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ON NORTHCOTT-REES THEOREM ON PRINCIPAL SYSTEMS

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§1. Introduction

Let R be a local ring with maximal ideal $\mathfrak{m}$ and let us make the following definition according to the paper [NR] of Northcott and Rees, which is essentially due to F. S. Macaulay.

DEFINITION. A proper ideal $\mathfrak{a}$ of R is said to be a principal system if, for any integer N, there exists an irreducible $\mathfrak{m}$-primary ideal $\mathfrak{q}$ satisfying $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{m}^N + \mathfrak{a}$.

It will be worth noting that this definition is equivalent to saying that the ring $R/\mathfrak{a}$ is approximately Gorenstein, in the terminology of M. Hochster in [H].

In their paper [NR] Northcott and Rees obtained some fundamental properties of principal systems and proved the following

THEOREM [NR; Theorem 6]. *Let R be a homomorphic image of a Gorenstein local ring. Then every ideal of R is the intersection of a finite number of principal systems.* (In [NR] it is assumed that R is a homomorphic image of a *regular* local ring. However one can easily see that the same proof as in [NR] works successfully also in the case of a homomorphic image of a *Gorenstein* ring.)

The aim of this paper is to determine a perfect condition for rings to satisfy the conclusion of Northcott-Rees theorem. Our main result is;

THEOREM. *The following conditions are equivalent for a local ring R.*

(1) *Every ideal of R is an intersection of a finite number of principal systems.*

(2) *Every irreducible ideal of R is a principal system.*

(3) *Every prime ideal of R is a principal system.*

(4) *If $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{Q} \in \operatorname{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$ such that $\dim(\hat{R}/\mathfrak{Q}) = 1$, then*

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$$\dim_{k(\mathfrak{Q})}\mathrm{Hom}_{\hat{R}_{\mathfrak{Q}}}(k(\mathfrak{Q}), (\hat{R}/\mathfrak{p}\hat{R})_{\mathfrak{Q}}) = 1$$

where $k(\mathfrak{Q})$ *denotes* $(\hat{R}/\mathfrak{Q})_{\mathfrak{Q}}$.

Indeed the equivalence of (1) and (2), and the implication from (2) to (3) are trivial. Other implications in Theorem will be discussed in section 2. ((3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2) will be obtained from Proposition 4 and 5 respectively.)

By our theorem one can see that Northcott-Rees theorem holds in acceptable rings. (For the definition of acceptable rings, see [S].) In fact they satisfy the condition (4) in Theorem. But there exists an example of one-dimensional local domain in which Northcott-Rees theorem *does not* hold. We also have an example of two dimensional local domain which is not acceptable but in which Northcott-Rees theorem holds. Those examples are constructed in [FR] and we have only to check the condition (4) of Theorem for them. See section 3 for more discussion.

Throughout this paper all rings will be commutative Noetherian rings with identity.

§ 2. Basic results on principal systems

The name of "principal system" is probably derived from the following

PROPOSITION 1. *For a local ring* $(R, \mathfrak{m}, k)$, *the following are equivalent.*

(1) *R is an approximately Gorenstein ring.*

(2) *There exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *of elements in* $E_R(k)$, *such that* $Rx_1 \subset Rx_2 \subset \cdots \subset Rx_n \subset Rx_{n+1} \subset \cdots \subset E_R(k)$ *and* $\bigcup_{n=1}^{\infty} Rx_n = E_R(k)$, *where* $E_R(k)$ *denotes the injective envelope of the residue field k of R.*

(3) *For any* $x, y \in E_R(k)$, *there exists an element z of* $E_R(k)$ *such that* $Rz \supset Rx + Ry$.

Proof. (1) $\Rightarrow$ (2): If R is approximately Gorenstein, then by the definition there is a set $\{\mathfrak{q}_n\}_{n=1}^{\infty}$ of irreducible $\mathfrak{m}$-primary ideals satisfying $\mathfrak{q}_n \subset \mathfrak{m}^n$ and $\mathfrak{q}_{n+1} \subset \mathfrak{q}_n$ for every n. Since $R/\mathfrak{q}_n$ is a Gorenstein ring of dimension 0, the R-module $E_{(R/\mathfrak{q}_n)}(k) \cong [0: \mathfrak{q}_n]_{E_R(k)}$ is generated by one element, say $[0: \mathfrak{q}_n]_{E_R(k)} = Rx_n$ $(n = 1, 2, 3, \cdots)$. Then it follows that $Rx_n \subset Rx_{n+1}$. Furthermore we have $Rx_n \supset [0: \mathfrak{m}^n]_{E_R(k)}$ for every n. Hence $E_R(k) = \bigcup_{n=1}^{\infty} [0: \mathfrak{m}^n]_{E_R(k)} = \bigcup_{n=1}^{\infty} Rx_n$.

(2) $\Rightarrow$ (1): If we denote $\mathfrak{q}_n = [0: x_n]_R$, then we have $R/\mathfrak{q}_n \cong Rx_n \subset E_R(k)$. Thus every ideal $\mathfrak{q}_n$ is $\mathfrak{m}$-primary irreducible, for the submodule (0) of $E_R(k)$ is irreducible. It remains to prove that for each n there exists an

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integer N such that $\mathfrak{q}_N \subset \mathfrak{m}^n$. Since $[0:\mathfrak{m}^n]_{E_R(k)}$ is an R-module of finite length and $E_R(k) = \bigcup_{i=1}^{\infty} Rx_i$, we have $[0:\mathfrak{m}_n]_{E_R(k)} \subset Rx_N$ for sufficiently large N. Then $\mathfrak{q}_N \subset [0:[0:\mathfrak{m}^n]_{E_R(k)}]_R = \mathfrak{m}^n$.

The equivalence of (2) and (3) will be immediate if one notices that $E_R(k)$ is countably generated over R. (See [M: Theorem 3.11].)

A finitely generated module K over a local ring $(R, \mathfrak{m}, k)$ is said to be a canonical module of R if there is an isomorphism;

$$K \otimes_R \hat{R} = \mathrm{Hom}_{\hat{R}}(H^d_{\mathfrak{m}}(R), E(k))$$

where $d = \dim(R)$. Such a module K, if it exists, is uniquely determined by R up to isomorphism. We refer the reader to [A] and [HK] for further information and details.

PROPOSITION 2. *If R is a local ring which possesses the canonical module K, then $[0:x]_R$ is a principal system for any $x \in K$.*

Proof. We may assume that R is complete. If we denote $I = [0:x]_R$ for fixed $x \in K$, we have an injective map $f: R/I \to K$ by $f(1) = x$. Applying $\mathrm{Hom}_R(\ , E_R(k))$ to f, we get a *surjective* homomorphism $g: H^d_{\mathfrak{m}}(R) \to \mathrm{Hom}_R(R/I, E_R(k))$ where one should notice that $\mathrm{Hom}_R(R/I, E_R(k)) \simeq E_{(R/I)}(k)$.

On the other hand it is known that local cohomology modules are obtained by taking cohomology of Ceck complex, that is, if $\{a_1, a_2, \cdots, a_d\}$ is a system of parameters of R, then $H^i_{\mathfrak{m}}(R)$ is an i-th cohomology module of the following complex;

$$0 \longrightarrow R \longrightarrow \prod_{i=1}^{d} R_{a_i} \longrightarrow \cdots \longrightarrow \prod_{j=1}^{d} R_{a_1 \cdots \hat{a}_j \cdots a_d} \longrightarrow R_{(a_1 a_2 \cdots a_d)} \longrightarrow 0\,.$$

In particular there is a *surjective* homomorphism $h: R_y \to H^d_{\mathfrak{m}}(R)$ where $y = a_1 a_2 \cdots a_d$. If we consider the composition map $g \cdot h$, we also have a surjection of R_y to $E_{(R/I)}(k)$. Let us denote $x_n = g \cdot h(1/y^n) \in E_{(R/I)}(k)$ for $n = 1, 2, 3, \cdots$. Since $R(1/y) \subset R(1/y^2) \subset \cdots \subset R(1/y^n) \subset \cdots \subset R_y$ and $\bigcup_{n=1}^{\infty} R(1/y^n) = R_y$, we also have $(R/I)x_1 \subset (R/I)x_2 \subset \cdots \subset (R/I)x_n \subset \cdots \subset E_{(R/I)}(k)$ and $\bigcup_{n=1}^{\infty} (R/I)x_n = E_{(R/I)}(k)$. Hence by Proposition 1 we see that I is a principal system.

COROLLARY. *Let R be a local ring possessing the canonical module K. If R is unmixed and generically Gorenstein, then R is approximately Gorenstein.*

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Proof. Since R is unmixed, we see that $\mathrm{Supp}_R(K) = \mathrm{Spec}(R)$. (See [A; (1.7)]. Thus [A; Corollary 4.3] shows that $K_{\mathfrak{p}}$ is a canonical module of $R_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$. Then by the assumption one sees that $S^{-1}K \simeq S^{-1}R$ where S is the set of all non-zero divisors of R. Thus one can find an element x of K satisfying $[0: x]_R = (0)$. Hence R is approximately Gorenstein by Proposition 2.

Next we would like to clarify the approximate Gorensteinness in one dimensional case. The following lemma will be useful for this purpose.

LEMMA 1. *Let $(R, \mathfrak{m}, k)$ be a local integral domain of dimension* 1 *which possesses a dualizing complex, and let T be a finitely generated torsion-free R-module. If there exist an irreducible submodule J of T and an element $x(\neq 0)$ of R such that*

$$xT \supset J \supset x^n T$$

for some n, then rank$(T) = 1$.

Proof. Since R has a dualizing complex, it also has a fundamental one (cf. [S_2]), thus there is an exact sequence;

$$(*) \qquad 0 \longrightarrow K \longrightarrow Q(R) \longrightarrow E_R(k) \longrightarrow 0$$

where $Q(R)$ is the field of fractions of R and the finitely generated R-module K is nothing but the canonical module of R. By the assumption T/J is an R-module of finite length in which the zero submodule is irreducible. Hence it can be embedded into $E_R(k)$, and we have the following diagram;

$$\begin{array}{ccccccccc} & & & & T & \longrightarrow & T/J & \longrightarrow & 0 \\ & & & & & \searrow^{g} & \downarrow \cap & & \\ 0 & \longrightarrow & K & \longrightarrow & Q(R) & \longrightarrow & E_R(k) & \longrightarrow & 0 \end{array}$$

Applying $\mathrm{Hom}_R(T, \ \ )$ to the sequence (*), we obtain the exact sequence;

$$\mathrm{Hom}_R(T, Q(R)) \longrightarrow \mathrm{Hom}_R(T, E_R(k)) \longrightarrow \mathrm{Ext}^1_R(T, K),$$

where $\mathrm{Ext}^1_R(T, K)$ should vanish by the local duality theorem, for T is a Cohen-Macaulay module over a Cohen-Macaulay ring R. Therefore we can see that there exists $f \in \mathrm{Hom}_R(T, Q(R))$ which lifts g. Thus we have a commutative diagram;

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$$\begin{array}{ccc} T & \longrightarrow & T/J \\ f\downarrow & & \downarrow\cap \\ Q(R) & \longrightarrow & E_R(k) \end{array}$$

To prove the lemma, it suffices to see that f is injective. For this purpose let us denote $\mathrm{Ker}(f)$ by S. Then by the above diagram $S \subset J \subset xT$. We claim that $S \subset xS$. In fact if $s \in S$, then s can be written as a product $x \cdot t$ for some $t \in T$. Then $xf(t) = f(xt) = f(s) = 0$, from which we have $f(t) = 0$ since x is a unit in $Q(R)$. Thus $t \in S$, hence $s = x \cdot t \in xS$. Therefore we get $S \subset xS$. Then Nakayama's lemma shows that $S = 0$ as required.

PROPOSITION 3. *Let R be a Cohen-Macaulay local ring of dimension 1 which has a dualizing complex. Then the following conditions are equivalent.*

(1) *R is a generically Gorenstein ring.*

(2) *R is approximately Gorenstein.*

(3) *There exist an $\mathfrak{m}$-primary irreducible ideal I and an element x of $\mathfrak{m}$ satisfying $I \subset xR$.*

Proof. (1) $\Rightarrow$ (2) is already shown in Corollary to Proposition 2. It is also proved by M. Hochster in [H; (4.8b)].

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1); Let $\mathfrak{p}$ be an arbitrary element of $\mathrm{Ass}(R)$ and put $T = [0 : \mathfrak{p}]_R$. Then T is a torsion-free $\bar{R}$-module, where $\bar{R} = R/\mathfrak{p}$. (In fact if T has torsion over $\bar{R}$, then there exists $z (\neq 0) \in T \subset R$ such that $\mathfrak{m}^N z = 0$ for large N. It therefore contradicts the fact $\mathrm{depth}(R) = 1$.) Moreover if we denote $J = I \cap T$, then J is an irreducible $\bar{R}$-submodule of T and $x^n T \subset J$ for large n since there is an injection of T/J into R/I. On the other hand we can see the equality $xR \cap T = xT$. In fact if $x \cdot r \in xR \cap T$ $(r \in R)$ then $\mathfrak{p}xr = 0$, hence $\mathfrak{p}r = 0$ since x is not a zero divisor on R. Thus $r \in T$, and we have $x \cdot r \in xT$. In particular one sees that $J \subset xT$. Applying Lemma 1 to the $\bar{R}$-module T we know that T has rank 1 over $\bar{R}$, equivalently $T_{\mathfrak{p}} = \mathrm{Hom}_R(R/\mathfrak{p}, R)_{\mathfrak{p}} \simeq k(\mathfrak{p})$. This implies the Gorensteinness of $R_{\mathfrak{p}}$. (Q.E.D.)

COROLLARY 1. *Let R be a Cohen-Macaulay local ring of dimension 1 which may not have a dualizing complex. Then R is approximately Gorenstein if and only if $\hat{R}$ is generically Gorenstein.*

For the proof of this corollary we have only to notice that R is approximately Gorenstein if and only if $\hat{R}$ is, and apply Proposition 1 to $\hat{R}$.

COROLLARY 2. *Let* $(R, \mathfrak{m})$ *be a local ring and assume that, for every prime ideal* $\mathfrak{p}$ *of coheight* 1, *the natural ring homomorphism of* $R/\mathfrak{p}$ *to* $(R/\mathfrak{p})^\wedge$ *is a Gorenstein homomorphism. If* $\{\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n\}$ *is a set of prime ideals of coheight* 1 *such that* $\mathfrak{p}_i \neq \mathfrak{p}_j$ $(i \neq j)$, *and* $\mathfrak{a}_i$ *is a* $\mathfrak{p}_i$*-primary irreducible ideal* $(i = 1, 2, \cdots, n)$, *then the ideal* $\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n$ *is a principal system.*

Proof. We may assume $\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n = (0)$, i.e. R can be supposed to be a Cohen-Macaulay ring of dimension 1. Then we see that R is generically Gorenstein if and only if $\hat{R}$ is generically Gorenstein by Lemma 2 below. Thus the conclusion is obtained from Corollary 1. (Notice that a $\mathfrak{p}$-primary ideal $\mathfrak{a}$ is irreducible if and only if $\dim_{k(\mathfrak{p})} \mathrm{Hom}_R (R/\mathfrak{p}, R/\mathfrak{a})_\mathfrak{p} = 1$.)

LEMMA 2. *Let* $(R, \mathfrak{m})$ *be a local ring and assume the following.*

(1) *For every* $\mathfrak{p} \in \mathrm{Ass}(R)$, $\dim_{k(\mathfrak{p})} \mathrm{Hom}_R(R/\mathfrak{p}, R)_\mathfrak{p} = 1$.

(2) *For every* $\mathfrak{p} \in \mathrm{Ass}(R)$ *and* $\mathfrak{P} \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ *satisfying* $\dim(\hat{R}/\mathfrak{P}) = 1$,

$$\dim_{k(\mathfrak{P})} \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{P}, \hat{R}/\mathfrak{p}\hat{R})_\mathfrak{P} = 1 .$$

Then for every $\mathfrak{Q} \in \mathrm{Ass}(\hat{R})$ *satisfying* $\dim(\hat{R}/\mathfrak{Q}) = 1$, *we have an equality;*

$$\dim_{k(\mathfrak{Q})} \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R})_\mathfrak{Q} = 1 .$$

In particular if $\dim(R) = 1$, *then* $\hat{R}$ *is a generically Gorenstein ring.*

Proof. Let $\mathfrak{Q}$ be a prime ideal associated to $\hat{R}$ and $\dim(\hat{R}/\mathfrak{Q}) = 1$. Then we see that $\mathfrak{Q} \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ for some $\mathfrak{p} \in \mathrm{Ass}(R)$ since $\mathrm{Ass}(\hat{R}) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$. By $\mathfrak{Q} \supset \mathfrak{p}\hat{R}$ we have an injection;

$$\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R}) \hookrightarrow \mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R}, \hat{R}) \simeq \mathrm{Hom}_R(R/\mathfrak{p}, R) \otimes_R \hat{R} .$$

If we localize these $\hat{R}$-modules by $\mathfrak{Q}$, we get

$$\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R})_\mathfrak{Q} \hookrightarrow \mathrm{Hom}_R(R/\mathfrak{p}, R)_\mathfrak{p} \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{Q} \simeq k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\mathfrak{Q} \simeq (\hat{R}/\mathfrak{p}\hat{R})_\mathfrak{Q}$$

by the first assumption in the lemma. Thus the $\hat{R}$-module $\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R})_\mathfrak{Q}$ can be embedded into $\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R}/\mathfrak{p}\hat{R})_\mathfrak{Q} \simeq k(\mathfrak{Q})$, and hence it completes the proof.

PROPOSITION 4. *If a prime ideal* $\mathfrak{p}$ *of* R *is a principal system and*

$\mathfrak{Q} \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ *such that* $\dim(\hat{R}/\mathfrak{Q}) = 1$, *then*

$$\dim_{k(\mathfrak{Q})}\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R}/\mathfrak{p}\hat{R})_{\mathfrak{Q}} = 1 .$$

Proof. We may assume that $\mathfrak{p} = (0)$, i.e. R may be supposed to be an approximately Gorenstein domain. Let us denote $\bar{R} = \hat{R}/\mathfrak{Q}$ and $T = [0:\mathfrak{Q}]_{\hat{R}}$. Then T is a torsion-free module over a one-dimensional local domain $\bar{R}$ as in the proof of Proposition 3. For every integer N, there exists an $\mathfrak{m}$-primary irreducible ideal I_N of $\hat{R}$ such that $I_N \subset \mathfrak{m}^N\hat{R}$. If we denote $J_N = I_N \cap T$ and if we take arbitrary $x \in \mathfrak{m}\hat{R}\backslash\mathfrak{Q}$, then J_N is irreducible in T and T/J_N is of finite length over $\bar{R}$ for every N. Moreover, if N is sufficiently large then $\mathfrak{m}^N\hat{R} \cap T \subset xT$ by Artin-Rees lemma, therefore J_N is contained in xT. Thus Lemma 1 shows that the rank of T over $\bar{R}$ is one, as required.

By virtue of Lemma 2, in order to prove the implication from (4) to (2) in Theorem, it suffices to verify the following

PROPOSITION 5. *Let R be a local ring which has a dualizing complex and* $\mathrm{depth}(R) > 0$, *and assume that for every* $\mathfrak{p} \in \mathrm{Ass}(R)$ *of* $\dim(R/\mathfrak{p}) = 1$, $\dim_{k(\mathfrak{p})}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p}) = 1$. *Then R is approximately Gorenstein.*

Remark. This result is contained in a theorem of M. Hochster in [H; (1.6)]. But for the completeness of this paper we shall show a brief proof below, using our proposition 1.

Proof. We proceed by induction on $\dim(R)$. If $\dim(R) = 1$, then R is Cohen-Macaulay and generically Gorenstein by the hypothesis. The consequence is hence obtained from Corollary to Proposition 2.

Assume that $\dim(R) \geqq 2$. Notice that one can assume R is a *complete* local ring. In fact, for any $\mathfrak{Q} \in \mathrm{Ass}(\hat{R})$ such that $\dim(\hat{R}/\mathfrak{Q}) = 1$ there is a prime ideal $\mathfrak{p} \in \mathrm{Ass}(R)$ satisfying $\mathfrak{Q} \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$. Since $R/\mathfrak{p}$ is an acceptable ring ([S_2]), if $\dim(R/\mathfrak{p}) \geqq 2$, then no prime ideal in $\mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ has coheight 1. For the Gorensteinness of the formal fibers shows that every prime ideal in $\mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ is a minimal prime of $\hat{R}/\mathfrak{p}\hat{R}$ and therefore if there were a prime ideal $\mathfrak{Q}' \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$ such that $\dim(\hat{R}/\mathfrak{Q}') = 1$, then $R/\mathfrak{p}$ would not be quasi-unmixed unless $\dim(R/\mathfrak{p}) = 1$. However it never happen since R is universally catenary. (See [R].) Thus one obtains $\dim(R/\mathfrak{p}) = 1$ and hence $\dim_{k(\mathfrak{p})}\mathrm{Hom}_R(R/\mathfrak{p}, R)_\mathfrak{p} = 1$ by the assumption. On the other hand one can see that the formal fiber $(\hat{R}/\mathfrak{p}\hat{R})_{\mathfrak{Q}}$ is a Gorenstein ring of dimension 0, since R is acceptable. In other words,

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$\dim_{k(\mathfrak{Q})}\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R}/\mathfrak{p}\hat{R})_{\mathfrak{Q}} = 1$. Then in the same way of the proof of Lemma 2, we see that $\dim_{k(\mathfrak{Q})}\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{Q}, \hat{R})_{\mathfrak{Q}} = 1$. Thus $\hat{R}$ satisfies the same condition as R and we may hence assume the completeness of R.

By Proposition 1 it is sufficient to see that for every finitely generated R-submodule M of $E_R(k)$ there exists an element x of $E_R(k)$ such that $M \subset Rx$. For this purpose let $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$ be an irredundant decomposition of (0) in R into *irreducible* ideals and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ $(i = 1, 2, \cdots, n)$. By the assumption if $\dim(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}_j) = 1$ for some $i \neq j$, then $\mathfrak{p}_i \neq \mathfrak{p}_j$. Thus we can find a set $\{\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_n\}$ of prime ideals satisfying the following conditions;

(1) $$\dim(R/\mathfrak{P}_i) = 1 \qquad (i = 1, 2, \cdots, n)\,.$$

(2) $$\mathfrak{P}_i \neq \mathfrak{P}_j \qquad \text{if } i \neq j\,.$$

(3) $$\mathfrak{P}_i \supset \mathfrak{p}_i \qquad (i = 1, 2, \cdots, n)\,.$$

Since $\bigcap_{i=1}^{n} \bigcap_{j=1}^{\infty} \{(\mathfrak{P}_i^j R_{\mathfrak{P}_i} + \mathfrak{q}_i R_{\mathfrak{P}_i}) \cap R\} = \bigcap_{i=1}^{n} (\mathfrak{q}_i R_{\mathfrak{P}_i} \cap R) = (0)$ and since M is an R-module of finite length, we have that $\bigcap_{i=1}^{n} \{(\mathfrak{P}_i^N R_{\mathfrak{P}_i} + \mathfrak{q}_i R_{\mathfrak{P}_i}) \cap R\} \cdot M = (0)$ for large enough N by the theorem (30.1) in [N]. Each ring $\bar{R}_i := R_{\mathfrak{P}_i}/\mathfrak{q}_i R_{\mathfrak{P}_i}$ has a dualizing complex and the zero ideal of $\bar{R}_i$ is irreducible. Thus the induction hypothesis shows that $\bar{R}_i$ is approximately Gorenstein for $i = 1, 2, \cdots, n$. It follows that there exists a $\mathfrak{P}_i \bar{R}_i$-primary and irreducible ideal $\mathfrak{Q}_i'$ of $\bar{R}_i$ such that $\mathfrak{Q}_i' \subset \mathfrak{P}_i^N \bar{R}_i$ $(i = 1, 2, \cdots, n)$. If we denote $\mathfrak{Q}_i = \mathfrak{Q}_i' \cap R$ and $\mathfrak{Q} = \mathfrak{Q}_1 \cap \mathfrak{Q}_2 \cap \cdots \cap \mathfrak{Q}_n$, then each $\mathfrak{Q}_i$ is $\mathfrak{P}_i$-primary and irreducible and $\mathfrak{Q}_i \subset (\mathfrak{P}_i^N R_{\mathfrak{P}_i} + \mathfrak{q}_i R_{\mathfrak{P}_i}) \cap R$ $(i = 1, 2, \cdots, n)$. Hence we see that $\mathfrak{Q} \cdot M = (0)$ and $R/\mathfrak{Q}$ is a Cohen-Macaulay local ring of dimension 1 which is generically Gorenstein (and has a dualizing complex). In particular $R/\mathfrak{Q}$ is approximately Gorenstein by Corollary to Proposition 2. Notice that there exists an isomorphism $E_{(R/\mathfrak{Q})}(k) \simeq [0 : \mathfrak{Q}]_{E_R(k)}$, and therefore M can be considered as a submodule of $E_{(R/\mathfrak{Q})}(k)$. Since $R/\mathfrak{Q}$ is approximately Gorenstein, we can find an element $x \in [0 : \mathfrak{Q}]_{E_R(k)} \subset E_R(k)$ satisfying $Rx \supset M$ by Proposition 1. This completes the proof.

§ 3. Examples

We shall give two examples below. Such bad Noetherian local rings are constructed by Ferrand and Raynaud in [FR]. Hence for the detail of construction we refer the reader to their paper.

EXAMPLE 1. There exists a local integral domain of dimension 1, in

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ch Northcott-Rees theorem does not hold.

In fact for an arbitrary integer $r \geqq 0$, there is a one-dimensional l domain R, such that $\hat{R}$ possesses a unique minimal prime ideal $\mathfrak{P}$ ch satisfies $\mathfrak{P}^2 = (0)$ and $\mathfrak{P} \simeq (R/\mathfrak{P})^r$. [FR; Proposition 3.1]. If $r \geqq 2$, ı such a ring R does not satisfy the condition (4) in Theorem, since $_{k(\mathfrak{P})}\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{P}, \hat{R}) = r$.

EXAMPLE 2. There exists a local integral domain of dimension 2, which Northcott-Rees theorem holds, but whose completion has an edded prime ideal. In particular it is neither acceptable nor excellent.

Proposition 3.3 in [FR] and its proof show that there is a local domain f dimension 2 such that

$$\hat{R} = C[[X, Y, Z]]/(Z^2, tZ)$$

re $t = X + Y + Y^2 s$ for some $s \in C[[Y]]\backslash C\{Y\}$. Let us denote $\mathfrak{P} =$ $t)\hat{R}$ and $\mathfrak{Q} = Z\hat{R}$. Notice that they are prime and $\mathfrak{Q} \subsetneqq \mathfrak{P}$. Since $= (Z) \cap (Z^2, t)$ is a primary decomposition of (0) in $\hat{R}$, we have $\mathrm{Ass}(\hat{R})$ $\mathfrak{P}, \mathfrak{Q}\}$. Thus $\hat{R}$ has an embedded prime ideal $\mathfrak{P}$. Moreover we have $_{k(\mathfrak{P})}\mathrm{Hom}_{\hat{R}}(\hat{R}/\mathfrak{P}, \hat{R})_{\mathfrak{P}} = 1$, for (Z^2, t) is $\mathfrak{P}$-primary and irreducible.

In order to prove that R satisfies the condition (4) in Theorem, it ices to see that, for every prime ideal $\mathfrak{p}$ of R of height 1, $\hat{R}/\mathfrak{p}\hat{R}$ is erically Gorenstein. If we take $\mathfrak{P}' \in \mathrm{Ass}(\hat{R}/\mathfrak{p}\hat{R})$, then it can be seen $\not\supset (Z^2, t)$. In fact if $\mathfrak{P}' \supset (Z^2, t)$, then $\mathfrak{P}' = \mathfrak{P}$ and $(0) = \mathfrak{P} \cap R = \mathfrak{P}' \cap R$ $\mathfrak{p}$, which is a contradiction. From this fact we obtain that $\hat{R}_{\mathfrak{P}'}$ is a ular local ring, in particular it is Gorenstein. Therefore $(\hat{R}/\mathfrak{p}\hat{R})_{\mathfrak{P}'}$ is o Gorenstein, since it is a fiber of a faithfully flat homomorphism of to $\hat{R}_{\mathfrak{P}'}$. This is what we wanted.

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