $h(W) = S$ and $h(\mathcal{D})$ contains the disk $\{\xi \in \mathbb{C}\colon |\xi - h(w_1)| < r\}$, where

$$r = \min\{|h(\zeta) - h(w_1)|\colon \zeta \in \partial\mathcal{D}\}.$$

Notice that if $\zeta \in \partial\mathcal{D}$, then $|\zeta| \leq |w_1| + 3/4 < 2|w_1|$. Then

$$r = \min\left\{\left|\frac{\zeta - w_1}{\zeta w_1}\right|\colon \zeta \in \partial\mathcal{D}\right\} > \frac{\frac{3}{4}\sin(\theta/2)}{2|w_1|^2} = \frac{\frac{3}{8}\sin(\theta/2)}{|w_1|^2}.$$

Thus, since

$$1/|w_1|^2 = |z_1|^2 \geq |z|^2(1 - |z|)^2 \geq |z|^2(1 - \kappa(\theta))^2 \geq |z|^2(1 - 1/8)^2 > \tfrac{2}{3}|z|^2,$$

we see that $r > \frac{1}{4}\sin(\theta/2)|z|^2 > \kappa(\theta)|z|^2$. Therefore we conclude that S contains the disk $\{\zeta \in \mathbb{C}\colon |\zeta - z_1| < \kappa(\theta)|z|^2\}$, which concludes the proof.

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Suppose now that $S = \widetilde{S}(1, \varepsilon, \theta)$, take $\kappa(\theta) = \frac{1}{8}\sin(\pi/4 - \theta/2)$ and assume that $\varepsilon \leq \kappa(\theta)$. Then the map $z \mapsto w = 1/z$ transforms S into

$$\widetilde{W} = W \cup \{w \in \mathbb{C}\colon\ \mathrm{Re}(we^{-i\theta}) > 1/\varepsilon\} \cup \{w \in \mathbb{C}\colon\ \mathrm{Re}(we^{i\theta}) > 1/\varepsilon\}.$$

Note that $S_w(\pi/4 - \theta/2) \subset S_w(\pi/2 - \theta) \subset W$. Then the proof follows exactly as in previous case, with $\pi/2 - \theta$ instead of θ. Finally, to unify the selection of κ we can take $\kappa(\theta) = \frac{1}{8}\min\{\sin(\theta/2), \sin(\pi/4 - \theta/2)\}$, which works for both cases. □

Proof of Proposition 2.2. From the expression for $x_1^m y_1^n$ we obtain that

$$(x_1^m y_1^n)^d = (x^m y^n)^d - (x^m y^n)^{2d} + (x^m y^n)^{2d}\sigma(x, y) \tag{1}$$

and

$$\frac{1}{(x_1^m y_1^n)^d} = \frac{1}{(x^m y^n)^d} + 1 + \sigma_1(x, y), \tag{2}$$

where $\sigma = O(x, y)$ and $\sigma_1 = O(x, y)$. Moreover, notice that $x_1^m y_1^n/(x^m y^n) = 1 + O(x, y)$ is arbitrarily close to 1 provided x and y are small enough. Then, given $\theta \in (0, \pi/2)$, we can find $\delta_G(\theta) > 0$ such that for all (x, y) with $|x| < \delta_G(\theta)$ and $|y| < \delta_G(\theta)$ we have that

$$|\sigma(x, y)| < \kappa(\theta), |\sigma_1(x, y)| < \frac{1}{4} \quad \text{and} \quad \left|\arg\frac{x_1^m y_1^n}{x^m y^n}\right| < \frac{\pi - 2\theta}{d},$$

where $\kappa(\theta)$ is given by Lemma 2.3. Set $\varepsilon_G(\theta) = \kappa(\theta)$.

Let us prove assertion 1 of the proposition. Take $\varepsilon \leq \varepsilon_G(\theta)$, let $\mathcal{S}$ be one of the components of $S(d, \varepsilon, \theta)$ or $\widetilde{S}(d, \varepsilon, \theta)$ and take (x, y) such that $x^m y^n \in \mathcal{S}$, $|x| < \delta_G(\theta)$ and $|y| < \delta_G(\theta)$. Set $f(z) = z - z^2$. Note that $S = \{z^d : z \in \mathcal{S}\}$ is one of the sets $S(1, \varepsilon, \theta)$ or $\widetilde{S}(1, \varepsilon, \theta)$, so by equation (1) and Lemma 2.3 we have that

$$|(x_1^m y_1^n)^d - f((x^m y^n)^d)| < \kappa(\theta)|x^m y^n|^{2d} \leq \mathrm{dist}(f((x^m y^n)^d), \partial S),$$

which means that $(x_1^m y_1^n)^d \in S$. Since the components of $S(d, \varepsilon, \theta)$ or $\widetilde{S}(d, \varepsilon, \theta)$ are separated by a sector of opening at least $(\pi - 2\theta)/d$ and $|\arg(x_1^m y_1^n/(x^m y^n))| < (\pi - 2\theta)/d$, we conclude that $(x_1^m y_1^n)^d$ belongs to the same component $\mathcal{S}$.

Now, let (x, y) be such that $x^m y^n \in \mathcal{S}$ with $|x_j| < \delta_G(\theta)$ and $|y_j| < \delta_G(\theta)$ for all $j \geq 0$. From (2) we obtain that

$$\frac{1}{(x_j^m y_j^n)^d} = \frac{1}{(x^m y^n)^d} + j + \sum_{l=0}^{j-1}\sigma_1(x_l, y_l),$$

so

$$\frac{1}{|x_j^m y_j^n|^d} \leq \frac{1}{|x^m y^n|^d} + j + \frac{1}{4}j \leq 2\left(\frac{1}{|x^m y^n|^d} + j\right)$$

for all $j \geq 0$, which gives the lower bounds in the inequalities of assertion 1. On the other hand, from the equation above we have that

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$$\frac{1}{|x_j^m y_j^n|^d} \geq \operatorname{Re} \frac{1}{(x_j^m y_j^n)^d} \geq \operatorname{Re} \frac{1}{(x^m y^n)^d} + j - \sum_{l=0}^{j-1} |\sigma_1(x_l, y_l)|$$
$$\geq \operatorname{Re} \frac{1}{(x^m y^n)^d} + j - \frac{1}{4} j = \operatorname{Re} \frac{1}{(x^m y^n)^d} + \frac{3}{4} j,$$

for all $j \geq 0$. If $\mathcal{S}$ is a component of $S(d, \varepsilon, \theta)$, since $|\arg(1/(x^m y^n)^d| < \theta$, we have that $\operatorname{Re}(1/(x^m y^n)^d) \geq \cos\theta / |x^m y^n|^d$ and therefore

$$\frac{1}{|x_j^m y_j^n|^d} \geq \frac{\cos\theta}{|x^m y^n|^d} + \frac{3}{4} j \geq \frac{\cos\theta}{2}\left(\frac{1}{|x^m y^n|^d} + j\right)$$

for all $j \geq 0$, which gives the upper bound in the first inequality of assertion 1. If $\mathcal{S}$ is a component of $\widetilde{S}(d, \varepsilon, \theta)$ we obtain that for all $j \geq 6/|x^m y^n|^d$,

$$\frac{1}{|x_j^m y_j^n|^d} \geq \operatorname{Re} \frac{1}{(x^m y^n)^d} + \frac{3}{4} j \geq -\frac{1}{|x^m y^n|^d} + \frac{3}{4} j \geq \frac{1}{2}\left(\frac{1}{|x^m y^n|^d} + j\right),$$

which concludes the proof of assertion 1.

Let us prove assertion 2. From the expression for $x_1^m y_1^n$ we can write $x_1^m y_1^n = x^m y^n (1 - \zeta)$, where $\zeta = (1/d + \tau)(x^m y^n)^d$ and $\tau = O(x, y)$. Then, up to reducing δ_ν, we have that if $|x| < \delta_\nu$ and $|y| < \delta_\nu$ then

$$|\zeta| < 1, \quad |\arg(1 - \zeta)| < \theta/d, \quad |1/d + \tau| > \delta_\nu, \quad |\arg(1/d + \tau)| < \theta^*,$$

where $\theta^* = \min\{\theta/2, \pi/4 - \theta/2\}$. Set

$$\tilde{\mu} = e^{-2\pi\nu/(d\delta_\nu \sin\theta^*)} \min\{\mu, \delta_\nu, \delta_G\}.$$

Take $x^m y^n \in \widetilde{S}_k(d, \varepsilon, \theta)$ such that $|x| < \tilde{\mu}$ and $|y| < \tilde{\mu}$. If $|\arg(x^m y^n)^d| < \theta$, we have that $x^m y^n \in S_k(d, \varepsilon, \theta)$ and, since $\tilde{\mu} \leq \mu$, we can take $j_0 = 0$ and we are done. Thus we can assume that $|\arg(x^m y^n)^d| \in [\theta, \theta + \pi/2)$. Moreover, we assume that $\arg(x^m y^n)^d \in [\theta, \theta + \pi/2)$; the other case is analogous. Then

$$\arg\zeta = \arg(1/d + \tau) + \arg(x^m y^n)^d \in (\theta - \theta^*, \theta + \pi/2 + \theta^*) \subset (\theta/2, \theta/2 + 3\pi/4)$$

and hence

$$\operatorname{Im}\zeta > |\zeta| \min\{\sin(\theta/2), \sin(\theta/2 + 3\pi/4)\} = |\zeta| \sin\theta^*.$$

Then, since $|\zeta| < 1$, $\sin(\arg(1 - \zeta)) = -\operatorname{Im}\zeta / |1 - \zeta| < -\frac{1}{2}\sin\theta^* |\zeta|$ and therefore

$$-\theta < \arg(1 - \zeta)^d < d\sin(\arg(1 - \zeta)) < -\frac{d}{2}\sin\theta^* |\zeta|.$$

It follows that

$$0 \leq \arg(x^m y^n)^d - \theta < \arg(x^m y^n)^d + \arg(1 - \zeta)^d < \arg(x^m y^n)^d - \frac{d}{2}\sin\theta^* |\zeta|$$
$$< \arg(x^m y^n)^d - \frac{d}{2}\delta_\nu \sin\theta^* |x^m y^n|^d;$$

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that is, $\arg(x_1^m y_1^n)^d = \arg(x^m y^n)^d + \arg(1-\zeta)^d$ satisfies

$$0 < \arg(x_1^m y_1^n)^d < \arg(x^m y^n)^d - \frac{d}{2}\delta_\nu \sin\theta^* |x^m y^n|^d. \tag{3}$$

Then $|x^m y^n|^d < (2/(d\delta_\nu \sin\theta^*)) \arg(x^m y^n)^d < 2\pi/(d\delta_\nu \sin\theta^*)$ so

$$|x_1| \le |x|(1+\nu|x^m y^n|^d) < |x|e^{\nu|x^m y^n|^d} < \tilde{\mu}e^{2\pi\nu/(d\delta_\nu \sin\theta^*)} = \min\{\mu, \delta_\nu, \delta_G\}$$

and, analogously, $|y_1| < \min\{\mu, \delta_\nu, \delta_G\}$. Thus, if $\arg(x_1^m y_1^n)^d < \theta$, since $x_1^m y_1^n \in \widetilde{S}_k(d, \varepsilon, \theta)$ because of assertion 1, we have that $x_1^m y_1^n \in S_k(d, \varepsilon, \theta)$, we can choose $j_0 = 1$ and we are done. We assume then that $\arg(x_1^m y_1^n)^d \in [\theta, \theta + \pi/2)$. Proceeding exactly as above, we obtain that

$$0 < \arg(x_2^m y_2^n)^d < \arg(x_1^m y_1^n)^d - \frac{d}{2}\delta_\nu \sin\theta^* |x_1^m y_1^n|^d$$

and, in view of (3),

$$0 < \arg(x_2^m y_2^n)^d < \arg(x^m y^n)^d - \frac{d}{2}\delta_\nu \sin\theta^* |x^m y^n|^d - \frac{d}{2}\delta_\nu \sin\theta^* |x_1^m y_1^n|^d.$$

Then

$$|x^m y^n|^d + |x_1^m y_1^n|^d < \frac{2}{d\delta_\nu \sin\theta^*} \arg(x^m y^n)^d < \frac{2\pi}{d\delta_\nu \sin\theta^*}$$

so

$$\begin{aligned}|x_2| &\le |x|(1+\nu|x^m y^n|^d)(1+\nu|x_1^m y_1^n|^d) < |x|e^{\nu|x^m y^n|^d + \nu|x_1^m y_1^n|^d} \\ &< \tilde{\mu}e^{2\pi\nu/(d\delta_\nu \sin\theta^*)} = \min\{\mu, \delta_\nu, \delta_G\}\end{aligned}$$

and, analogously, $|y_2| < \min\{\mu, \delta_\nu, \delta_G\}$. Thus, if $\arg(x_2^m y_2^n)^d < \theta$, we have that $x_2^m y_2^n \in S_k(d, \varepsilon, \theta)$, we can take $j_0 = 2$ and we are done. We assume then that $\arg(x_2^m y_2^n)^d \in [\theta, \theta + \pi/2)$ and repeat the argument. If assertion 2 were false, this process would continue indefinitely and we would obtain, for every $j \ge 0$, that $|x_j| < \delta_G$, $|y_j| < \delta_G$ and

$$|x^m y^n|^d + |x_1^m y_1^n|^d + \cdots + |x_j^m y_j^n|^d < \frac{2\pi}{d\delta_\nu \sin\theta^*}.$$

But the inequalities in assertion 1 show that $\sum_{j\ge 0} |x_j^m y_j^n|^d$ diverges, which is a contradiction.

Let us prove assertion 3. Let $\theta \in (0, \pi/2)$. Choose $\delta_1 > 0$ such that for all (x, y) with $|x| < \delta_1$ and $|y| < \delta_1$ we have $|\sigma_1(x, y)| < \frac{1}{2}\tan\theta$ and set

$$\tilde{\delta}_G(\theta) = \min\{\delta_1, \delta_G, \varepsilon_G^{1/((m+n)d)}\}.$$

Take (x, y) such that $|x_j| < \tilde{\delta}_G(\theta)$ and $|y_j| < \tilde{\delta}_G(\theta)$ for all $j \ge 0$. From (2) we have

$$\left|\operatorname{Im}\frac{1}{(x_j^m y_j^n)^d}\right| \le \left|\operatorname{Im}\frac{1}{(x^m y^n)^d}\right| + \sum_{l=0}^{j-1} |\sigma_1(x_l, y_l)| \le \left|\operatorname{Im}\frac{1}{(x^m y^n)^d}\right| + \frac{1}{2} j \tan\theta.$$

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On the other hand, as we showed in the proof of assertion 1,

$$\operatorname{Re}\frac{1}{(x_j^m y_j^n)^d} \geq \operatorname{Re}\frac{1}{(x^m y^n)^d} + \frac{3}{4}j.$$

Therefore, for j big enough,

$$0 < \frac{|\operatorname{Im}(1/(x_j^m y_j^n)^d)|}{\operatorname{Re}\,(1/(x_j^m y_j^n)^d)} \leq \frac{|\operatorname{Im}(1/(x^m y^n)^d)| + (1/2)j\tan\theta}{\operatorname{Re}(1/(x^m y^n)^d) + (3/4)j} < \tan\theta.$$

This means that $|\arg(1/(x_j^m y_j^n)^d)| < \theta$ for j big enough, so we can fix $j_0 \geq 0$ such that $|\arg(x_{j_0}^m y_{j_0}^n)^d| < \theta$. Moreover, since $|(x_{j_0}^m y_{j_0}^n)^d| < \tilde{\delta}_G(\theta)^{(m+n)d} \leq \varepsilon_G(\theta)$, we have that $x_{j_0}^m y_{j_0}^n \in S(d, \varepsilon_G(\theta), \theta)$. Thus $x_{j_0}^m y_{j_0}^n \in S_k(d, \varepsilon_G(\theta), \theta)$ for some $k \in \{0, \ldots, d-1\}$ and it follows from assertion 1 that $x_j^m y_j^n \in S_k(d, \varepsilon_G(\theta), \theta)$ for all $j \geq j_0$ and $x_j^m y_j^n \to 0$. Clearly, given $\varepsilon > 0$, up to increasing j_0, we have that $x_j^m y_j^n \in S_k(d, \varepsilon, \theta)$ for all $j \geq j_0$. Moreover, arguing exactly as in the proof of assertion 1 we have that the two last inequalities of that assertion hold. This concludes the proof of Proposition 2.2. □

3. *Existence of parabolic domains*

In this section we show the existence of parabolic domains for biholomorphisms of the form (B) satisfying condition (B⋆). Actually, we consider a slightly larger class of biholomorphisms, since we allow $N \geq 0$ (note that if $N = 0$ condition (B⋆) is precisely condition (A⋆)).

Consider a biholomorphism of the form (B), with $N \geq 0$, satisfying condition (B⋆). Set $d = M$ if $N = 0$ and $d = \gcd(M, N)$ otherwise. Applying a linear change of coordinates of the form $(x, y) \mapsto (\alpha x, \beta y)$, we obtain the same expression for F but with a and b respectively replaced by $\tilde{a} = -a/(aM + bN)$ and $\tilde{b} = -b/(aM + bN)$, so hypothesis (B⋆) becomes $\operatorname{Re}\tilde{a} < 0$ and $\operatorname{Re}\tilde{b} < 0$, and we have $\tilde{a}M + \tilde{b}N = -1$. Thus, we directly assume that

$$\operatorname{Re} a < 0, \quad \operatorname{Re} b < 0 \quad \text{and} \quad aM + bN = -1.$$

Set $m = M/d$ and $n = N/d$. Given a point (x, y) in the domain of definition of F, we denote $(x_j, y_j) = F^j(x, y)$ for all $j \geq 0$. Notice that, from the expression for F, we easily obtain that

$$x_1^m y_1^n = x^m y^n - \frac{1}{d}(x^m y^n)^{d+1} + (x^m y^n)^{d+1} O(x, y). \tag{4}$$

Definition 3.1. Fix $\gamma \in (0, 1)$ such that

$$\gamma(m+n) \leq 1, \quad \operatorname{Re}\left(a + \frac{\gamma}{d}\right) < 0 \quad \text{and} \quad \operatorname{Re}\left(b + \frac{\gamma}{d}\right) < 0$$

and set $\iota = 0$ if $n = 0$ and $\iota = 1$ if $n \geq 1$. Given $\theta \in (0, \pi/2)$ and $\varepsilon, \delta \in (0, 1]$, we consider the sets $D_k = D_k(\varepsilon, \theta, \delta)$, $\widetilde{D}_k = \widetilde{D}_k(\varepsilon, \theta, \delta)$ and $U_k = U_k(\varepsilon, \theta)$, for $k \in \{0, \ldots, d-1\}$, defined by

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$$D_k = \{(x, y) \in \mathbb{C}^2 \colon x^m y^n \in S_k(d, \varepsilon, \theta), |\iota x| < \delta, |y| < \delta\},$$

$$\widetilde{D}_k = \{(x, y) \in \mathbb{C}^2 \colon x^m y^n \in \widetilde{S}_k(d, \varepsilon, \theta), |\iota x| < \delta, |y| < \delta\},$$

where $S_k(d, \varepsilon, \theta)$ and $\widetilde{S}_k(d, \varepsilon, \theta)$ are the sets of Definition 2.1, and

$$U_k = \{(x, y) \in \mathbb{C}^2 \colon x^m y^n \in S_k(d, \varepsilon, \theta), |\iota x| \leq |x^m y^n|^\gamma, |y| \leq |x^m y^n|^\gamma\}.$$

Remark 3.2. If $n \geq 1$, the condition $x^m y^n \in S_k(d, \varepsilon, \theta)$ or $x^m y^n \in \widetilde{S}_k(d, \varepsilon, \theta)$ implies that the sets D_k, $\widetilde{D}_k$ and U_k are disjoint from the coordinate axes $\{xy = 0\}$. If $n = 0$, we have that $x^m y^n = x$ so the sets D_k, $\widetilde{D}_k$ and U_k are disjoint from $\{x = 0\}$ but they intersect $\{y = 0\}$. On the other hand, if $n = 0$ notice that $U_k = \{x \in S_k(d, \varepsilon, \theta), |y| \leq |x|^\gamma\}$ and, since $\varepsilon, \gamma \leq 1$, the inequality $|x| \leq |x|^\gamma$ holds in U_k. Therefore, even if $n = 0$ we can write

$$U_k = \{(x, y) \in \mathbb{C}^2 : x^m y^n \in S_k(d, \varepsilon, \theta), |x| \leq |x^m y^n|^\gamma, |y| \leq |x^m y^n|^\gamma\}.$$

Finally note that, because of the condition $\gamma(m + n) \leq 1$, the set U_k is non-empty.

Since $|\arg(-a)| < \pi/2$ and $|\arg(-b)| < \pi/2$ and, by the choice of γ, $|\arg(-a - \gamma/d)| < \pi/2$ and $|\arg(-b - \gamma/d)| < \pi/2$, we can fix $\theta_0 \in (0, \pi/2)$ such that

$$|\arg(-a)| + 2\theta_0 < \pi/2, \quad |\arg(-a - \gamma/d)| + 2\theta_0 < \pi/2,$$
$$|\arg(-b)| + 2\theta_0 < \pi/2, \quad |\arg(-b - \gamma/d)| + 2\theta_0 < \pi/2.$$

PROPOSITION 3.3. *Let F be a biholomorphism of the form (B) with $N \geq 0$ satisfying condition ($B^\star$) and fix θ_0 as above. Given $\theta \in (0, \theta_0)$, there exist $\varepsilon_\theta, \delta_\theta \in (0, 1]$ such that for $\varepsilon \leq \varepsilon_\theta$ and $\delta \leq \delta_\theta$ the following properties hold.*

(1) *For every $k \in \{0, \ldots, d - 1\}$ we have that*

$$F(D_k(\varepsilon, \theta, \delta)) \subset D_k(\varepsilon, \theta, \delta), \quad F(U_k(\varepsilon, \theta)) \subset U_k(\varepsilon, \theta),$$

$F^j \to 0$ as $j \to +\infty$ uniformly on $U_k(\varepsilon, \theta)$, $D_k(\varepsilon, \theta, \delta)$ is contained in the basin of attraction of $U_k(\varepsilon, \theta)$, and every point $(x, y) \in U_k(\varepsilon, \theta) \cup D_k(\varepsilon, \theta, \delta)$ satisfies

$$\frac{1}{2} \frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \leq |x_j^m y_j^n|^d \leq \frac{2}{\cos\theta} \frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \tag{5}$$

for all $j \geq 0$. Moreover, any orbit of F that converges to 0 eventually lies in $U_k(\varepsilon, \theta)$ for some $k \in \{0, \ldots, d - 1\}$.

(2) *There exists $\tilde{\delta} \in (0, 1]$ such that $\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})$ is contained in the basin of attraction of $U_k(\varepsilon, \theta)$ for all $k \in \{0, \ldots, d - 1\}$ and*

$$F^j(\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})) \subset \widetilde{D}_k(\varepsilon, \theta, \delta)$$

for all $j \geq 0$.

Proof. From the expression for F, we can write $x_1 = x(1 - \zeta)$, where $\zeta = (-a + \sigma)(x^m y^n)^d$ and $\sigma = O(x, y)$. By the choice of θ_0, we can take $\delta_0 > 0$ such that for all (x, y) with $|x| < \delta_0$ and $|y| < \delta_0$ we have

$$\delta_0 < |-a + \sigma| < 1/\delta_0, \quad |\arg(-a + \sigma)| + 2\theta_0 < \pi/2.$$

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Set $\eta = \frac{1}{2}\delta_0\cos(\pi/2 - \theta_0) > 0$ and $\varepsilon_0 = \delta_0 \cos(\pi/2 - \theta_0)$. Let us show that if $\varepsilon \le \varepsilon_0$ and $k \in \{0, \ldots, d-1\}$ then for all $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ we have

$$|x_1| \le |x|(1 - \eta|x^m y^n|^d) \le |x|. \tag{6}$$

Consider $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ with $\varepsilon \le \varepsilon_0$. We have that

$$|\zeta| < |x^m y^n|^d/\delta_0 < \varepsilon/\delta_0 < 1$$

and

$$|\arg \zeta| \le |\arg(-a + \sigma)| + |\arg(x^m y^n)^d| < (\pi/2 - 2\theta_0) + \theta < \pi/2 - \theta_0.$$

Thus $|\zeta| < \varepsilon/\delta_0 \le \cos(\pi/2 - \theta_0) < \cos(\arg \zeta)$ and so $|\zeta|^2 < \operatorname{Re} \zeta$. Then

$$|1 - \zeta|^2 = 1 - 2 \operatorname{Re} \zeta + |\zeta|^2 < 1 - \operatorname{Re} \zeta < 1 - \operatorname{Re} \zeta + \tfrac{1}{4}(\operatorname{Re} \zeta)^2 = (1 - \tfrac{1}{2} \operatorname{Re} \zeta)^2$$

and therefore $|1 - \zeta| < |1 - \frac{1}{2} \operatorname{Re} \zeta| = 1 - \frac{1}{2} \operatorname{Re} \zeta$, so

$$|1 - \zeta| < 1 - \tfrac{1}{2} \operatorname{Re} \zeta = 1 - \tfrac{1}{2} \cos(\arg \zeta)|\zeta| < 1 - \tfrac{1}{2} \cos(\pi/2 - \theta_0)|\zeta|$$

and hence $|1 - \zeta| < 1 - \frac{1}{2}\delta_0 \cos(\pi/2 - \theta_0)|x^m y^n|^d$, which proves (6). Proceeding in the same way, up to reducing δ_0 if necessary, we get that for all $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ with $\varepsilon \le \varepsilon_0$,

$$|y_1| \le |y|(1 - \eta|x^m y^n|^d) \le |y|. \tag{7}$$

From the expression for F, if $x, y \in \mathbb{C}^*$ are small enough, we have that

$$\begin{aligned}\frac{|y_1|}{|x_1^m y_1^n|^\gamma} &= \frac{|y||1 + b(x^m y^n)^d(1 + O(x, y))|}{|x^m y^n|^\gamma|1 - (\gamma/d)(x^m y^n)^d(1 + O(x, y))|} \\ &= \frac{|y|}{|x^m y^n|^\gamma}\left|1 + (x^m y^n)^d\left[b + \frac{\gamma}{d} + O(x, y)\right]\right|,\end{aligned}$$

so we can write

$$\frac{|y_1|}{|x_1^m y_1^n|^\gamma} = \frac{|y|}{|x^m y^n|^\gamma}|1 - \zeta_1|,$$

where $\zeta_1 = (-b - \gamma/d + \tau)(x^m y^n)^d$ with $\tau = O(x, y)$. By the choice of θ_0, reducing δ_0 if necessary, we get that for all (x, y) with $|x| < \delta_0$ and $|y| < \delta_0$ we have

$$\delta_0 < |-b - \gamma/d + \tau| < 1/\delta_0, \quad |\arg(-b - \gamma/d + \tau)| + 2\theta_0 < \pi/2.$$

Let us show that if $\varepsilon \le \varepsilon_0$ and $k \in \{0, \ldots, d-1\}$, then for all $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ we have

$$\frac{|y_1|}{|x_1^m y_1^n|^\gamma} \le \frac{|y|}{|x^m y^n|^\gamma}(1 - \eta|x^m y^n|^d) \le \frac{|y|}{|x^m y^n|^\gamma}. \tag{8}$$

Consider $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ with $\varepsilon \le \varepsilon_0$. We have that

$$|\zeta_1| < |x^m y^n|^d/\delta_0 < \varepsilon/\delta_0 \le 1$$

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and

$$|\arg \zeta_1| \le |\arg(-b - \gamma/d + \tau)| + |\arg(x^m y^n)^d| < (\pi/2 - 2\theta_0) + \theta < \pi/2 - \theta_0,$$

so proceeding exactly as above, we obtain that $|1 - \zeta_1| < 1 - \eta|x^m y^n|^d$ which proves (8). If $n \ge 1$, analogously we get, up to reducing δ_0 if necessary, that for $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$ with $\varepsilon \le \varepsilon_0$ we have

$$\frac{|x_1|}{|x_1^m y_1^n|^\gamma} \le \frac{|x|}{|x^m y^n|^\gamma}(1 - \eta|x^m y^n|^d) \le \frac{|x|}{|x^m y^n|^\gamma}. \tag{9}$$

Notice that, because of equation (4), F satisfies the hypotheses of Proposition 2.2. Set

$$\delta_\theta = \min\{1, \delta_0, \delta_F(\theta)\} \quad \text{and} \quad \varepsilon_\theta = \min\{1, \varepsilon_0, \varepsilon_F(\theta), \delta_0^{d/\gamma}, \delta_F(\theta)^{d/\gamma}\},$$

where $\varepsilon_F(\theta)$ and $\delta_F(\theta)$ are the constants given by Proposition 2.2. Now, consider $\varepsilon \le \varepsilon_\theta$, $\delta \le \delta_\theta$ and $k \in \{0, \dots, d-1\}$ and let us show that $D_k(\varepsilon, \theta, \delta)$ and $U_k(\varepsilon, \theta)$ are invariant by F and that estimates (5) hold. First, take $(x, y) \in D_k(\varepsilon, \theta, \delta)$. Then, it follows from (6) and (7) that $|x_1| \le |x| < \delta$ and $|y_1| \le |y| < \delta$. Moreover, since $\varepsilon \le \varepsilon_F(\theta)$, $\delta \le \delta_F(\theta)$ and $x^m y^n \in S_k(d, \varepsilon, \theta)$, we have by assertion 1 of Proposition 2.2 that $x_1^m y_1^n \in S_k(d, \varepsilon, \theta)$. Therefore $(x_1, y_1) \in D_k(\varepsilon, \theta, \delta)$ and so $D_k(\varepsilon, \theta, \delta)$ is invariant by F. Now, take $(x, y) \in U_k(\varepsilon, \theta)$. Then $|y| < |x^m y^n|^\gamma < \varepsilon^{\gamma/d} \le \delta_0$ and in the same way $|x| < \delta_0$, so $(x, y) \in D_k(\varepsilon, \theta, \delta_0)$. It follows from (8) and (9) that $|y_1| < |x_1^m y_1^n|^\gamma$ and analogously $|x_1| < |x_1^m y_1^n|^\gamma$. Moreover, since $x^m y^n \in S_k(d, \varepsilon, \theta)$, $\varepsilon \le \varepsilon_F(\theta)$, $|y| < \varepsilon^{\gamma/d} < \delta_F(\theta)$ and $|x| < \varepsilon^{\gamma/d} < \delta_F(\theta)$, we have by assertion 1 of Proposition 2.2 that $x_1^m y_1^n \in S_k(d, \varepsilon, \theta)$, which proves that $U_k(\varepsilon, \theta)$ is invariant by F. Moreover, if $(x, y) \in U_k(\varepsilon, \delta) \cup D_k(\varepsilon, \delta, \theta)$ we have estimates (5) directly from assertion 1 of Proposition 2.2.

Let us prove now that $F^j \to 0$ uniformly on $U_k(\varepsilon, \theta)$. Take $(x, y) \in U_k(\varepsilon, \theta)$. Since $(x_j, y_j) \in U_k(\varepsilon, \theta)$ for all $j \ge 0$, by (5) we have

$$|x_j^m y_j^n|^d \le \frac{2}{\cos\theta} \frac{|x^m y^n|^d}{1 + j|x^m y^n|^d} \le \frac{2}{\cos\theta} \frac{1}{j}$$

for all $j \ge 0$, which shows that $x_j^m y_j^n \to 0$ uniformly on $U_k(\varepsilon, \theta)$. Then, since $|y_j| < |x_j^m y_j^n|^\gamma$ and $|x_j| < |x_j^m y_j^n|^\gamma$, we have that $x_j \to 0$ and $y_j \to 0$ uniformly on $U_k(\varepsilon, \theta)$.

Consider now an orbit (x_j, y_j) converging to 0 and let us show that it eventually lies in $U_k = U_k(\varepsilon, \theta)$ for some k. Let $\tilde{\delta}_F(\theta)$ be the constant given by Proposition 2.2. Since $(x_j, y_j) \to 0$, there exists $j_0 \ge 0$ such that $|x_j| < \tilde{\delta}_F(\theta)$ and $|y_j| < \tilde{\delta}_F(\theta)$ for all $j \ge j_0$. By assertion 3 of Proposition 2.2, up to increasing j_0, we have that $x_j^m y_j^n \in S_k(d, \varepsilon, \theta)$ for all $j \ge j_0$ and for some $k \in \{0, \dots, d-1\}$ so, increasing j_0 again if necessary, we get that $(x_j, y_j) \in D_k(\varepsilon, \theta, \delta_0)$ for all $j \ge j_0$. Then, by iterated applications of (8),

$$\frac{|y_j|}{|x_j^m y_j^n|^\gamma} \le \frac{|y_{j_0}|}{|x_{j_0}^m y_{j_0}^n|^\gamma} \prod_{l=j_0}^{j-1} (1 - \eta|x_l^m y_l^n|^d)$$

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for all $j \geq j_0$. The estimate

$$|x_j^m y_j^n|^d \geq \frac{1}{2} \frac{|x^m y^n|^d}{1 + j|x^m y^n|^d}$$

from (5) implies that $\prod_{l=j_0}^{j-1}(1 - \eta|x_l^m y_l^n|^d)$ tends to 0 as $j \to +\infty$ and so does $|y_j|/|x_j^m y_j^n|^\gamma$. Hence $|y_j| \leq |x_j^m y_j^n|^\gamma$ for j big enough and, if $n \geq 1$, analogously $|x_j| \leq |x_j^m y_j^n|^\gamma$ for j big enough, which proves that eventually $(x_j, y_j) \in U_k$. Then, to prove that $D_k(\varepsilon, \theta, \delta)$ is contained in the basin of attraction of $U_k(\varepsilon, \theta)$ it suffices to show that the orbit of any point in $D_k(\varepsilon, \theta, \delta)$ converges to 0. Take $(x, y) \in D_k(\varepsilon, \theta, \delta)$. Since $(x_j, y_j) \in D_k(\varepsilon, \theta, \delta)$ for all $j \geq 0$, we can apply (6) and (7) for each (x_j, y_j) and we have

$$|x_j| \leq |x_0| \prod_{l=0}^{j-1}(1 - \eta|x_l^m y_l^n|^d), \quad |y_j| \leq |y_0| \prod_{l=0}^{j-1}(1 - \eta|x_l^m y_l^n|^d).$$

We have as before, by estimates (5), that the product above tends to zero and so do x_j and y_j. This concludes the proof of assertion 1.

Let us prove assertion 2. From the expression for F we have in a neighborhood of $0 \in \mathbb{C}^2$ that $|x_1| \leq |x|(1 + \nu|x^m y^n|^d)$ and $|y_1| \leq |y|(1 + \nu|x^m y^n|^d)$ for some $\nu > 0$. Let $\tilde{\delta} = \min\{1, \tilde{\mu}\}$, where $\tilde{\mu}$ is given by assertion 2 of Proposition 2.2 for $\mu = \delta$. If $x^m y^n \in \widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})$, by Proposition 2.2 there exists $j_0 \geq 0$ such that $|x_j| < \delta$, $|y_j| < \delta$ and $x_j^m y_j^n \in \widetilde{S}_k(d, \varepsilon, \theta)$ for all $j \leq j_0$ and $x_{j_0}^m y_{j_0}^n \in S_k(d, \varepsilon, \theta)$. In particular, $(x_{j_0}, y_{j_0}) \in D_k(\varepsilon, \theta, \delta)$ so by assertion 1 $(x_j, y_j) \in D_k(\varepsilon, \theta, \delta)$ for all $j \geq j_0$ so it eventually lies in $U_k(\varepsilon, \theta)$. Moreover, since $|x_j| < \delta$, $|y_j| < \delta$ and $x_j^m y_j^n \in \widetilde{S}_k(d, \varepsilon, \theta)$ for all $j \geq 0$, we have that $F^j(\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})) \subset \widetilde{D}_k(\varepsilon, \theta, \delta)$ for all $j \geq 0$. This concludes the proof of the proposition. □

4. *Existence of invariant functions*

In this section we show the existence of invariant functions for F on the domains $U_k = U_k(\varepsilon, \theta)$ given by Proposition 3.3 (up to reducing ε).

Since F is close to the time-one flow of the vector field

$$x^M y^N \left(ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \right)$$

and the vector field $ax(\partial/\partial x) + by(\partial/\partial y)$ has the Liouvillian first integrals $x^{\eta b} y^{-\eta a}$, $\eta \in \mathbb{C}^*$, our aim is to find an invariant function close to one of these first integrals, for which we start by defining a suitable branch $g(x, y)$ of $x^{db} y^{-da}$ on U_k, where $0 \leq k \leq d - 1$. From now on, if $z \in \mathbb{C}\backslash[-\infty, 0]$ and $\lambda \in \mathbb{C}\backslash\mathbb{Z}$, we denote $z^\lambda = e^{\lambda \log z}$, where log is the main branch of the logarithm. Note that, since m and n are coprime if $n \geq 1$, there exist $p, q \in \mathbb{N}$ such that $qm - pn = 1$; if $n = 0$, we set $p = 0, q = 1$. Denote $\lambda = d(ap + bq)$ and define $g : U_k \to \mathbb{C}$ as

$$g(x, y) = x^p y^q (x^m y^n)^\lambda, \tag{10}$$

which is well defined since $x^m y^n$ belongs to $\mathbb{C}\backslash[-\infty, 0]$ for all $(x, y) \in U_k$. We point out that if $n \geq 1$, since U_k is disjoint from $\{xy = 0\}$, the function g is non-vanishing whereas if

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$n = 0$, since $g(x, y) = yx^{Mb}$ and U_k intersects $\{y = 0\}$, the function g has $U_k \cap \{y = 0\}$ as zero set.

If x and y belong to a small sector bisected by $\mathbb{R}^+$ we have, in view of the identity $dma + dnb = -1$, that

$$g(x, y) = x^p y^q (x^m y^n)^\lambda = x^{p+m\lambda} y^{q+n\lambda} = x^{db} y^{-da},$$

so g is a branch of $x^{db} y^{-da}$ on U_k.

PROPOSITION 4.1. *Let F be a biholomorphism of the form (B) with $N \geq 0$ satisfying condition ($B^\star$) and fix θ_0 as in §3. Consider $\theta \in (0, \theta_0)$ and let ε_θ be the constant given by Proposition 3.3. Then there exist $\tilde{\varepsilon}_\theta \in (0, \varepsilon_\theta]$ and functions $\psi_k \in \mathcal{O}(U_k)$ for all $k \in \{0, \ldots, d-1\}$, where $U_k = U_k(\tilde{\varepsilon}_\theta, \theta)$, such that $\psi_k \circ F = \psi_k$. Moreover, $\psi_k = ug$, where g is the function defined above and $u \in \mathcal{O}(U_k)$ satisfies $|u(x, y) - 1| < 1/2$ for all $(x, y) \in U_k$; in particular, ψ_k is non-vanishing if $n \geq 1$ and has $U_k \cap \{y = 0\}$ as zero set if $n = 0$.*

Proof. Take $\tilde{\varepsilon}_\theta \leq \varepsilon_\theta$. Note that the domains $U_k = U_k(\tilde{\varepsilon}_\theta, \theta)$ for $k \in \{0, \ldots, d-1\}$ satisfy the conclusion of Proposition 3.3.

We define ψ_k as

$$\psi_k(x, y) = \lim_{j \to \infty} g(x_j, y_j), \quad (x, y) \in U_k,$$

where $(x_j, y_j) = F^j(x, y)$. It is clear that this function, if well defined, will be invariant by F. Let us show that it is well defined and holomorphic. Using the expression for F and equation (4), we have that

$$\frac{x_1^p y_1^q}{x^p y^q} = 1 + (x^m y^n)^d [ap + bq + O(x, y)]$$

and

$$\frac{(x_1^m y_1^n)^\lambda}{(x^m y^n)^\lambda} = 1 - (x^m y^n)^d \left[\frac{\lambda}{d} + O(x, y)\right]$$

for all (x, y), so

$$\frac{g(x_1, y_1)}{g(x, y)} = 1 + \ell(x, y), \qquad \text{with } \ell(x, y) = (x^m y^n)^d O(x, y). \tag{11}$$

Since $(x_j, y_j) \in U_k$ for all j, we have that $|y_j| \leq |x_j^m y_j^n|^\gamma$ and $|x_j| \leq |x_j^m y_j^n|^\gamma$ so

$$|\ell(x_j, y_j)| \leq K |x_j^m y_j^n|^{d+\gamma}$$

for some $K > 0$. Therefore, by estimates (5), the product $\prod_{j \geq 0}(g(x_{j+1}, y_{j+1})/g(x_j, y_j))$ converges uniformly for $(x, y) \in U_k$ and defines a holomorphic function $u \in \mathcal{O}(U_k)$. Then $g(x_j, y_j) \to u(x, y)g(x, y)$ uniformly on U_k, so ψ_k is well defined and holomorphic in U_k, and we have $\psi_k = ug$. Note that the function u is arbitrarily close to 1 if we suppose $|x^m y^n|$ to be small enough: if $n \in \mathbb{N}$ is large enough the finite product $\prod_{0 \leq j \leq n}(g(x_{j+1}, y_{j+1})/g(x_j, y_j))$ is arbitrarily uniformly close to u and we have from (11) that this finite product is arbitrarily uniformly close to 1 if $|x^m y^n|$ is small enough.

Therefore, since $|x^m y^n| < \tilde{\varepsilon}_\theta^{1/d}$ in U_k, we get that $|u(x,y) - 1| < 1/2$ for all $(x,y) \in U_k$, provided $\tilde{\varepsilon}_\theta$ is small enough. □

5. *Approximate Fatou coordinates*

In this section we find a change of coordinates ϕ_k on each of the domains $U_k(\varepsilon, \theta)$ provided by Propositions 3.3 and 4.1 (up to reducing ε and θ) which gives a first approximation of Fatou coordinates (that is, conjugations with $(z, w) \mapsto (z+1, w)$).

Definition 5.1. Given $\theta \in (0, \pi/2)$ and $\varepsilon \in (0, 1]$ such that Proposition 4.1 holds in $U_k = U_k(\varepsilon, \theta)$, we consider the map $\phi_k \colon U_k \to \mathbb{C}^2$ given by

$$\phi_k(x, y) = \left(\frac{1}{(x^m y^n)^d}, \psi_k(x, y) \right),$$

where $\psi_k \in \mathcal{O}(U_k)$ is the invariant function for F given by Proposition 4.1. Note that $\phi_k(U_k) \subset \mathbb{C} \times \mathbb{C}^*$ if $n \geq 1$. We also define the set $V = V(\varepsilon, \theta, r)$, for $0 < r < 1$, as

$$V = \{(z, w) \in \mathbb{C}^2 : |z| > \varepsilon^{-1}, \ |\arg z| < \theta, \ |w| < r|z|^{-\operatorname{Re} b/m - \gamma/(dm)}\}$$

if $n = 0$ and

$$V = \{(z, w) \in \mathbb{C}^2 : |z| > \varepsilon^{-1}, \ |\arg z| < \theta, \\ r^{-1}|z|^{\operatorname{Re} a/n + \gamma/(dn)} < |w| < r|z|^{-\operatorname{Re} b/m - \gamma/(dm)}\}$$

if $n \geq 1$. Since $\operatorname{Re} a + \gamma/d < 0 < -\operatorname{Re} b - \gamma/d$, V is non-empty. Notice also that V is homeomorphic to $\mathbb{C}^2$ if $n = 0$ and to $\mathbb{C} \times \mathbb{C}^*$ if $n \geq 1$.

PROPOSITION 5.2. *Let F be a biholomorphism of the form (B) with $N \geq 0$ satisfying condition ($B^\star$) and fix θ_0 as in §3. There exist $\theta_1 \in (0, \theta_0)$ and $\varepsilon_1 \in (0, \tilde{\varepsilon}_{\theta_1}]$, where $\tilde{\varepsilon}_{\theta_1}$ is the constant given by Proposition 4.1, such that if $\varepsilon < \varepsilon_1$, $\theta < \theta_1$ and r is small enough and we denote $V = V(\varepsilon, \theta, r)$ then the following properties hold for each $k \in \{0, \ldots, d-1\}$.*

(1) *$V \subset \phi_k(U_k)$ and $\phi_k : \phi_k^{-1}(V) \to V$ is a biholomorphism.*
(2) *$U_k(\varepsilon, \theta)$ is in the basin of attraction of $\phi_k^{-1}(V)$.*
(3) *The map $\tilde{F} = \phi_k \circ F \circ \phi_k^{-1}$ maps V into V and has the form $\tilde{F}(z, w) = (z + 1 + h(z, w), w)$ with*

$$|z + 1 + h(z, w)| > |z| + \frac{1}{2}, \ \ |h(z, w)| < K|z|^{-\gamma/d} \ \text{ and } \ \left| \frac{\partial h}{\partial z}(z, w) \right| < K'|z|^{-1-\gamma/d}$$

for some $K, K' > 0$ and for all $(x, y) \in V$.

Proof. Consider $\theta_1 \in (0, \theta_0)$ with $\cos \theta_1 > 2/3$ and let $\varepsilon_1 = \tilde{\varepsilon}_{\theta_1}$ be the constant given by Proposition 4.1. From equation (4) we have that

$$\frac{1}{(x_1^m y_1^n)^d} = \frac{1}{(x^m y^n)^d} + 1 + O(x, y).$$

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Thus, since $|x| \leq |x^m y^n|^\gamma$ and $|y| \leq |x^m y^n|^\gamma$ for all $(x, y) \in U_k(\varepsilon_1, \theta_1)$, there exists $K > 0$ such that

$$\left| \frac{1}{(x_1^m y_1^n)^d} - \frac{1}{(x^m y^n)^d} - 1 \right| < K|x^m y^n|^\gamma < K\varepsilon_1^{\gamma/d} \tag{12}$$

for all $(x, y) \in U_k(\varepsilon_1, \theta_1)$ and all $k \in \{0, \ldots, d-1\}$. Then, up to reducing ε_1 if necessary, we can assume that

$$\left| \frac{1}{(x_1^m y_1^n)^d} - \frac{1}{(x^m y^n)^d} - 1 \right| < \frac{1}{6} \tag{13}$$

for all $(x, y) \in U_k(\varepsilon_1, \theta_1)$ and all $k \in \{1, \ldots, d-1\}$. We now fix $\theta < \theta_1$ and $\varepsilon < \varepsilon_1$ and we denote $U_k = U_k(\varepsilon, \theta)$.

Let us prove assertion 1; without loss of generality we assume that $k = 0$. Let g be the function defined by (10). Using the fact that $qm - pn = 1$ and $adm + bdn = -1$, a straightforward computation shows that the map $\varphi : U_0 \to \mathbb{C}^2$ given by

$$\varphi(x, y) = \left(\frac{1}{(x^m y^n)^d}, g(x, y) \right)$$

is injective and its inverse is given by

$$\varphi^{-1}(z, w) = (z^a w^{-n}, z^b w^m).$$

Consider a point $(z_0, w_0) \in V$, and set

$$A_{z_0} = \{(x, y) \in \mathbb{C}^2 : x^m y^n = z_0^{-1/d},\ |\iota x| < |z_0|^{-\gamma/d},\ |y| < |z_0|^{-\gamma/d}\}.$$

Notice that $A_{z_0} \subset U_0$ and $\overline{A_{z_0}}$ is a Riemann surface with boundary. If we set $x = z_0^a w_0^{-n}$ and $y = z_0^b w_0^m$, then $x^m y^n = z_0^{-1/d}$ and $g(x, y) = w_0$. Moreover, $|y| = |z_0^b w_0^m| < r^m |z_0^b||z_0|^{-\operatorname{Re} b - \gamma/d} \leq r^m e^{|b|\theta} |z_0|^{-\gamma/d}$ and, if $n \geq 1$, analogously $|x| = |z_0^a w_0^{-n}| < r^n e^{|a|\theta} |z_0|^{-\gamma/d}$, so $(x, y) \in A_{z_0}$ if r is small enough. Therefore, since φ is injective, $g|_{A_{z_0}}$ assumes the value w_0 once. If we show that

$$|\psi_0(x, y) - g(x, y)| < |g(x, y) - w_0| \quad \text{whenever } (x, y) \in \partial A_{z_0} \tag{14}$$

then, by Rouché's theorem, the function $\psi_0|_{A_{z_0}}$ will also assume the value w_0 exactly once, showing that $V \subset \phi_0(U_0)$ and that ϕ_0 is injective in $\phi_0^{-1}(V)$. Let us prove that inequality (14) holds. If $n \geq 1$, the boundary ∂A_{z_0} of A_{z_0} is composed by two connected components,

$$\partial_1 A_{z_0} = \{(x, y) \in \mathbb{C}^2 : x^m y^n = z_0^{-1/d},\ |x| = |z_0|^{-\gamma/d}\},$$
$$\partial_2 A_{z_0} = \{(x, y) \in \mathbb{C}^2 : x^m y^n = z_0^{-1/d},\ |y| = |z_0|^{-\gamma/d}\},$$

whereas if $n = 0$ we have $\partial A_{z_0} = \partial_2 A_{z_0}$. Consider a point $(x, y) \in \partial A_{z_0} \subset U_0$. Since $x^m y^n = z_0^{-1/d}$, it follows from the computation of φ^{-1} that

$$g(x, y)^n = x^{-1} z_0^a \quad \text{and} \quad g(x, y)^m = y z_0^{-b}.$$

We suppose first that $(x, y) \in \partial_2 A_{z_0}$. Then

$$|g(x, y)| = |y|^{1/m} |z_0^{-b/m}| = |z_0|^{-\gamma/(dm)} |z_0^{-b/m}| \geq e^{-|b|\theta/m} |z_0|^{-\operatorname{Re} b/m - \gamma/(dm)}$$

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so $|g(x, y)| > e^{-|b|\theta/m} r^{-1} |w_0|$. Then

$$|g(x, y) - w_0| \geq |g(x, y)| - |w_0| > (1 - e^{|b|\theta/m} r)|g(x, y)| \geq \tfrac{1}{2}|g(x, y)|$$

if r is small enough. This relation, together with the fact that

$$|\psi_0(x, y) - g(x, y)| < \tfrac{1}{2}|g(x, y)|$$

for all $(x, y) \in U_0$, as was shown in Proposition 4.1, implies (14). Analogously, if $(x, y) \in \partial_1 A_{z_0}$ (so $n \geq 1$) then

$$|g(x, y)| = |x|^{-1/n} |z_0^{a/n}| = |z_0|^{\gamma/(dn)} |z_0^{a/n}| \leq e^{|a|\theta/n} |z_0|^{\operatorname{Re} a/n + \gamma/(dn)}$$

so $|g(x, y)| < e^{|a|\theta/n} r |w_0|$ and hence

$$|g(x, y) - w_0| \geq |w_0| - |g(x, y)| > (e^{-|a|\theta/n} r^{-1} - 1)|g(x, y)| \geq \tfrac{1}{2}|g(x, y)|$$

if r is sufficiently small, which again implies (14) and assertion 1 is proved.

Now take $(x, y) \in U_k$. If we set $(z_j, w_j) = \phi_k(x_j, y_j)$, then clearly $|z_j| > \varepsilon^{-1}$ and $|\arg z_j| < \theta$; moreover, since ψ_k is invariant by F we have that w_j is constant for all j while $z_j \to +\infty$ because of (5), so for j large enough we get $|w_j| < r|z_j|^{-\operatorname{Re} b/m - \gamma/(dm)}$ and if $n \geq 1$, since $w_j \neq 0$, $|w_j| > r^{-1}|z_j|^{\operatorname{Re} a/n + \gamma/(dn)}$, so $(z_j, w_j) \in V$ and assertion 2 is proved.

Let us prove assertion 3. For $r_1 \in (0, 1)$, consider the set $V_1 = V(\varepsilon_1, \theta_1, r_1)$. Proceeding exactly as in the proof of assertion 1, we can choose $r_1 \in (0, 1)$ such that $V_1 \subset \phi_k(U_k(\varepsilon_1, \theta_1))$ and $\phi_k : \phi_k^{-1}(V_1) \to V_1$ is a biholomorphism for every k. Then ϕ_k^{-1} is well defined on V_1 and takes values in $U_k(\varepsilon_1, \theta_1)$, so $\tilde{F} = \phi_k \circ F \circ \phi_k^{-1}$ is well defined on V_1. Since ψ_k is invariant by F, we can express $\tilde{F}(z, w) = (f(z, w), w)$. Then, if we write $\phi_k^{-1}(z, w) = (x, y) \in U_k(\varepsilon_1, \theta_1)$ and $F(x, y) = (x_1, y_1)$, we have from equation (4) that

$$f(z, w) = \frac{1}{(x_1^m y_1^n)^d} = \frac{1}{(x^m y^n)^d} + 1 + \tilde{h}(x, y),$$

where $\tilde{h}(x, y) = O(x, y)$. That is,

$$f(z, w) = z + 1 + h(z, w),$$

where by (12) and (13)

$$|h(z, w)| < K|z|^{-\gamma/d} \quad \text{and} \quad |h(z, w)| < \tfrac{1}{6}$$

for all $(z, w) \in V_1$. Thus, since $\cos\theta_1 > 2/3$, we have

$$\begin{aligned} |f(z, w)|^2 &= |z + 1 + h(z, w)|^2 = |z|^2 + 2\operatorname{Re} z + 2\operatorname{Re}(z\overline{h}(z, w)) + |1 + h(z, w)|^2 \\ &\geq |z|^2 + 2\cos\theta_1 |z| - 2|z||h(z, w)| + |1 + h(z, w)|^2 \\ &> |z|^2 + 4|z|/3 - |z|/3 + (5/6)^2 > (|z| + 1/2)^2, \end{aligned}$$

so $|f(z, w)| > |z| + 1/2$ for all $(z, w) \in V_1$.

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We now fix r satisfying assertion 1 and such that $r < r_1$. Consider a point $(z, w) \in V$. Since $(f(z, w), w) \in \phi_k(U_k)$, it is clear that $|f(z, w)| > \varepsilon^{-1}$ and $|\arg(f(z, w))| < \theta$ and, since $|w| < r|z|^{-\operatorname{Re} b/m-\gamma/(dm)}$ and $|f(z, w)| > |z|$, we also have that

$$|w| < r|f(z, w)|^{-\operatorname{Re} b/m-\gamma/(dm)}.$$

Analogously, if $n \geq 1$, since $|w| > r^{-1}|z|^{\operatorname{Re} a/n+\gamma/(dn)}$ and $|f(z, w)| > |z|$, we have $|w| > r^{-1}|f(z, w)|^{\operatorname{Re} a/n+\gamma/(dn)}$, so $\tilde{F}(z, w) \in V$.

To prove the bound for $\partial h/\partial z$ let us first show that there exists $\rho > 0$ such that if $(z_0, w_0) \in V$ and $|z - z_0| < \rho|z_0|$ then $(z, w_0) \in V_1$. Consider $(z_0, w_0) \in V$ and assume that $|z - z_0| < \rho|z_0|$. Then

$$|z| > (1-\rho)|z_0| > (1-\rho)\varepsilon^{-1},$$

so $|z| > \varepsilon_1^{-1}$ for ρ sufficiently small. Since $|z/z_0 - 1| < \rho$, we have $|\arg(z/z_0)| < \arcsin \rho$, so

$$|\arg z| \leq |\arg z_0| + \arcsin \rho < \theta + \arcsin \rho,$$

hence $|\arg z| < \theta_1$ if ρ is small enough. Since $|z_0| < (1-\rho)^{-1}|z|$, it follows that

$$|w_0| < r|z_0|^{-\operatorname{Re} b/m-\gamma/(dm)} < r[(1-\rho)^{-1}|z|]^{-\operatorname{Re} b/m-\gamma/(dm)},$$

so $|w_0| < r_1|z|^{-\operatorname{Re} b/m-\gamma/(dm)}$ if ρ is small enough and, if $n \geq 1$,

$$|w_0| > r^{-1}|z_0|^{\operatorname{Re} a/n+\gamma/(dn)} > r^{-1}[(1-\rho)^{-1}|z|]^{\operatorname{Re} a/n+\gamma/(dn)},$$

so $|w_0| > r_1^{-1}|z|^{\operatorname{Re} a/n+\gamma/(dn)}$ if ρ is small enough. Hence, $(z, w_0) \in V_1$ if ρ is small enough. Now, take a point $(z_0, w_0) \in V$. As we have seen, if $D \subset \mathbb{C}$ is the disk of radius $\rho|z_0|$ centered at z_0, then $D \times \{w_0\}$ is contained in V_1, so the function

$$h_{w_0} : z \in D \mapsto h(z, w_0)$$

is well defined and

$$|h_{w_0}(z)| < K|z|^{-\gamma/d} < K(1-\rho)^{-\gamma/d}|z_0|^{-\gamma/d}.$$

Thus, it follows from Cauchy's inequality that

$$\left|\frac{\partial h}{\partial z}(z_0, w_0)\right| = |(h_{w_0})'(z_0)| \leq K(1-\rho)^{-\gamma/d}|z_0|^{-\gamma/d}(\rho|z_0|)^{-1} = K'|z_0|^{-1-\gamma/d},$$

which finishes the proof of assertion 3. □

6. *Existence of Fatou coordinates*

In this section we construct Fatou coordinates for F on the domains $\phi_k^{-1}(V) \subset U_k$ given by Proposition 5.2.

PROPOSITION 6.1. *Let F be a biholomorphism of the form (B) with $N \geq 0$ satisfying condition ($B^\star$), let ε, θ and r be as in Proposition 5.2 and denote $U_k = U_k(\varepsilon, \theta)$ and $V = V(\varepsilon, \theta, r)$. Then, for each $k \in \{0, \ldots, d-1\}$, there exists a biholomorphism*

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$\varphi_k^+ \colon \phi_k^{-1}(V) \to W \subset \mathbb{C}^2$, *with* $W \subset \mathbb{C} \times \mathbb{C}^*$ *if* $n \geq 1$, *conjugating* F *with the map* $(z, w) \mapsto (z+1, w)$ *and satisfying*

$$\bigcup_{j \in \mathbb{N}} [W - (j, 0)] = \mathbb{C}^2 \quad \textit{if } n = 0; \quad \bigcup_{j \in \mathbb{N}} [W - (j, 0)] = \mathbb{C} \times \mathbb{C}^* \textit{ if } n \geq 1. \tag{15}$$

Proof. By Proposition 5.2, we have that $\phi_k : \phi_k^{-1}(V) \to V$ is a biholomorphism conjugating F with $\widetilde{F} = \phi_k \circ F \circ \phi_k^{-1} \colon V \to V$. Thus, it is enough to find a biholomorphism $\Phi \colon V \to W \subset \mathbb{C}^2$, with $W \subset \mathbb{C} \times \mathbb{C}^*$ if $n \geq 1$, conjugating $\tilde{F}$ with the map $(z, w) \mapsto (z+1, w)$. Since $\tilde{F} \colon V \to V$ is written $\tilde{F}(z, w) = (f(z, w), w)$, each function $f_w : z \mapsto f(z, w)$ maps the domain $V_w = \{z \in \mathbb{C} : (z, w) \in V\}$ into itself. Thus, we start considering $w \in \mathbb{C}$ fixed ($w \in \mathbb{C}^*$ if $n \geq 1$) and we will find, following the ideas in [**13**, Lemma 10.10], a map $\beta_w : V_w \to \mathbb{C}$ conjugating f_w with $z \mapsto z+1$. We will also show, arguing as in [**13**, Lemma 10.11], that

$$\bigcup_{j \in \mathbb{N}} (\beta_w(V_w) - j) = \mathbb{C}. \tag{16}$$

Finally, from the maps β_w we will construct a global map $\beta : V \to \mathbb{C}$ such that $\beta \circ \tilde{F}(z, w) = \beta(z, w) + 1$, so the function $\Phi : V \to \mathbb{C}^2$ given by $\Phi(z, w) = (\beta(z, w), w)$ is a Fatou coordinate for $\tilde{F}$.

From the definition of V we have

$$V_w = \{z \in \mathbb{C} \colon |z| > R_w,\ |\arg(z)| < \theta\},$$

where

$$R_w = \max\{\varepsilon^{-1},\ (r^{-1}|w|)^{-dm/(d \operatorname{Re} b + \gamma)},\ \iota(r|w|)^{dn/(d \operatorname{Re} a + \gamma)}\}$$

(recall that $\iota = 0$ if $n = 0$ and $\iota = 1$ if $n \geq 1$).

Take a base point $p \in V_w$. The map β_w conjugating f_w with $z \mapsto z+1$ will be constructed as the limit of the functions

$$\beta_j(z) = f_w^j(z) - f_w^j(p), \quad j \in \mathbb{N}.$$

In order to simplify the proof of the convergence of these functions we assume p to be large enough so that for all $z \in V_w$ the Euclidean segment $[z, p]$ is contained in V_w, which is possible because $\theta < \pi/2$. Since $|f_w(p)| \geq |p| + 1/2$ by Proposition 5.2, the sequence $|f_w^j(p)|$ is increasing, hence the property above also holds for $f_w^j(p)$. In particular we have, for all $z \in V_w$ and all $j \in \mathbb{N}$,

$$[f_w^j(z), f_w^j(p)] \subset V_w.$$

Since $f_w(z) = z + 1 + h(z, w)$ and $|\partial h/\partial z(z, w)| < K'|z|^{-1-\gamma/d}$ by Proposition 5.2, it follows from the mean value inequality that, if $[z_1, z_2] \subset V_w$, then

$$\left| \frac{f_w(z_1) - f_w(z_2)}{z_1 - z_2} - 1 \right| = \left| \frac{h(z_1, w) - h(z_2, w)}{z_1 - z_2} \right| \leq \max_{z \in [z_1, z_2]} \frac{K'}{|z|^{1+\gamma/d}}.$$

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Since the angle between $z_1\mathbb{R}^+$ and $z_2\mathbb{R}^+$ is bounded by $2\theta < \pi$, there is a constant $\tau > 0$ depending only on θ such that $\min\{|z| : z \in [z_1, z_2]\} \geq \tau \min\{|z_1|, |z_2|\}$, so

$$\left|\frac{f_w(z_1) - f_w(z_2)}{z_1 - z_2} - 1\right| \leq \frac{K'}{(\tau \min\{|z_1|, |z_2|\})^{1+\gamma/d}}.$$

In particular, setting $z_1 = f_w^j(z)$ and $z_2 = f_w^j(p)$, we obtain

$$\left|\frac{\beta_{j+1}(z)}{\beta_j(z)} - 1\right| = \left|\frac{f_w^{j+1}(z) - f_w^{j+1}(p)}{f_w^j(z) - f_w^j(p)} - 1\right| \leq \frac{K'}{(\tau \min\{|f_w^j(z)|, |f_w^j(p)|\})^{1+\gamma/d}}$$

for all $z \in V_w$ and $j \in \mathbb{N}$. Therefore, since $|f_w^j(z)| \geq j/2$ and $|f_w^j(p)| \geq j/2$, we obtain

$$\left|\frac{\beta_{j+1}(z)}{\beta_j(z)} - 1\right| \leq K'(2\tau^{-1})^{1+\gamma/d} j^{-1-\gamma/d}$$

for all $z \in V_w$ and $j \in \mathbb{N}$. This shows that the product $\prod(\beta_{j+1}(z)/\beta_j(z))$ is uniformly convergent in V_w and therefore β_j converges uniformly to a function $\beta_w \in \mathcal{O}(V_w)$. Let us show that β_w is one-to-one and conjugates f_w with the map $z \mapsto z+1$. Since $f_w(z) = z + 1 + h(z, w)$ and $|h(z, w)| < K|z|^{-\gamma/d}$ by Proposition 5.2, we have

$$|f_w^{j+1}(p) - f_w^j(p) - 1| = |h(f_w^j(p), w)| \leq \frac{K}{|f_w^j(p)|^{\gamma/d}} \to 0 \quad \text{as } j \to \infty.$$

Thus, since $\beta_j(f_w(z)) = \beta_{j+1}(z) + f_w^{j+1}(p) - f_w^j(p)$, we obtain, taking $j \to \infty$,

$$\beta_w(f_w(z)) = \beta_w(z) + 1, \quad z \in V_w.$$

Finally, since β_j is injective for all j and β_w is not constant, we conclude that β_w is injective.

Now, as in [13, Lemma 10.11], we prove that β_w satisfies (16). We show first that $\lim_{z\to\infty}(\beta_w(z)/z) = 1$. Since β_j tends uniformly to β_w, for some $l \in \mathbb{N}$ we have that $|\beta_w - \beta_l|$ is bounded, whence

$$|\beta_w - f_w^l| \leq |\beta_w - \beta_l| + |\beta_l - f_w^l|$$

is bounded. Then, since $f_w(z) = z + 1 + h(z, w)$ and $|h(z, w)| < K|z|^{-\gamma/d}$,

$$\lim_{z\to\infty} \frac{\beta_w(z)}{z} = \lim_{z\to\infty} \frac{f_w^l(z)}{z} = 1.$$

Consider $\zeta \in \mathbb{C}$. In order to prove (16) we will show that for $j \in \mathbb{N}$ large enough the point $\zeta_j = \zeta + j$ belongs to $\beta_w(V_w)$. Since V_w is essentially a sector of opening 2θ, if we take a positive number $\rho < \sin\theta$, it is not difficult to see that, for j large enough, the closed disk D_j of radius $r_j = \rho|\zeta_j|$ centered at ζ_j is contained in V_w. By Rouché's theorem, if

$$|\beta_w(z) - z| < r_j$$

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for all $z \in \partial D_j$, then $\zeta_j \in \beta_w(D_j) \subset \beta_w(V_w)$. Since $\beta_w(z)/z \to 1$ when $z \to \infty$, for z large enough we have $|\beta_w(z) - z| < \tau|z|$, where $\tau > 0$ is taken such that $\tau(1+\rho) < \rho$. Then, if $z \in \partial D_j$ and j is large enough,

$$|\beta_w(z) - z| < \tau|z| \le \tau(|\zeta_j| + r_j) = \tau(1+\rho)|\zeta_j| < \rho|\zeta_j| = r_j$$

and (16) follows.

The function β_w that we have constructed depends on the choice of the base point $p \in V_w$, but its derivative does not, as we can check from the definition of β_j:

$$(\beta_w)'(z) = \lim_{j\to\infty} (f_w^j)'(z).$$

It is easy to see that the same choice of p also works for any w' in a neighborhood of w and the function $\beta_{w'}$ will depend holomorphically on w'. That is, $\beta_{w'}(z)$ is a holomorphic function of (z, w'). Thus, we can find an open covering $\mathbb{C} = \bigcup_{i\in I} W_i$ if $n = 0$ or $\mathbb{C}^* = \bigcup_{i\in I} W_i$ if $n \ge 1$ and, for each $i \in I$, a holomorphic function

$$\beta_i(z, w), \quad \text{for } z \in V_w,\ w \in W_i$$

such that, for each $w \in W_i$, the map $z \in V_w \mapsto \beta_i(z, w) \in \mathbb{C}$ is univalent and satisfies $\beta_i(f(z, w), w) = \beta_i(z, w) + 1$. Moreover, from the observation above, the partial derivative $\partial\beta_i/\partial z$ does not depend on $i \in I$, that is,

$$\frac{\partial\beta_i}{\partial z}(z, w) = \frac{\partial\beta_j}{\partial z}(z, w) \quad \text{for } z \in V_w,\ w \in W_i \cap W_j,\ i, j \in I.$$

Therefore, if $W_i \cap W_j \neq \emptyset$, there is a function $g_{ij} \in \mathcal{O}(W_i \cap W_j)$ such that

$$\beta_j(z, w) - \beta_i(z, w) = g_{ij}(w) \quad \text{for } z \in V_w,\ w \in W_i \cap W_j, \tag{17}$$

hence $g_{ij} + g_{jk} + g_{ki} = 0$ on $W_i \cap W_j \cap W_k$, for $i, j, k \in I$. Then, since the first Cousin problem can be solved in $\mathbb{C}$ and $\mathbb{C}^*$, there exist functions $g_i \in \mathcal{O}(W_i)$, $i \in I$, such that $g_{ij} = g_i - g_j$ on $W_i \cap W_j$ and it follows from (17) that

$$\beta_j(z, w) + g_j(w) = \beta_i(z, w) + g_i(w) \quad \text{for } z \in V_w,\ w \in W_i \cap W_j.$$

Therefore we can define a global function $\beta \in \mathcal{O}(V)$ by

$$\beta(z, w) = \beta_i(z, w) + g_i(w) \quad \text{for } z \in V_w,\ w \in W_i,$$

and we can see that for each $w \in \mathbb{C}^*$, the map

$$z \in V_w \mapsto \beta(z, w) \in \mathbb{C}$$

is univalent and $\beta(f(z, w), w) = \beta(z, w) + 1$ for every $(z, w) \in V$. Now it is easy to check that the holomorphic function

$$\Phi(z, w) = (\beta(z, w), w), \quad (z, w) \in V,$$

is univalent and satisfies $\Phi \circ \tilde{F}(z, w) = \Phi(z, w) + (1, 0)$ for every $(z, w) \in V$. To show that $W = \Phi(V)$ satisfies (15), consider a point $(z_0, w_0) \in \mathbb{C}^2$, with $(z_0, w_0) \in \mathbb{C} \times \mathbb{C}^*$ if $n \ge 1$. If $w_0 \in W_i$, we have

$$\beta(z, w_0) = \beta_i(z, w_0) + g_i(w_0) = \beta_{w_0}(z) + g_i(w_0)$$

for all $z \in V_{w_0}$. By (16) there exist $z \in V_{w_0}$ and $j \in \mathbb{N}$ such that $[z_0 - g_i(w_0)] + j = \beta_{w_0}(z)$, thus

$$\Phi(z, w_0) = (\beta(z, w_0), w_0) = (z_0 + j, w_0)$$

and therefore $(z_0, w_0) \in W - (j, 0)$. □

7. *The flower theorem*

In this section we prove Theorem 1.2. We do this first in the case we have been dealing with in the previous sections: a biholomorphism F of the form (B) with $N \geq 0$ and satisfying condition ($\mathrm{B}^\star$). At the end of the section we deal with biholomorphisms of the form (A).

Let ε, θ be as in Proposition 5.2 and denote $U_k = U_k(\varepsilon, \theta)$. Let $\tilde{\delta} > 0$ be the constant given by assertion 2 of Proposition 3.3 and consider the sets $\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})$. Since $\gcd(m, n) = 1$, it is easy to see that these sets are connected. For each $k \in \{0, \dots, d-1\}$ we define

$$\Omega_k^+ = \bigcup_{j \geq 0} F^j(\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta})).$$

This set is connected and invariant by F. Moreover, in view of Proposition 3.3, we have that Ω_k^+ is in the basin of attraction of U_k and

$$\widetilde{D}_k(\varepsilon, \theta, \tilde{\delta}) \subset \Omega_k^+ \subset \widetilde{D}_k(\varepsilon, \theta, \delta_\theta).$$

It is easy to see that the diffeomorphism F^{-1} is also of the form (B), with the same pair (M, N) and $(-a, -b)$ instead of (a, b). Thus, if we work with F^{-1} instead of F, our constructions allow us to find connected open sets $\Omega_0^-, \dots, \Omega_{d-1}^-$, which play for F^{-1} the role of the sets Ω_k^+ in the case of F. We can assume that the constructions are done with the same constants $\varepsilon, \delta, \tilde{\delta}$, etc. Thus, the sets Ω_k^- are defined by

$$\Omega_k^- = \bigcup_{j \geq 0} F^{-j}(\widetilde{D}_k^-(\varepsilon, \theta, \tilde{\delta})),$$

where

$$\widetilde{D}_k^-(\varepsilon, \theta, \tilde{\delta}) = \{(x, y) \in \mathbb{C}^2 : x^m y^n \in \widetilde{S}_k^-(d, \varepsilon, \theta), |\iota x| < \tilde{\delta}, |y| < \tilde{\delta}\},$$

in which $\widetilde{S}_k^-(d, \varepsilon, \theta)$ is one of the connected components of

$$\{z \in \mathbb{C} : |z^d| < \varepsilon, |\arg(z^d) - \pi| < \theta\} \cup \left\{\left|z^d - \frac{\varepsilon}{2} e^{i(\pi - \theta)}\right| < \frac{\varepsilon}{2}\right\} \cup \left\{\left|z^d - \frac{\varepsilon}{2} e^{i(\pi + \theta)}\right| < \frac{\varepsilon}{2}\right\}.$$

In each Ω_k^-, we have that $F^{-j} \to 0$ and

$$\widetilde{D}_k^-(\varepsilon, \theta, \tilde{\delta}) \subset \Omega_k^- \subset \widetilde{D}_k^-(\varepsilon, \theta, \delta_\theta).$$

Since the opening of the sets $\widetilde{S}_k(d, \varepsilon, \theta)$ and $\widetilde{S}_k^-(d, \varepsilon, \theta)$ is greater than π/d, it is clear that the domains $\Omega_0^+, \dots, \Omega_{d-1}^+, \Omega_0^-, \dots, \Omega_{d-1}^-$, together with the fixed point set $\{x y^\iota = 0\}$, cover the open set

$$\{(x, y) \in \mathbb{C}^2 : |x^m y^n| < \varepsilon^{1/d}, |\iota x| < \tilde{\delta}, |y| < \tilde{\delta}\},$$

so assertions 1 and 2 of Theorem 1.2 are proved.

For each k, let $\varphi_k^+ : \phi_k^{-1}(V) \to W \subset \mathbb{C}^2$ be the biholomorphism given by Proposition 6.1, which conjugates F with the map $(z, w) \mapsto (z + 1, w)$. It is straightforward to extend φ_k^+ as a biholomorphism

$$\varphi_k^+ : \Omega_k^+ \to W_k^+ \subset \mathbb{C}^2,$$

with $W_k^+ \subset \mathbb{C} \times \mathbb{C}^*$ if $n \geq 1$, defining, for each $p \in \Omega_k^+$,

$$\varphi_k^+(p) = \varphi_k^+(F^j(p)) - (j, 0)$$

for any $j \geq 0$ such that $F^j(p) \in \phi_k^{-1}(V)$; this is possible because Ω_k^+ is in the basin of attraction of U_k and, by Proposition 5.2, U_k is in the basin of attraction of $\phi_k^{-1}(V)$. This shows assertion 3a of Theorem 1.2 for the domains $\Omega_0^+, \ldots, \Omega_{d-1}^+$; property 3b follows from (15). We can proceed analogously with the sets $\Omega_0^-, \ldots, \Omega_{d-1}^-$ and this finishes the proof of Theorem 1.2 for F of the form (B) with $N \geq 0$.

Suppose now that F is of the form (A) satisfying (A*). Up to a linear change of coordinates of the form $(x, y) \mapsto (\alpha x, y)$, we can assume that $a \in \mathbb{R}^-$. In this case, Hakim proved in [8] that if r is small enough then for any $k \in \{0, \ldots, M - 1\}$ there exists a holomorphic map $u_k : D_{r,k} \to \mathbb{C}$, with $|u_k(x)| \leq K|x \log x|$ for some $K > 0$ and for all $x \in D_{r,k}$, such that $u_k(F_1(x, u_k(x))) = F_2(x, u_k(x))$, where $D_{r,k}$ is the component of $\{x \in \mathbb{C} : |x^M - r| < r\}$ bisected by $e^{2\pi ik/M}\mathbb{R}^+$. Moreover, with the small modification of her proof introduced in [11, Lemma 4.4], we can enlarge the domain of definition of u_k to the set $\widetilde{S}_k(M, \varepsilon_0, \theta)$ for any $\theta \in (0, \pi/2)$ and for ε_0 small enough. Then, making the sectorial change of coordinates

$$(x, y) \in \widetilde{S}_k(M, \varepsilon_0, \theta) \times \mathbb{C} \mapsto (x, z^k) = (x, y - u_k(x)),$$

we can write

$$F(x, z^k) = (x + x^{M+1}[a + O_1(x, z^k)], z^k + x^M z^k[b + O_1(x, z^k)]),$$

where we use the notation $O_1(x, z^k) = O(x, x \log x, z^k)$. The key point to note is that all the constructions we made in the previous sections to obtain the invariant sets Ω_k^+ for a map F of the form (B) with $N = 0$ were performed in $\widetilde{S}_k(M, \varepsilon, \theta) \times \mathbb{C}$, and all the calculations involved work similarly if we have $O_1(x, y)$ instead of $O(x, y)$. Then, for some $\varepsilon \leq \varepsilon_0$, the domains Ω_k^+ can be defined in the same way, but in sectorial coordinates (x, z^k) depending on k, and the same holds for Ω_k^-. Assertions 1 and 3 of Theorem 1.2, since referred to a fixed $k \in \{0, \ldots, M - 1\}$, follow exactly as above if we work in the corresponding sectorial coordinates. For the proof of assertion 2, it is enough to show that each Ω_k^+ contains, in the original coordinates (x, y), a set of the form

$$D_k' = \{(x, y) \in \mathbb{C}^2 : x \in \widetilde{S}_k(M, \varepsilon', \theta), |y| < \delta'\}$$

for some $\varepsilon', \delta' > 0$, and the same for each Ω_k^-. By definition, Ω_k^+ contains the set

$$\widetilde{D}_k = \{(x, z^k) \in \mathbb{C}^2 : x \in \widetilde{S}_k(M, \varepsilon, \theta), \ |z^k| < \tilde{\delta}\}.$$

Take $\delta' \leq \tilde{\delta}/2$ and let $\varepsilon' \leq \varepsilon$ be such that $|u_k(x)| < \tilde{\delta}/2$ for all $x \in S_k(M, \varepsilon', \theta)$ and $k \in \{0, \ldots, d - 1\}$. Take $(x, y) \in D_k'$. In the sectorial coordinates this point is given by (x, z^k) with $z^k = y - u_k(x)$. Then $|z^k| \leq |y| + |u_k(x)| < \delta' + \tilde{\delta}/2 \leq \tilde{\delta}$, so $(x, z^k) \in \widetilde{D}_k$.

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This proves that Ω_k^+ contains D_k' and clearly the analogous property holds for each Ω_k^-, so Theorem 1.2 is proved.

8. *Proof of Theorem 1.3*

Suppose first that F is of the form (B). As in §3, up to a linear change of coordinates, we can write F as

$$F(x, y) = (x + x^{M+1}y^N[a + O(x, y)], y + x^M y^{N+1}[b + O(x, y)]),$$

with $M \geq 1$, $N \geq 1$ and $aM + bN = -1$, so the hypothesis of Theorem 1.3 means that either $\operatorname{Re} a > 0$ or $\operatorname{Re} b > 0$. We assume without loss of generality that $\operatorname{Re} b > 0$. Also as in §3, put $d = \gcd(M, N)$ and set $m = M/d$, $n = N/d$. Since $aM + bN = -1$, we have that

$$x_1^m y_1^n = x^m y^n - \frac{1}{d}(x^m y^n)^{d+1} + (x^m y^n)^{d+1} O(x, y).$$

Since $y_1 = y(1 + [b + O(x, y)](x^m y^n)^d)$ and $\operatorname{Re} b > 0$, with a similar argument to the one we used for equation (6) we find $\varepsilon, \delta, \nu > 0$ and $\theta \in (0, \pi/2)$ such that, for each $k \in \{0, \ldots, d-1\}$, if $x^m y^n \in S_k(d, \varepsilon, \theta)$ with $|x| < \delta$ and $|y| < \delta$ then

$$|y_1| \geq |y|(1 + \nu|x^m y^n|^d). \tag{18}$$

Take $\delta' = \min\{\delta, \tilde{\delta}_F(\theta)\}$, where $\tilde{\delta}_F(\theta)$ is given by Proposition 2.2, and set

$$\mathcal{U} = \{(x, y) \in \mathbb{C}^2 : |x| < \delta', |y| < \delta'\}.$$

Consider a point $(x, y) \in \mathcal{U}$ outside the fixed set $\{xy = 0\}$ and let us show that there exists $j \in \mathbb{N}$ such that $(x_j, y_j) \notin \mathcal{U}$. Assume by contradiction that $(x_j, y_j) \in \mathcal{U}$ for all j. Then, by assertion 3 of Proposition 2.2, we find $k \in \{0, \ldots, d-1\}$ and $j_0 \geq 0$ such that $x_j^m y_j^n \in S_k(d, \varepsilon, \theta)$ for all $j \geq j_0$. Thus, by iterated applications of (18) we obtain, for all $j \geq j_0$, that

$$|y_j| \geq |y_{j_0}| \prod_{l=j_0}^{j-1} (1 + \nu|x_l^m y_l^n|^d).$$

But the estimate

$$|x_j^m y_j^n|^d \geq \frac{1}{2} \frac{|x^m y^n|^d}{1 + j|x^m y^n|^d}$$

from assertion 3 of Proposition 2.2 shows that $\sum |x_l^m y_l^n|^d = +\infty$, so $|y_j| \to \infty$, which is a contradiction. In the same way, up to reducing δ', we can prove that the negative orbit of any $(x, y) \in \mathcal{U} \backslash \{xy = 0\}$ leaves $\mathcal{U}$.

Suppose now that F is of the form (A). Again as in §3, up to a linear change of coordinates, we have

$$F(x, y) = (x + x^{M+1}[-1/M + O(x, y)], y + x^M[by + O(x, y^2)]),$$

where $\operatorname{Re} b > 0$. As explained in the previous section, for some $\varepsilon_0 > 0$ there exists a holomorphic map $u_k : S_k(M, \varepsilon_0, \pi/4) \to \mathbb{C}$ such that in the sectorial coordinates $(x, z^k) = (x, y - u_k(x))$ defined in $S_k(M, \varepsilon_0, \pi/4) \times \mathbb{C}$ we can write

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$$F(x, z^k) = (x + x^{M+1}[-1/M + O_1(x, z^k)], z^k + x^M z^k[b + O_1(x, z^k)]).$$

Since $z_1^k = z^k(1 + [b + O_1(x, z^k)]x^M)$ and $\operatorname{Re} b > 0$, as in the previous case we can find constants $\varepsilon, \delta, \nu > 0$ and $\theta \in (0, \pi/4)$ such that, for each $k \in \{0, \ldots, M-1\}$, if $x \in S_k(M, \varepsilon, \theta)$ with $|x| < \delta$ and $|z^k| < \delta$ then

$$|z_1^k| \geq |z^k|(1 + \nu|x|^M). \tag{19}$$

Take $\delta_0 \leq \delta/2$ such that $|u_k(x)| < \delta/2$ for all $|x| < \delta_0$ and $k \in \{0, \ldots, M-1\}$. Notice that the equation

$$x_1 = x + x^{M+1}[-1/M + O(x, y)]$$

satisfies the hypothesis of Proposition 2.2 with $m = 1$, $n = 0$ and $d = M$, so there exists a constant $\tilde{\delta}_F(\theta)$ satisfying assertion 3. Set $\delta' = \min\{\delta_0, \tilde{\delta}_F(\theta)\}$ and

$$\mathcal{U} = \{(x, y) \in \mathbb{C}^2 : |x| < \delta', |y| < \delta'\}.$$

Consider a point $(x, y) \in \mathcal{U}$ outside the fixed set $\{x = 0\}$ and let us show that there exists $j \in \mathbb{N}$ such that $(x_j, y_j) \notin \mathcal{U}$. Assume by contradiction that $(x_j, y_j) \in \mathcal{U}$ for all j. Then, by assertion 3 of Proposition 2.2, we find $k \in \{0, \ldots, M-1\}$ and $j_0 \geq 0$ such that $x_j \in S_k(M, \varepsilon, \theta)$ for all $j \geq j_0$. Since $|x_j| < \delta' \leq \delta_0 \leq \delta$ for all $j \geq 0$, we have that

$$|z_j^k| \leq |y_j| + |u_k(x_j)| < \delta' + \delta/2 \leq \delta$$

for all $j \geq 0$. Thus, by iterated applications of (19) we obtain, for all $j \geq j_0$, that

$$|z_j^k| \geq |z_{j_0}^k| \prod_{l=j_0}^{j-1} (1 + \nu|x_l|^M).$$

But the estimate

$$|x_j|^M \geq \frac{1}{2} \frac{|x|^M}{1 + j|x|^M}$$

from assertion 3 of Proposition 2.2 shows that if $z_{j_0}^k \neq 0$ then $z_j^k \to +\infty$, which is a contradiction. Hence $z_{j_0}^k = 0$, which means that the orbit of (x, y) eventually lies in the parabolic curve $z^k = 0$ of F in the domain $S_k(M, \varepsilon, \pi/4) \times \mathbb{C}$. For each $k \in \{0, \ldots, M-1\}$, let $\mathcal{P}_k$ be the set of points in $\mathcal{U}$ attracted by $F|_{\mathcal{U}}$ into the parabolic curve of F in $S_k(M, \varepsilon, \pi/4) \times \mathbb{C}$; it is not difficult to see that $\mathcal{P}_k$ is a one-dimensional complex submanifold of $\mathcal{U}$, and we have shown that every point in $\mathcal{U}$ outside the fixed set $\{x = 0\}$ and outside the manifold $\mathcal{P}^+ = \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{M-1}$ has a finite positive orbit in $\mathcal{U}$. In the same way, up to reducing δ', if $\mathcal{P}_k^-$ is the one-dimensional complex submanifold of $\mathcal{U}$ of points attracted by $F^{-1}|_{\mathcal{U}}$ into the parabolic curve of F^{-1} in the domain $S_k^-(M, \varepsilon, \pi/4) \times \mathbb{C}$, then every point in $\mathcal{U}$ outside the fixed set $\{x = 0\}$ and outside the manifold $\mathcal{P}^- = \mathcal{P}_0^- \cup \cdots \cup \mathcal{P}_{M-1}^-$ has a finite negative orbit in $\mathcal{U}$. This concludes the proof of Theorem 1.3.

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A. *Appendix. Resolution theorem for biholomorphisms*

The resolution theorem for two-dimensional biholomorphisms stated in the introduction is valid not only for biholomorphisms tangent to the identity, but more generally for unipotent biholomorphisms, and is based on the corresponding theorem for vector fields and foliations in $\mathbb{C}^2$. A formal vector field X in $(\mathbb{C}^2, 0)$ can be written, in a unique way up to multiplication by a unit, as $X = f(A(\partial/\partial x) + B(\partial/\partial y))$, where $f, A, B \in \mathbb{C}[[x, y]]$ and A and B have no common factor. The vector field $\operatorname{Sat} X = A(\partial/\partial x) + B(\partial/\partial y)$ is called the saturation of X. If f is not a unit, we say that $\operatorname{Sing} X = \sqrt{(f)}$ is the singular locus of X; if f is a unit, we say that X is saturated and define $\operatorname{Sing} X = \{0\}$ if A and B are not units and $\operatorname{Sing} X = \emptyset$ otherwise. An irreducible formal curve (g) in $(\mathbb{C}^2, 0)$ is said to be a separatrix of a formal vector field X if $X(g) \in (g)$. If X is not singular, its formal integral curve through the origin is its only separatrix. The branches of the singular locus of X are separatrices, which are called fixed.

We say that a saturated singular vector field X in $(\mathbb{C}^2, 0)$ is reduced if the eigenvalues λ_1, λ_2 of its linear part satisfy $\lambda_1 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_{>0}$; if $\lambda_2 \neq 0$ we say that X is non-degenerate, otherwise X is called a saddle-node. A reduced vector field X has exactly two formal separatrices, which are non-singular and transverse, and each one is tangent to an eigenspace of the linear part of X. The resolution theorem for vector fields (see [4, 14]) asserts the following (throughout this section, if $\mathfrak{g}$ is any analytic or formal object, we denote by $\mathfrak{g}_p$ its germ at the point p).

THEOREM A.1. *Let X be a singular formal vector field in* $(\mathbb{C}^2, 0)$. *There exist a finite composition of blow-ups $\pi : (M, E) \to (\mathbb{C}^2, 0)$, a formal vector field $\tilde{X}$ along E with $\pi_*\tilde{X} = X$, and finitely many points $p_1, \ldots, p_k \in E$ such that* $\operatorname{Sat} \tilde{X}_{p_1}, \ldots, \operatorname{Sat} \tilde{X}_{p_k}$ *are reduced and* $\operatorname{Sat} \tilde{X}_p$ *is not singular for any $p \in E \backslash \{p_1, \ldots, p_k\}$.*

The set E, called the exceptional divisor, is a finite union of smooth rational curves with normal crossings; we say that a point in E is a corner if it is the intersection of two components of D. Up to composing π with some additional blow-ups, we can assume that the family of separatrices of X, even if it is infinite, is desingularized by π:

(a) the strict transform of each separatrix is a non-singular curve at a non-corner point of E and is transverse to E;

(b) the strict transforms of different separatrices are curves at different points of E.

Any map π as above is called a resolution of X and we also say that $\tilde{X}$ is a resolution of X. Any further blow-up at a singular point of $\tilde{X}$ in E gives another resolution of X. As may be expected, there exists a unique minimal resolution of X in the sense that any other resolution is obtained from the minimal one by performing finitely many additional blow-ups. If X is a nilpotent vector field in $(\mathbb{C}^2, 0)$ (that is, its linear part is nilpotent) and $\tilde{X}$ is a resolution of X, then $\tilde{X}_p$ is nilpotent for any point $p \in E$. In particular, if X is nilpotent then the singular points of $\tilde{X}$ in E are not isolated: an isolated singularity p of $\tilde{X}$ would be

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saturated and reduced, so $\tilde{X}_p$ would not be nilpotent. Therefore the set $\mathrm{Sing}_E\,\tilde{X}$ of singular points of $\tilde{X}$ in E is a union of components of E. We define the singular locus $\mathrm{Sing}\,\tilde{X}$ of $\tilde{X}$ as the union of $\mathrm{Sing}_E\,\tilde{X}$ with the strict transform of $\mathrm{Sing}\,X$ by π. It is easy to see that $(\mathrm{Sing}\,\tilde{X})_p = \mathrm{Sing}\,\tilde{X}_p$ for all $p \in E$. Observe that $E \cup \mathrm{Sing}\,\tilde{X}$ is the transform of $\mathrm{Sing}\,X$, so it is a finite union of smooth curves with normal crossings, and so is $\mathrm{Sing}\,\tilde{X}$.

LEMMA A.2. *Let $\tilde{X}$ be a resolution of X and let $p \in E$ such that* $\mathrm{Sat}\,\tilde{X}_p$ *is singular (hence a reduced singularity). Then each branch of $(E \cup \mathrm{Sing}\,\tilde{X})_p$ is one of the two separatrices of* $\mathrm{Sat}\,\tilde{X}_p$.

Proof. Let S_1 and S_2 be the separatrices of $\mathrm{Sat}\,\tilde{X}_p$. If both S_1 and S_2 were not contained in E, they would be the strict transforms of different separatrices of X passing through the same point in E, contradicting (a), so we can assume that S_1 is contained in a component of E. Suppose that there is a branch S of $(E \cup \mathrm{Sing}\,\tilde{X})_p$ different from S_1 and S_2. If S is a component of E then p is a corner and S_2 is the strict transform of a separatrix of X passing through p, which contradicts (a). If S is a branch of $\mathrm{Sing}\,\tilde{X}_p$ not contained in E then S is the strict transform of a separatrix of X, so by (a) p is not a corner and then S_2 is also the strict transform of a separatrix of X passing through p, contradicting (b). □

If $\tilde{X}$ is a resolution of X, we classify the components of the exceptional divisor into two types.

(1) A component D of E is invariant if for some point $p \in D$ the germ D_p is a separatrix of $\mathrm{Sat}\,\tilde{X}_p$. In this case the same happens for any other point in D.
(2) If a component D of E is not invariant, we say that it is dicritical. In this case $D \subset \mathrm{Sing}\,\tilde{X}$ and, as we will see next, the vector field $\mathrm{Sat}\,\tilde{X}_p$ is non-singular and transverse to D for all $p \in D$, and any other component of E intersecting D is invariant.

Let us show the assertions in (2). By Lemma A.2, if $\mathrm{Sat}\,\tilde{X}_p$ were singular for some $p \in D$ then D would be invariant, so $\mathrm{Sat}\,\tilde{X}_p$ is not singular and its formal integral curve C is different from D_p because D is not invariant. If p is not a corner, then C is the strict transform of a separatrix of X, so C is transverse to D; if p is a corner, from (a) we conclude that $C = D'_p$, where D' is the other component of E through p and therefore $\mathrm{Sat}\,\tilde{X}_p$ is transverse to D. This also shows that any component D' of E intersecting D is invariant.

Consider now a unipotent biholomorphism F, that is, $DF(0) = I + N$ where I is the identity and N is nilpotent. In a formal sense, F is the time-one flow of a unique formal vector field in $(\mathbb{C}^2, 0)$, denoted $\log F$, which is singular at the origin and has N as linear part. In particular, if F is tangent to the identity then $\log F$ has order at least two. Moreover, the fixed point set of F coincides with the singular locus of $\log F$, which is therefore convergent. If π is the blow-up at the origin, the map $\tilde{F} = \pi^{-1} \circ F \circ \pi$ is a biholomorphism in a neighborhood of $E = \pi^{-1}(0)$ which leaves E invariant, and satisfies $\log \tilde{F}_p = \tilde{X}_p$ for any fixed point $p \in E$ of $\tilde{F}$.

THEOREM A.3. *Let F be a unipotent biholomorphism in $(\mathbb{C}^2, 0)$, let π be a resolution of $\log F$ and let $\tilde{F} = \pi^{-1} \circ F \circ \pi$ be the transform of F by π. Then, if $p \in E = \pi^{-1}(0)$ is a*

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fixed point of $\tilde{F}$, the germ $\tilde{F}_p$ is reduced, according to Definition 1.1. If F is tangent to the identity and π is the minimal resolution of $\log F$*, then E is fixed pointwise by $\tilde{F}$.*

Proof. Let $\tilde{X}$ be the transform of $X = \log F$ by π and let $p \in E$ be a fixed point of $\tilde{F}$, so $p \in \operatorname{Sing} \tilde{X}$.

Suppose first that p belongs to a dicritical component D of E. Then $D \subset \operatorname{Sing} \tilde{X}_p$ and $\operatorname{Sat} \tilde{X}_p$ is non-singular and transverse to D. Since $(D \cup \operatorname{Sing} \tilde{X})_p$ is smooth or has two smooth transverse branches, we can take holomorphic coordinates (x, y) at p such that $D = \{x = 0\}$ and such that $\{y = 0\}$ is the other branch of $\operatorname{Sing} \tilde{X}_p$ if it exists, so $\operatorname{Sing} \tilde{X}_p \subset \{xy = 0\}$. Then, up to rescaling the coordinates, we have

$$\tilde{X}_p = x^M y^N \bigg[(1 + \tilde{A}(x, y)) \frac{\partial}{\partial x} + \tilde{B}(x, y) \frac{\partial}{\partial y} \bigg],$$

where $\operatorname{ord} \tilde{A} \geq 1$, $M \geq 1$, $N \geq 0$ and $(M, N) \neq (1, 0)$ because $\tilde{X}_p$ is nilpotent. Therefore its time-one flow $\tilde{F}_p$ will be of the form (i) once we show that $\tilde{B} \in (y)$ if $N \geq 1$. Suppose that $N \geq 1$, so $\{y = 0\} \subset \operatorname{Sing} \tilde{X}_p$, and let C be the formal integral curve of $(1 + \tilde{A})(\partial/\partial x) + \tilde{B}(\partial/\partial y)$ through the origin. If $\{y = 0\}$ is a component of E then p is a corner and, in view of (a), necessarily $C = \{y = 0\}$ and therefore $\tilde{B} \in (y)$. If $\{y = 0\}$ is not a component of E, then it is the strict transform of a (fixed) separatrix of X so, in view of (b), it has to coincide with C and again $\tilde{B} \in (y)$.

Suppose now that p does not belong to a dicritical component of E. We assume first that $\operatorname{Sat} \tilde{X}_p$ is not singular. Take a component D of E such that $p \in D$. As in the previous case, we have holomorphic coordinates (x, y) at p such that $D = \{y = 0\}$ and $\operatorname{Sing} \tilde{X}_p \subset \{xy = 0\}$. Since D is invariant, $\{y = 0\}$ is the formal integral curve of $\operatorname{Sat} \tilde{X}_p$ through p, so we can write $\operatorname{Sat} \tilde{X}_p = (1 + \hat{A})(\partial/\partial x) + \hat{B}(\partial/\partial y)$ with $\hat{B} \in (y)$. Then, up to rescaling the coordinates, we have

$$\tilde{X}_p = x^M y^N \bigg[(1 + \tilde{A}) \frac{\partial}{\partial x} + \tilde{B} \frac{\partial}{\partial y} \bigg],$$

where $M, N \geq 0$ and $M, N \notin \{(0, 0), (1, 0)\}$ because $\tilde{X}_p$ is singular and nilpotent, so $\tilde{F}_p$ is of the form (i).

Assume now that $\operatorname{Sat} \tilde{X}_p$ is singular, hence reduced. We suppose first that it is a saddle-node. Let S be the separatrix of $\operatorname{Sat} \tilde{X}_p$ that is tangent to the eigenspace associated to the non-zero eigenvalue of the linear part of $\operatorname{Sat} \tilde{X}_p$, and let S_0 be the other separatrix. Let (x, y) be holomorphic coordinates at p such that $\{x = 0\}$ and $\{y = 0\}$ are respectively tangent to S_0 and S, so we can write $\operatorname{Sat} \tilde{X}_p = (x + \hat{A})(\partial/\partial x) + \hat{B}(\partial/\partial y)$, where $\operatorname{ord} \hat{A}, \operatorname{ord} \hat{B} \geq 2$ and $x + \hat{A}$ and $\hat{B}$ have no common factors. We know that $\operatorname{Sing} \tilde{X}_p$ has one or two branches, which by Lemma A.2 are contained in $\{S, S_0\}$. Thus, we can assume that the coordinates are chosen in such a way that if $S_0 \subset \operatorname{Sing} \tilde{X}_p$ then $S_0 = \{x = 0\}$, so $\hat{A} \in (x)$, and if $S \subset \operatorname{Sing} \tilde{X}_p$ then $S = \{y = 0\}$, so $\hat{B} \in (y)$; hence $\operatorname{Sing} \tilde{X}_p \subset \{xy = 0\}$. Thus, up to rescaling the coordinates, we have

$$\tilde{X}_p = x^M y^N \bigg[(x + \tilde{A}) \frac{\partial}{\partial x} + \tilde{B} \frac{\partial}{\partial y} \bigg],$$

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where $M+N\geq 1$, ord $\tilde{A}$, ord $\tilde{B}\geq 2$, $\tilde{A}\in(x)$ if $M\geq 1$, $\tilde{B}\in(y)$ if $N\geq 1$ and $x+\tilde{A}$ and $\tilde{B}$ have no common factors. Hence, $\tilde{F}_p$ is of the form (iii).

Finally, suppose that Sat $\tilde{X}_p$ is non-degenerate. Since Sing $\tilde{X}_p$ contains at least one branch, in suitable coordinates (x, y) we have $\{x=0\}\subset$ Sing $\tilde{X}_p$ so, by Lemma A.2, the curve $\{x=0\}$ is a separatrix of Sat $\tilde{X}_p$. As in the previous case, we can assume that the other separatrix S of Sat $\tilde{X}_p$ is tangent to $\{y=0\}$ and that $S=\{y=0\}\subset$ Sing $\tilde{X}_p$ if Sing $\tilde{X}_p$ has two branches, so Sing $\tilde{X}_p\subset\{xy=0\}$. Therefore we have

$$\tilde{X}_p=x^My^N\bigg[(ax+\tilde{A}x)\frac{\partial}{\partial x}+(by+\tilde{B})\frac{\partial}{\partial y}\bigg],$$

where $M\geq 1$, $N\geq 0$, $a, b\in\mathbb{C}^*$, $a/b\notin\mathbb{Q}_{>0}$, ord $\tilde{A}\geq 1$, ord $\tilde{B}\geq 2$ and $\tilde{B}\in(y)$ if $N\geq 1$, so $\tilde{F}_p$ is of the form (ii).

In order to prove the last assertion of the theorem it suffices to show that if a vector field X has order at least two and π is its minimal resolution, then $\pi^{-1}(0)\subset$ Sing $\tilde{X}$. Suppose that this property holds when the minimal resolution of X is achieved with fewer than $n\in\mathbb{N}$ blow-ups, and let X be a formal vector field with ord $X\geq 2$ whose minimal resolution is obtained with n blow-ups. Let σ be the blow-up at the origin and let $\hat{X}$ be the transform of X by σ. Since ord $X\geq 2$, we have that $\hat{X}$ vanishes on $D=\sigma^{-1}(0)$ with order $\nu\geq 1$ if D is invariant or $\nu\geq 2$ if D is dicritical. So D will be in the singular locus of the resolution of X and, in view of the inductive hypothesis, it is enough to show that $\hat{X}$ has order at least two at each point in D that is blown up in the resolution. Let $p\in D$ be one such point. Since ord $\hat{X}_p\geq\nu$, it suffices to consider the case where D is invariant and $\nu=1$. We can also assume that Sing $\hat{X}_p=D_p$ and that $\hat{X}_p$ vanishes on D_p with multiplicity one, because otherwise ord $\hat{X}_p\geq 2$. Then, if (x, y) are holomorphic coordinates at p such that $D_p=\{y=0\}$, we have

$$\hat{X}_p=y\bigg[(a+\tilde{A})\frac{\partial}{\partial x}+(b+\tilde{B})\frac{\partial}{\partial y}\bigg],$$

where ord $\tilde{A}$, ord $\tilde{B}\geq 1$, and necessarily $b=0$ because $\hat{X}_p$ is nilpotent. If $a\neq 0$, we see that $\hat{X}_p$ is actually in final form, so no further blow-up at p would be necessary. Therefore $a=0$ and ord $\hat{X}_p\geq 2$. □

Remark A.4. The reduced models (i), (ii) and (iii) of Definition 1.1 correspond to the standard final models for vector fields, and are not exactly the same ones that appear in the resolution theorem in [1]. Our set of reduced fixed points is stable under blow-ups, in the sense that any further blow-up will produce only reduced fixed points (for example, the dicritical fixed points considered as final models in [1] can be reduced by additional blow-ups to non-dicritical models). In our final models, the blow-up of a fixed point of the form (i) produces fixed points of the form (i), the blow-up of a fixed point of the form (ii) produces fixed points of the form (i) and (ii), and the blow-up of a fixed point of the form (iii) produces fixed points of the form (i), (ii) and (iii).

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