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# OSCILLATION CRITERIA FOR CERTAIN DAMPED PDE'S WITH $p$-LAPLACIAN

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**Abstract.** Some oscillation criteria are obtained for the damped PDE with $p$-Laplacian

$$\sum_{i,j=1}^{N} D_i( a_{ij}(x)\|Dy\|^{p-2}D_jy ) + \langle b(x), \|Dy\|^{p-2}Dy \rangle + c(x)|y|^{p-2}y = 0.$$

The results established here are extensions of some classical oscillation theorems due to Fite-Wintner and Kamenev for second order ordinary differential equations, and improve and complement recent results of Mařík and Usami.

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**1. Introduction.** In this paper we will study the following damped PDE with $p$-Laplacian

$$\sum_{i,j=1}^{N} D_i( a_{ij}(x)\|Dy\|^{p-2}D_jy ) + \langle b(x), \|Dy\|^{p-2}Dy \rangle + c(x)|y|^{p-2}y = 0 \qquad (1.1)$$

in the exterior domain $\Omega(r_0) = \{x \in \mathbb{R}^N : \|x\| \geq r_0\}$ for some $r_0 > 0$, where $x = (x_i)_{i=1}^N \in \Omega(r_0) \subset \mathbb{R}^N$, $N \geq 2$, $p > 1$, $a_{ij} \in \mathbf{C}^{1+\mu}(\Omega(r_0), \mathbb{R}^+)$, $\mu \in (0, 1)$, $\mathbb{R}^+ = (0, \infty)$, and $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix, $b(x) = (b_i(x))_{i=1}^N$, $b_i, c \in \mathbf{C}^{\mu}_{loc}(\Omega(r_0), \mathbb{R})$, $Dy = (D_iy)_{i=1}^N$, $D_iy = \partial y/\partial x_i$, and where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the scalar product in $\mathbb{R}^N$, respectively.

As usual, by a solution (classical solution) of (1.1) we mean a function $y \in \mathbf{C}^{1+\mu}(\Omega(r_0), \mathbb{R})$ which has the property $a_{ij}(x)\|Dy\|^{p-2}D_jy \in \mathbf{C}^{1+\mu}(\Omega(r_0), \mathbb{R})$ and which satisfies (1.1) at each $x \in \Omega(r_0)$. Regarding the questions of existence and uniqueness of solution of (1.1), see [**3**]. In what follows, our attention is restricted to those solutions which do not vanish identically in any neighborhood of $\infty$. A solution $y(x)$ of (1.1) is said to be oscillatory if it has arbitrarily large zeros, i.e., the set $\{x \in \Omega(r_0) : y(x) = 0\}$ is unbounded. Equation (1.1) is said to be oscillatory if all its solutions (if any exists) are oscillatory. Conversely, Equation (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

Equation (1.1) appears for examples in the study of non-Newtonian fluids, nonlinear elasticity and in glaciology (see, for example, [**3**]). There are some special cases of the equation (1.1) as follows:

- the undamped PDE with $p$-Laplacian ($A \equiv I$, identity matrix, $b(x) \equiv 0$)

$$\operatorname{div}(\,\|Dy\|^{p-2}Dy\,) + c(x)|y|^{p-2}y = 0, \tag{1.2}$$

- the damped PDE with $p$-Laplacian ($A \equiv I$)

$$\operatorname{div}(\|Dy\|^{p-2}Dy) + \langle\, b(x), \|Dy\|^{p-2}Dy\,\rangle + c(x)|y|^{p-2}y = 0. \tag{1.3}$$

- the second order linear ordinary differential equation ($p \equiv 2$, $N \equiv 1$, $a_{11}(x) \equiv 1$, $b(x) \equiv 0$)

$$y''(t) + c(t)y(t) = 0. \tag{1.4}$$

In this paper we deal with extending some classical oscillation criteria for (1.4) to that of (1.1). As we know, concerning the oscillation of (1.4) there exists well-elaborated theory, and the most important simple oscillation criterion is the well-known Fite-Wintner theorem [**5**, **20**] which states that if $c \in \mathbf{C}([t_0, \infty), \mathbb{R})$ and satisfies

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \int_{t_0}^{t} c(s)\, ds = \infty, \tag{1.5}$$

then (1.4) is oscillatory. In fact, Fite [**5**] assumed in addition that $c(t)$ is nonnegative, while Wintner [**20**] proved a stronger result which required a weaker condition involving the integral average of $C(t)$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} C(s)\, ds = \infty. \tag{1.6}$$

Obviously, condition (1.5) is not necessary for the oscillation of (1.4). Actually, suppose that

$$\lim_{t\to\infty} \int_{t_0}^{t} c(s)\, ds < \infty$$

holds; (1.4) may be still oscillatory (see, for example, [**7**, **17**]). By constructing function sequences, in 1977 Kamenev [**10**] gave some beautiful oscillation criteria for (1.4) under the assumption that $c(t)$ is an "integrally small" coefficient, that is,

$$C(t) = \int_{t}^{\infty} c(s)\, ds \quad \text{converges.} \tag{1.7}$$

Note that Kamenev studied the more general equation $y''(t) + c(t)f(y) = 0$, where $f'(y) \geq k > 0$, but condition (1.7) is the base one. The results of Fite-Wintner and Kamenev have been later developed also for various type of equations, namely, discrete equations, half-linear differential equations, functional differential equations, semilinear elliptic differential equations, et al. (see, for example, [**1**, **2**, **9**, **16**, **18**, **21**, **26**]).

In 1998, employing an $N$-dimensional vector Riccati transformation developed by Noussair and Swanson [**16**], Usami [**19**, Theorem 4] first extended the Fite theorem to (1.2). Recently, Mařík [**13**, Theorem 3.8] further extended the Fite theorem to (1.3). For (1.2) and (1.3), for later work in this direction we refer the reader to the papers [**4**, **8**, **11–15**, **19**, **22–25**] and references therein.

However, as far as the author knows, the Fite-Wintner and Kamenev theorems have not been well developed in existing literature even for (1.2) and (1.3), let alone for

(1.1). In view of this fact, it is therefore of interest to study the oscillation of damped PDE with $p$-Laplacian (1.1).

The aim of this paper is to study oscillation properties of (1.1) via modified Riccati technique and obtain extensions of Fite-Wintner [**5**, **20**] and Kamenev [**10**] for this equation, thereby improving results of Mařík [**13**] and Usami [**19**]. It is emphasized that the oscillation criteria obtained here are new even for (1.2) and (1.3). Examples are also given in the text to illustrate the relevance of our main theorems.

**2. Notations and Lemmas.** It will be convenient to make use of the following notations in the remainder of this paper. For $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$, and $l > 1$, define

$$c_{\phi,l}(x) = \phi(x)c(x) - \frac{1}{p}\left(\frac{l}{p}\right)^{p-1} \frac{\|A\|^p}{(\phi(x)\lambda_{\min}(x))^{p-1}} \|\phi(x)b(x)A^{-1} - \nabla\phi\|^p,$$

$$g(r) = \left(\rho(r)\int_{S_r} \frac{\phi(x)\|A\|^p}{\lambda_{\min}^{p-1}(x)} d\sigma\right)^{1/(1-p)}, \quad k = \frac{1}{2l}(p-1)(l-1),$$

$$C_{\phi,\rho,l}(r) = \rho(r)\int_{S_r} c_{\phi,l}(x)d\sigma - \frac{1}{p}(qk)^{1-p} g^{1-p}(r)\left|\frac{\rho'(r)}{\rho(r)}\right|^p,$$

where $\nabla\phi(x) = (D_i\phi)_{i=1}^N$, $S_r = \{x \in \mathbb{R}^N : \|x\| = r\}$, $\|A\|$ is the norm of the matrix $A$, i.e., $\|A\| = [\sum_{i=1}^N a_{ij}^2]^{1/2}$, $q$ denotes the conjugate number to $p$, i.e., $q = p/(p-1)$, and where $d\sigma$ and $\lambda_{\min}(x)$ denote the spherical element in $\mathbb{R}^N$ and the smallest eigenvalue of the matrix $A$, respectively.

The following two lemmas will be needed in the proofs of our results. The first can be founded in [**6**, Theorem 41]. The second is a modified version of Lemma 1 in [**16**] for (1.1).

LEMMA 2.1. *If $X$ and $Y$ are nonnegative, then the inequality*

$$(X + Y)^\lambda \geq X^\lambda + \lambda X^{\lambda-1}Y, \quad \lambda > 1$$

*holds.*

LEMMA 2.2. *Let $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$ and $l > 1$. Suppose that (1.1) has a nonoscillatory solution $y = y(x) \neq 0$ for all $x \in \Omega(r_1)$, $r \geq r_1$. Then the $N$-dimensional vector function $w(x)$ is well defined on $\Omega(r_1)$ by*

$$w(x) = \frac{A(x)\|Dy\|^{p-2}Dy}{y^{p-1}}, \tag{2.1}$$

*and satisfies the following inequality*

$$\operatorname{div}(\phi(x)w(x)) \leq -c_{\phi,l}(x) - 2k\frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q}\|w(x)\|^q. \tag{2.2}$$

*Proof.* Differentiation of $w(x)$ with respect to $x_i$, and summation over $i$, give

$$\operatorname{div} w(x) = \frac{1}{y^{p-1}} \sum_{i,j=1}^{N} D_i(a_{ij}(x)\|Dy\|^{p-2}D_j y) - (p-1)\frac{\|Dy\|^{p-2}}{y^p}(Dy)^T A Dy.$$

By (1.1), we find that

$$\operatorname{div} w(x) = -c(x) - (p-1)\frac{\|Dy\|^{p-2}}{y^p}(Dy)^T A Dy - \left\langle b(x), \frac{\|Dy\|^{p-2}Dy}{y^{p-1}} \right\rangle. \tag{2.3}$$

Note that

$$\|w(x)\| \leq \frac{\|A\|\|Dy\|^{p-1}}{|y|^{p-1}},$$

and

$$(Dy)^T A Dy \geq \lambda_{\min}(x)\|Dy\|^2.$$

Then from (2.3) it follows that

$$\operatorname{div} w(x) \leq -c(x) - (p-1)\frac{\lambda_{\min}(x)}{\|A\|^q}\|w(x)\|^q - \langle b(x)A^{-1}, w^T(x) \rangle. \tag{2.4}$$

Multiplying (2.4) by $\phi(x)$, we get

$$\begin{aligned} \operatorname{div}(\phi(x)w(x)) \leq{}& -\phi(x)c(x) - (p-1)\frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q}\|w(x)\|^q \\ & - \langle \phi(x)b(x)A^{-1} - \nabla\phi(x), w^T(x) \rangle. \end{aligned} \tag{2.5}$$

Application of Young's inequality ([**6**], Theorem 37) yields

$$\begin{aligned} &(p-1)\frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q}\|w(x)\|^q + \langle \phi(x)b(x)A^{-1} - \nabla\phi(x), w^T(x)\rangle \\ &= \frac{p\,\phi(x)\lambda_{\min}(x)}{l\,\|A\|^q}\left[\frac{1}{q}\|w(x)\|^q + \frac{l\,\|A\|^q}{p\,\phi(x)\lambda_{\min}(x)}\langle \phi(x)b(x)A^{-1}(x) - \nabla\phi, w^T(x)\rangle \right. \\ &\quad \left. + \frac{l-1}{q}\|w(x)\|^q\right] \\ &\geq \frac{p\,\phi(x)\lambda_{\min}(x)}{l\,\|A\|^q}\left[-\frac{1}{p}\left(\frac{l}{p}\right)^p \frac{\|A\|^{pq}}{(\phi(x)\lambda_{\min}(x))^p}\|\phi(x)b(x)A^{-1} - \nabla\phi(x)\|^p + \frac{l-1}{q}\|w(x)\|^q\right] \\ &= -\frac{1}{p}\left(\frac{l}{p}\right)^{p-1} \frac{\|A\|^p}{(\phi(x)\lambda_{\min}(x))^{p-1}}\|\phi(x)b(x)A^{-1} - \nabla\phi\|^p + 2k\frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q}\|w(x)\|^q. \end{aligned}$$

Combining the inequality above with (2.5), we obtain (2.2). □

**3. Main results.** In this section, we will establish some oscillation criteria for (1.1). First of all, we give Fite-Wintner type criteria (Theorems 3.1 and 3.2) for (1.1).

THEOREM 3.1. *Let* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$*,* $\rho \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$*, and* $l > 1$*. If*

$$\lim_{r\to\infty} \int_{r_0}^{r} C_{\phi,\rho,l}(s)ds = \infty, \tag{3.1}$$

*and*

$$\lim_{r\to\infty} \int_{r_0}^{r} g(s)ds = \infty, \tag{3.2}$$

*then* (1.1) *is oscillatory.*

*Proof.* Without loss of generality, suppose, by contradiction, that $y = y(x)$ is a solution of (1.1) which is positive on $\Omega(r_1)$ for some $r_1 \geq r_0$; then $w(x)$ is well defined by (2.1) on $\Omega(r_1)$. Let

$$Z(r) = \rho(r) \int_{S_r} \langle \phi(x)w(x),\ \nu(x)\rangle d\sigma \quad \text{for } r \geq r_1, \tag{3.3}$$

where $\nu(x) = x/\|x\|$, $\|x\| \neq 0$, denotes the outward unit normal to the sphere $S_r$. By means of the Green formula in (3.3), we obtain

$$\begin{aligned} Z'(r) &= \frac{\rho'(r)}{\rho(r)} Z(r) + \rho(r) \int_{S_r} \operatorname{div}\langle \phi(x)w(x),\ \nu(x)\rangle d\sigma \\ &\leq \frac{\rho'(r)}{\rho(r)} Z(r) - \rho(r) \int_{S_r} c_{\phi,l}(x)d\sigma - 2k\rho(r) \int_{S_r} \frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q} \|w(x)\|^q d\sigma. \end{aligned} \tag{3.4}$$

Hölder's inequality [**6**, Theorem 89] implies that

$$\begin{aligned} |Z(r)| &\leq \rho(r) \int_{S_r} \phi(x)\|w(x)\|\,\|\nu(x)\| d\sigma \\ &\leq \rho(r) \left( \int_{S_r} \frac{\phi(x)\|A\|^p}{\lambda_{\min}^{p-1}(x)} d\sigma \right)^{1/p} \left( \int_{S_r} \frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q} \|w(x)\|^q d\sigma \right)^{1/q}, \end{aligned}$$

equivalently,

$$\int_{S_r} \frac{\phi(x)\lambda_{\min}(x)}{\|A\|^q} \|w(x)\|^q d\sigma \geq \rho^{-q}(r) \left( \int_{S_r} \frac{\phi(x)\|A\|^p}{\lambda_{\min}^{p-1}(x)} d\sigma \right)^{1/(1-p)} |Z(r)|^q,$$

from which, by (3.4), it follows that

$$Z'(r) \leq -\rho(r) \int_{S_r} c_{\rho,l}(x)d\sigma + \frac{\rho'(r)}{\rho(r)} Z(r) - 2kg(r)|Z(r)|^q. \tag{3.5}$$

Young's inequality gives that

$$\frac{|\rho'(r)|}{\rho(r)}|Z(r)| \leq kg(r)|Z(r)|^q + \frac{1}{p}(qk)^{1-p}g^{1-p}(r)\left|\frac{\rho'(r)}{\rho(r)}\right|^p.$$

This inequality together with (3.5) yields

$$Z'(r) \leq -C_{\phi,\rho,l}(r) - kg(r)|Z(r)|^q. \tag{3.6}$$

Integrating (3.6) over $[r_1, r]$, we have

$$Z(r) + \int_{r_1}^{r} C_{\phi,\rho,l}(s)ds + k\int_{r_1}^{r} g(s)|Z(s)|^q ds \leq Z(r_1). \tag{3.7}$$

In view of (3.1), there exists a $r_2 \geq r_1$ such that for $r \geq r_2$,

$$\int_{r_1}^{r} C_{\phi,\rho,l}(s)ds - Z(r_1) \geq 0.$$

This and (3.7) imply that

$$|Z(r)| \geq k\int_{r_1}^{r} g(s)|Z(s)|^q ds := G(r).$$

So,

$$G'(r) = kg(r)|Z(r)|^q \geq kg(r)G^q(r),$$

and consequently

$$\frac{G'(r)}{G^q(r)} \geq kg(r).$$

Integration of this inequality over $[r_2, \infty)$ gives a divergent integral on the right hand side, according to (3.2), and a convergent integral on the left hand side. This contradiction completes the proof. □

COROLLARY 3.1. [Fite-type Theorem]. *Let $p \geq N$ and $l > 1$. If*

$$\int_{\Omega(r_0)} \left[c(x) - \frac{1}{p}\left(\frac{l}{p}\right)^{p-1} N^{p/2}\|b(x)\|^p\right]dx = \infty, \tag{3.8}$$

*then* (1.3) *is oscillatory.*

*Proof.* Follows from Theorem 3.1 for $\phi(x) \equiv 1$ and $\rho(r) \equiv 1$. □

COROLLARY 3.2. *If*

$$\int_{\Omega(r_0)} \left\{\|x\|^{p-N}\left[c(x) - \frac{1}{p}N^{p/2}\|b(x)\|^p\right] - \frac{1}{p}\left(\frac{2}{p-1}\right)^{p-1} N^{p/2}|p-N|^p\|x\|^{-N}\right\}dx = \infty, \tag{3.9}$$

*then* (1.3) *is oscillatory.*

*Proof.* Follows from Theorem 3.1 for $\phi(x) \equiv 1$, $\rho(r) \equiv r^{p-N}$, and $l = p$. □

REMARK 3.1. For (1.2), with $\phi(x) = 1$, Theorem 3.1 improves Theorem 4 of [**19**]. For (1.3), with $\rho(x) = 1$, Theorem 3.1 improves Theorem 3.8 of [**13**].

THEOREM 3.2. *Let* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$, *and* $l > 1$. *If*

$$\lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r} \int_{r_0}^{s} C_{\phi,\rho,l}(\tau)d\tau ds = \infty, \tag{3.10}$$

*and*

$$\lim_{r\to\infty} \int_{r_0}^{r} \left( \int_{r_0}^{s} g^{1-p}(\tau)d\tau \right)^{1/(1-p)} ds = \infty, \tag{3.11}$$

*then* (1.1) *is oscillatory.*

*Proof.* Following the proof of Theorem 3.1, we obtain that (3.7) holds. Integrating (3.7) from $r_1$ to $r$ and dividing through by $r$ yields

$$\frac{1}{r}\int_{r_1}^{r} Z(s)ds + \frac{1}{r}\int_{r_0}^{r}\int_{r_1}^{s} C_{\phi,\rho,l}(\tau)d\tau ds + \frac{k}{r}\int_{r_1}^{r}\int_{r_1}^{s} g(\tau)|Z(\tau)|^q d\tau ds \le Z(r_1)\left(1 - \frac{r_1}{r}\right). \tag{3.12}$$

By (3.10), we can choose $r_2$ sufficiently large so that, for $r \ge r_2$,

$$\frac{1}{r}\int_{r_1}^{r} Z(s)ds + \frac{k}{r}\int_{r_1}^{r}\int_{r_1}^{r} g(\tau)|Z(\tau)|^q d\tau ds \le 0. \tag{3.13}$$

Define

$$H(r) = \int_{r_1}^{r}\int_{r_1}^{s} g(\tau)|Z(\tau)|^q d\tau ds.$$

Using Hölder's inequality, we have

$$H(r) \le \frac{1}{k}\int_{r_1}^{r} |Z(s)|ds \le \frac{1}{k}\left(\int_{r_1}^{r} g^{1-p}(s)ds\right)^{1/p} \left(\int_{r_1}^{r} g(s)|Z(s)|^q ds\right)^{1/q},$$

and thus

$$\left(\int_{r_1}^{r} g^{1-p}(s)ds\right)^{1/(1-p)} \le \frac{1}{k^q}\frac{H'(r)}{H^q(r)}.$$

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Integrating the above inequality from $r_1$ to $r$, we get

$$\int_{r_1}^{r}\left(\int_{r_1}^{s} g^{1-p}(\tau)d\tau\right)^{1/(1-p)} ds \le \frac{1}{(q-1)k^q}\left(\frac{1}{H^{q-1}(r_1)}-\frac{1}{H^{q-1}(r)}\right)$$
$$< \frac{1}{(q-1)k^q}\frac{1}{H^{q-1}(r_1)} < \infty,$$

which gives a desired contradiction with (3.11) as $r \to \infty$. This completes the proof. □

It is clear that Theorem 3.1 cannot be applied in the following case,

$$\int^{\infty} C_{\phi,\rho,l}(s)ds < \infty. \tag{3.14}$$

Next, we shall discuss the oscillatory behavior of solutions of (1.1) satisfying (3.14), and establish Kamenev's theorem [**10**] for (1.1). For this, we start with a useful lemma which is similar to Hartman's Lemma ([**7**, p. 365]) for second order linear ordinary differential equations.

LEMMA 3.1. *Let* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0,\infty), \mathbb{R}^+)$, *and* $l > 1$ *such that* (3.2) *and* (3.14) *hold. Define*

$$\Theta_0(r) = \int_r^{\infty} C_{\phi,\rho,l}(s)ds < \infty, \quad r \ge r_0. \tag{3.15}$$

*If* (1.1) *is nonoscillatory, then there exist a constant* $r_1 > r_0$ *and a function* $Z \in \mathbf{C}([r_1,\infty), \mathbb{R})$ *such that for* $r \ge r_1$,

$$\int_r^{\infty} g(s)|Z(s)|^q ds < \infty, \tag{3.16}$$

*and*

$$Z(r) \ge \Theta_0(r) + k\int_r^{\infty} g(s)|Z(s)|^q ds. \tag{3.17}$$

*Proof.* As in the proof Theorem 3.1, there exist a constant $r_1 \ge r_0$ and a function $Z \in \mathbf{C}^1([r_1,\infty), \mathbb{R})$ satisfying (3.7) for $r \ge r_1 \ge r_0$. Now, we claim that (3.16) holds. To see this, suppose on the contrary that

$$\int_r^{\infty} g(s)|Z(s)|^q ds = \infty. \tag{3.18}$$

Note that from (3.7), (3.14) and (3.18), there is a $r_2 \ge r_1$ such that

$$Z(r) \le -k\int_{r_1}^{r} g(s)|Z(s)|^q ds, \quad r \ge r_2.$$

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As in the proof of Theorem 3.1, we can obtain $\int_{r_1}^{\infty} g(s)ds < \infty$, which contradicts (3.2). Hence, (3.16) holds. It follows from (3.7), (3.14) and (3.16) that

$$Z(r) \geq \limsup_{b\to\infty} Z(b) + \int_r^{\infty} C_{\phi,\rho,l}(s)ds + k\int_r^{\infty} g(s)|Z(s)|^q ds, \quad r \geq r_0. \tag{3.19}$$

If $\limsup_{b\to\infty} Z(b) < 0$, then there exist two numbers $\delta < 0$ and $r_2 \geq r_1$ such that $Z(b) < \delta$ for $b \geq r_2$. Thus, from (3.2), we have

$$\int_r^{\infty} g(s)|Z(s)|^q ds \geq \delta^q \int_r^{\infty} g(s)ds = \infty,$$

which contradicts (3.16). Thus, $\limsup_{b\to\infty} Z(b) \geq 0$. It follows from (3.7) that (3.17) holds for $r \geq r_1$. This completes the proof. □

Let $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0,\infty), \mathbb{R}^+)$, and $l > 1$ such that (3.2) and (3.14) hold. For $r \geq r_0$, we define a function sequence $\{\Theta_n(r)\}_{n=0}^{\infty}$ as follows.

$$\Theta_1(r) = \int_r^{\infty} g(s)\Theta_0^q(s)_+ ds;$$

$$\vdots$$

$$\Theta_{n+1}(r) = \int_r^{\infty} g(s)[\Theta_0(s) + k\Theta_n(s)]_+^q ds, \quad n = 1, 2, \ldots, \tag{3.20}$$

where $\Theta_0(r)$ is defined by (3.15) and $\varphi(r)_+ = [\varphi(r)]_+ = \max\{\varphi(r), 0\}$.

By induction method, it is easy to prove that (3.20) is a nondecreasing sequence; that is,

$$\Theta_{n+1}(r) \geq \Theta_n(r) \quad \text{for } r \geq r_0, \quad n = 1, 2 \ldots. \tag{3.21}$$

LEMMA 3.2. *Let $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0,\infty), \mathbb{R}^+)$, and $l > 1$ such that* (3.2) *and* (3.14) *hold. Suppose that* (1.1) *has a nonoscillatory solution $y = y(x) \neq 0$ for all $x \in \Omega(r_1)$, $r_1 \geq r_0$. Then* (3.20) *exists and converges; that is,*

$$\lim_{n\to\infty} \Theta_n(r) = \Theta(r) \quad \textit{for } r \geq r_1. \tag{3.22}$$

*Proof.* Without loss of generality, let us consider $y = y(x) > 0$ for $x \in \Omega(r_1)$; then the results of Lemma 3.1 hold. So (3.17) implies that $Z(r) \geq \Theta_0(r)$. Consequently, $|Z(r)| \geq \Theta_0(r)_+$ for $r \geq r_1$. Noting (3.16), we have

$$\Theta_1(r) \leq \int_r^{\infty} g(s)|Z(s)|^q ds < \infty. \tag{3.23}$$

Then, from (3.17) and (3.23), we get

$$\Theta_0(r) + k\Theta_1(r) \le Z(r).$$

Thus, by Lemma 3.1,

$$\Theta_2(r) \le \int_r^\infty g(s)|Z(s)|^q ds < \infty.$$

By induction we can easily obtain

$$\Theta_m(r) \le \int_r^\infty g(s)Z(s)|^q ds < \infty, \quad m = 1, 2 \dots . \tag{3.24}$$

In view of (3.21) and (3.24), we see that (3.20) is nondecreasing and bounded. Hence, (3.20) exists and converges, that is, (3.22) holds. Hence, Lemma 3.2 is proved. □

As an immediate consequence of Lemma 3.2, we have the following Kamenev-type oscillation criteria [**10**] for (1.1).

THEOREM 3.3. *Let* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$*,* $\rho \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$*, and* $l > 1$ *such that* (3.2) *and* (3.14) *hold. Suppose that, for sequence* (3.20)*, one of the following conditions is satisfied.*

(1). *There is a positive integer* $m \ge 1$ *such that* $\Theta_1(r), \dots, \Theta_{m-1}(r)$ *exist, but* $\Theta_m(r)$ *does not exist;*

(2). $\{\Theta_i(r)\}_{i=1}^\infty$ *exists, but there is a* $r^* \ge b$ *for an arbitrarily large* $b \ge r_0$ *such that* $\lim_{n\to\infty} \Theta_n(r^*) = \infty$.

*Then* (1.1) *is oscillatory.*

In what follows, we further assume that $r$ is sufficient large so that

$$\Theta_0(r) \ge 0. \tag{3.25}$$

LEMMA 3.3. *Let* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbb{R}^+)$*,* $\rho \in \mathbf{C}^1([r_0, \infty), \mathbb{R}^+)$*, and* $l > 1$ *such that* (3.2)*,* (3.14) *and* (3.25) *hold. Suppose that* (1.1) *has a nonoscillatory solution* $y = y(x) \ne 0$ *for all* $x \in \Omega(r_1)$*,* $r_1 \ge r_0$*. Then*

$$\limsup_{r\to\infty} \left\{ \Theta(r) \exp\left( kq \int_{r_1}^r g(s)\Theta_0^{q-1}(s)ds \right) \right\} < \infty, \tag{3.26}$$

*where* $\Theta(r)$ *is defined by* (3.22).

*Proof.* By Lemma 3.1, for $r \ge r_1$, we have

$$Z(r) \ge \Theta_0(r) + U(r) \ge 0,$$

where

$$U(r) = k \int_r^{\infty} g(s)|Z(s)|^q ds.$$

In view of Lemma 2.1, we find that

$$\begin{aligned} U'(r) &= -kg(r)|Z(r)|^q \\ &\le -kg(r)[\Theta_0(r) + U(r)]^q \\ &\le -kg(r)[\Theta_0^q(r) + q\Theta_0^{q-1}(r)U(r)] \\ &\le -kqg(r)\Theta_0^{q-1}(r)U(r). \end{aligned}$$

Thus it follows that

$$U(r) \le U(r_1)\exp\left(-kq\int_{r_1}^{r} g(s)\Theta_0^{q-1}(s)ds\right). \tag{3.27}$$

On the other hand, we showed, in the proof of Lemma 3.2, that (3.24) holds, that is,

$$U(r) \ge k\Theta_m(r), \quad m = 1, 2, \ldots.$$

This and (3.27) imply that

$$\Theta_m(r)\exp\left(kq\int_{r_1}^{r} g(s)\Theta_0^{q-1}(s)ds\right) \le \frac{U(r_1)}{k}, \quad m = 1, 2, \ldots. \tag{3.28}$$

By Lemma 3.2, $\{\Theta_m(r)\}_{m=1}^{\infty}$ converges, and then, by (3.28) it follows that

$$\begin{aligned} &\lim_{m\to\infty} \Theta_m(r)\exp\left(kq\int_{r_1}^{r} g(s)\Theta_0^{q-1}(s)ds\right) \\ &= \Theta(r)\exp\left(kq\int_{r_1}^{r} g(s)\Theta_0^{q-1}(s)ds\right) \le \frac{U(r_1)}{k}. \end{aligned}$$

Let limsup as $r \to \infty$ in above inequality, to get that (3.26) holds. □

By Lemma 3.3, we have

THEOREM 3.4. *Let* $\varphi \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$, $\rho \in \mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$ *and* $l > 1$ *such that* (3.2)*,* (3.14) *and* (3.25) *hold. Suppose that, for sequence* (3.20)*, one of the following conditions is satisfied.*

(1). $\Theta_i(r)$*,* $i = 1, 2, \ldots, m$*, exist, and*

$$\limsup_{r\to\infty}\left\{\Theta_m(r)\exp\left(kq\int_{r_0}^{r} g(s)\Theta_0^{q-1}(s)ds\right)\right\} = \infty; \tag{3.29}$$

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(2). (3.22) *holds, and*

$$\limsup_{r\to\infty}\left\{\Theta(r)\exp\left(kq\int_{r_0}^{r} g(s)\Theta_0^{q-1}(s)ds\right)\right\}=\infty, \tag{3.30}$$

*where* $\Theta(r)$ *is defined by* (3.22). *Then* (1.1) *is oscillatory.*

THEOREM 3.5. *Let* $\varphi\in\mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$, $\rho\in\mathbf{C}^1([r_0,\infty),\mathbb{R}^+)$ *and* $l>1$ *such that* (3.2), (3.14) *and* (3.25) *hold. If*

$$\lim_{r\to\infty}\int_{r_0}^{r}\exp\left(-kq\int_{r_0}^{s} g(\tau)\Theta_0^{q-1}(\tau)d\tau\right)ds<\infty, \tag{3.31}$$

*and there exists* $m\geq 1$ *such that*

$$\lim_{r\to\infty}\int_{r_0}^{r}\Theta_m(s)ds=\infty, \tag{3.32}$$

*where* $\Theta_m(r)$ *is defined by* (3.2), *then* (1.1) *is oscillatory.*

*Proof.* Proceeding as in the proof Lemma 3.3, we get that (3.28) holds, that is,

$$\Theta_m(r)\leq\frac{U(r_1)}{k}\exp\left(-kq\int_{r_1}^{r} g(s)\Theta_0^{q-1}(s)ds\right).$$

Noting (3.31) and (3.32), let $r\to\infty$ in the inequality above, to get a contradiction. This contradiction proves our theorem. □

In the following we illustrate our main theorems with three examples.

EXAMPLE 3.1. Consider (1.1) with

$$A=\operatorname{diag}(\|x\|,\|x\|),\quad b(x)=\left(\frac{x_1}{\|x\|^2},\frac{x_2}{\|x\|^2}\right),\quad c(x)=\frac{1+\varepsilon\sin\|x\|}{\|x\|^{\gamma}}, \tag{3.33}$$

where $x\in\Omega(1)$, $N=2$, $1<\gamma\leq 2$, $p=3$, and $\varepsilon\in\mathbb{R}$. For Theorem 3.1, let $\phi(x)=1$, $\rho(r)=1$ and $l=3$; then

$$C_{\phi,\rho,l}(r)=\frac{2\pi(1+\varepsilon\sin r)}{r^{\gamma-1}}-\frac{2^{5/2}\pi}{3r^4},\quad g(r)=(2^{5/2}\pi r^2)^{-1/2}.$$

It is easy to see that all conditions of Theorem 3.1 are satisfied, so (3.33) is oscillatory.

EXAMPLE 3.2. Consider (1.1) with

$$A=\operatorname{diag}(1,1),\quad b(x)=\left(\frac{x_1}{\|x\|^2},\frac{x_2}{\|x\|^2}\right),\quad c(x)=\frac{\nu}{\|x\|^{13/4}}, \tag{3.34}$$

where $x\in\Omega(1)$, $N=2$, $\nu>0$, $p=4$. For Theorem 3.3 (1), let $\phi(x)=\|x\|$, $\rho(r)=1$ and $l=4$, then

$$C_{\phi,\rho,l}(r)=\frac{2\pi\nu}{r^{5/4}},\quad g(r)=(8\pi r^2)^{-1/3}.$$

It follows, for $r \geq 1$, that

$$\Theta_0(r) = \frac{8\pi\nu}{r^{1/4}}, \quad \Theta_1(r) = 8\pi\nu^{4/3}\int_r^\infty \frac{1}{s}ds = \infty.$$

Thus, by Theorem 3.3 (1), (3.34) is oscillatory.

EXAMPLE 3.3. Consider (1.1) with

$$A = \operatorname{diag}\left(\frac{1}{\|x\|}, \frac{1}{\|x\|}\right), \quad b(x) = \left(\frac{2x_1}{\|x\|^3}, \frac{2x_2}{\|x\|^3}\right),$$
$$c(x) = \frac{\varepsilon(\|x\| \sin\|x\| + \cos\|x\|) + 1}{\|x\|^5}, \tag{3.35}$$

where $x \in \Omega(1), N = 2, 0 \leq \varepsilon < 1 - (1/2)(l/(l-1))^3$ for $1 < l < 2^{1/3}/(2^{1/3}-1), p = 4$. For Theorem 3.5, let $\phi(x) = 1$, $\rho(r) = \|x\|^2$; then

$$C_{\phi,\rho,l} = \frac{2\pi[\varepsilon(r\sin r + \cos r) + 1]}{r^2}, \quad g(r) = (8\pi r^2)^{-1/3}, \quad k = \frac{3(l-1)}{2l}, \quad q = \frac{4}{3}.$$

So

$$Q_0(r) = \frac{2\pi(1 + \varepsilon\cos r)}{r}$$

and

$$Q_1(r) = 2^{1/3}\pi\int_r^\infty \frac{(1+\varepsilon\cos s)^{4/3}}{s^2}ds \geq \frac{2^{1/3}\pi(1-\varepsilon)^{4/3}}{r}.$$

Thus,

$$\lim_{r\to\infty}\int_1^r Q_1(s)ds = \infty,$$

and

$$\begin{aligned}
&\lim_{r\to\infty}\int_1^r \exp\left(-kq\int_1^s g(\tau)\Theta_0^{q-1}(\tau)d\tau\right)ds \\
&= \lim_{r\to\infty}\int_1^r \exp\left(-\frac{2^{1/3}(l-1)}{l}\int_1^s \frac{(1+\varepsilon\cos\tau)^{1/3}}{\tau}d\tau\right)ds \\
&\leq \lim_{r\to\infty}\int_1^r s^{-2^{1/3}(l-1)/l(1-\varepsilon)^{1/3}}ds < \infty.
\end{aligned}$$

Hence all conditions of Theorem 3.5 hold, and so (3.35) is oscillatory.

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