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A LOCAL RATIO THEOREM

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1. Introduction. Let T_t, $t > 0$, be a strongly continuous semigroup of positive linear contractions on the L_1-space of a σ-finite measure space $(X, \mathscr{F}, \mu)$. We denote the integral $\int_0^t T_s f\, ds, f \in L_1$, by $S_0^t f$, which is defined as the limit of Riemann sums, in the norm topology of L_1. It is easy to see that, given $f \in L_1^+$, there exists a function F on the product space $X \times (0, \infty)$, measurable with respect to the usual product σ-field, such that for every $t \geqq 0$, $\int_0^t F(\cdot, s)\, ds$ gives a representation of $S_0^t f$. We write $S_0^t f(x)$ for $\int_0^t F(x, s)\, ds$, with a fixed choice of F.

Our aim in this article is to prove the existence of $\lim_{t\downarrow 0}(S_0^t f/S_0^t g)$ a.e., on a certain part of X and to use this result to show the existence of $\lim_{t\downarrow 0}(1/t)S_0^t f$ a.e., on X. We note that the existence of the latter limit has recently been proved independently by Krengel [**3**] and by Ornstein [**4**], under the additional hypothesis of continuity at $t = 0$. We will show that there are semigroups which do not satisfy this hypothesis.

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2. Preliminaries. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space, let L_p, $1 \leqq p \leqq \infty$, be the usual Banach space of functions on $(X, \mathscr{F}, \mu)$, and let L_p^+ denote the positive cone of L_p, consisting of the non-negative functions in L_p. Let, for every t and $s > 0$, $T_t: L_1 \rightarrow L_1$ be a linear operator with $||T_t|| \leqq 1$, $T_t L_1^+ \subset L_1^+$ and $T_t T_s = T_{t+s}$. Also, assume that for every $t > 0$ and $f \in L_1$, $\lim_{s\rightarrow t}||T_s f - T_t f|| = 0$.

We first show that T_t, $t > 0$, divides the space X into two sets, which can be called the initially conservative and dissipative parts of X.

Definition 2.1. Let $g \in L_1$, $g > 0$ a.e. and $C = \{x | S_0^t g(x) > 0, \forall\, t > 0\}$, $D = X - C$.

To justify this definition we prove the following result.

LEMMA 2.1. *C and D are uniquely determined up to sets of measure zero, and do not depend on the choice of g, $g \in L_1$, $g > 0$, a.e.*

Proof. It is clear that for a given g, C is determined up to a set of measure zero. Now, let $f \in L_1, f > 0$ a.e., and assume that there exists $E \in \mathscr{F}$, $\mu(E) > 0$,

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such that for all $t > 0$, $S_0^t f > 0$ a.e. on E, but for almost all (a.a.) $x \in E$, there exists $t = t(x) > 0$ such that $S_0^{t(x)} g(x) = 0$. Then for a.a. $x \in E$ one can find a rational number $r = r(x) > 0$ such that $S_0^{r(x)} g(x) = 0$. Let r_i, $i \geqq 1$, be a counting of the positive rational numbers and let

$$E_i = \{x \mid x \in E, S_0^{r_i} g(x) = 0\}.$$

Then there exists a rational number $r_i > 0$ such that $\mu(E_i) > 0$. To simplify the notation let $E_i = E$ and $r_i = \delta$. We then have $S_0^\delta f > 0$ a.e. on E and $S_0^\delta g = 0$ a.e. on E. Let $\epsilon > 0$ be fixed and choose $n > 0$ large enough so that $ng \geqq f$ a.e. except on a set H with $\int_H f\, d\mu < \epsilon$. Let $f_1 = \chi_{H^c} f$ and $f_2 = \chi_H f$. Therefore, $S_0^\delta f_1 = 0$ a.e. on E, and hence,

$$\int_E S_0^\delta f\, d\mu = \int_E S_0^\delta f_2\, d\mu = \int_0^\delta ds \int_E T_s f_2\, d\mu \leqq \delta \|f_2\| \leqq \delta\epsilon.$$

But this is a contradiction, since $\int_E S_0^\delta f\, d\mu$ is a fixed positive number and $\epsilon > 0$ is arbitrary. This completes the proof.

We also note that a similar argument shows that $S_0^t f = 0$ a.e. on D, for any $f \in L_1$ and $t \geqq 0$.

To prove the next result on C we first observe the following general fact.

LEMMA 2.2. *Let $T: L_1 \to L_1$ be a positive linear contraction, $f \in L_1^+$, $E \in \mathscr{F}$, and $f > 0$ a.e. on E, $T^n f = 0$ a.e. on E, for all n, $1 \leqq n \leqq N$. Then for any $g \in L_1^+$,*

$$\sum_{n=0}^{N} \int_E T^n g\, d\mu \leqq \|g\|.$$

Proof. A simple argument, similar to those used in the proof of Lemma 2.1 shows that $T^n \chi_E h = 0$ a.e. on E, for all n, $1 \leqq n \leqq N$, and for all $h \in L_1^+$. Now let $\{f_0, f_1, \ldots, f_N\}$ and $\{h_0, h_1, \ldots, h_N\}$ be defined as follows: $f_0 = \chi_E g$, $h_0 = \chi_{E^c} g$, $f_n = \chi_E T h_{n-1}$, $h_n = \chi_{E^c} T h_{n-1}$, $1 \leqq n \leqq N$. An induction argument shows that

$$T^n g = \sum_{k=0}^{n} T^{n-k} f_k + h_n, \qquad 0 \leqq n \leqq N,$$

and hence $\int_E T^n g\, d\mu = \int_E f_n\, d\mu = \|f_n\|$. But it is clear that $\sum_{n=0}^N \|f_n\| \leqq \|g\|$.

LEMMA 2.3. *Let $f \in L_1^+$ and $K = \{x \mid f(x) > 0\} \cap C$ (K is determined up to a set of measure zero). Then, for all $t > 0$, $S_0^t f > 0$ a.e. on K.*

Proof. Let $K_t = \{x \mid S_0^t f(x) > 0\} \cap K$, $t > 0$. Clearly $t < t'$ implies that $K_t \subset K_{t'}$ ($\subset K$). We would like to show that $K_t = K$ for all $t > 0$. If this is not true, there exists a $\delta > 0$ such that $\mu(K - K_\delta) > 0$. Let $E = K - K_\delta$. Then $T_s f = 0$ a.e. on E, for all s, $0 < s \leqq \delta$; in fact, otherwise there would exist a σ, $0 < \sigma \leqq \delta$, with $\int_E T_\sigma f\, d\mu > 0$. But this would imply that $\int_0^\delta ds \int_E T_s f\, d\mu > 0$, since the integrand of the second integral is a continuous function of s. Hence we would have $\int_E S_0^\delta f\, d\mu > 0$, which is a contradiction.

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Hence $T_s f = 0$ a.e. on E, for all s, $0 < s \leqq \delta$. Now assume that C is defined in terms of g. Then $\int_E S_0^\delta g\, d\mu = \int_0^\delta ds \int_E T_s g\, d\mu = \alpha > 0$. Hence if N_0 is sufficiently large,

$$\sum_{n=0}^{N-1} \frac{\delta}{N} \int_E T_{\delta/N}{}^n g\, d\mu \geqq \alpha/2 \quad \text{for all } N \geqq N_0.$$

Therefore, for a sufficiently large N,

$$\sum_{n=0}^{N-1} \int_E T_{\delta/N}{}^n g\, d\mu > ||g||.$$

By letting $T = T_{\delta/N}$, however, we see from the previous lemma that this is a contradiction.

3. The local ratio theorem. As mentioned before, our main purpose is to prove the following result.

THEOREM 3.1. *Let T_t, $t > 0$, be a strongly continuous semigroup of positive linear contractions on $L_1(X, \mathscr{F}, \mu)$ and let C be the initially conservative part of X. Then for all $f \in L_1$ and $g \in L_1^+$,*

$$\lim_{t \downarrow 0} \frac{S_0{}^t f}{S_0{}^t g} \quad \textit{exists a.e. on } K = \{x \mid g(x) > 0\} \cap C.$$

Before giving the proof we note the following theorem as a corollary.

THEOREM 3.2. *For all $f \in L_1$, $\lim_{t\downarrow 0}(1/t)S_0{}^t f$ exists a.e. on X.*

Proof of Theorem 3.2. It is clear that, assuming $f \in L_1^+$, $\lim_{t\downarrow 0}(1/t)S_0{}^t f = 0$ a.e. on D. Now let $g > 0$ a.e. on X, $g \in L_1$ and let, for example, $h = S_0{}^1 g$. If $S_0{}^t g$ is represented by $\int_0^t G(\cdot, s)\, ds$, then it is easy to see that

$$\int_0^t ds' \int_{s'}^{s'+1} G(\cdot, s)\, ds$$

represents $S_0{}^t h$. Hence $\lim_{t\downarrow 0}(1/t)S_0{}^t h = h$ a.e. and, since $h > 0$ a.e. on C,

$$\lim_{t\downarrow 0} \frac{1}{t} S_0{}^t f = h \cdot \lim_{t\downarrow 0} \frac{S_0{}^t f}{S_0{}^t h} \quad \text{exists a.e. on } C,$$

by the previous theorem.

The proof of Theorem 3.1 will be divided into several lemmas.

Definition 3.1. If $\alpha \in L_\infty$ and $t > 0$, then let $T_t^\alpha: L_1 \rightarrow L_1$ be defined as $T_t^\alpha f = \alpha f + T_t(1 - \alpha)f$, $f \in L_1$. If $f \in L_1^+$ and $t > 0$, then $f <^t f'$ means that there exists an integer $n \geqq 1$ and n functions $\alpha_i \in L_\infty$, $0 \leqq \alpha_i \leqq 1$, $i = 1, \ldots, n$, and n positive numbers t_i, $i = 1, \ldots, n$, with $\sum_{i=1}^n t_i \leqq t$ and $f' = T_{t_n}{}^{\alpha_n} \ldots T_{t_1}{}^{\alpha_1} f$. If $E \in \mathscr{F}$, $f \in L_1^+$, and $t > 0$, we let

$$\varphi_E{}^t f = \sup_{f <^t f'} \int_E f'\, d\mu \quad \left(\geqq \int_E f\, d\mu\right)$$

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and

$$\varphi_E f = \lim_{t \downarrow 0} \varphi_E{}^t f \quad \left(\geqq \int_E f\, d\mu \right).$$

LEMMA 3.1. *If* $f, g \in L_1^+$, $t_0 > 0$, *and* $\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g) > 0$ *a.e. on* $E \in \mathscr{F}$, *then* $\varphi_E{}^{t_0} f \geqq \int_E g\, d\mu$.

Proof. For a.a. $x \in E$ there exists a positive rational number $r = r(x) < t_0$, such that $S_0{}^{r(x)}(f - g)(x) > 0$. Let r_i, $i \geqq 1$, be a counting of the positive rational numbers less than t_0 and let $E_i = \{x | \, x \in E, S_0{}^{r_i}(f - g)(x) > 0\}$. Let $\epsilon > 0$ be fixed and choose N large enough so that

$$\int_{E - (\bigcup_{i=1}^N E_i)} g\, d\mu < \epsilon.$$

Also, for every $i \geqq 1$, choose an $\alpha_i > 0$ such that if

$$E_i' = \{x | \, x \in E_i, S_0{}^{r_i}(f - g)(x) > \alpha_i\},$$

then $\int_{E_i - E_i'} g\, d\mu < \epsilon_i$, where $\epsilon_i > 0$ and $\sum_{i=1}^\infty \epsilon_i < \epsilon$.

Now, for every $i = 1, \ldots, N$, there exists an integer Q_i, such that $q_i \geqq Q_i$ implies that

$$\left\| \frac{r_i}{q_i} \sum_{k=0}^{q_i - 1} T_{r_i/q_i}{}^k (f - g) - S_0{}^{r_i}(f - g) \right\| < \alpha_i \delta(\epsilon_i),$$

where, for every $\beta > 0$, $\delta(\beta) > 0$ denotes a number with the property that $\mu(G) < \delta(\beta)$ implies that $\int_G g\, d\mu < \beta$.

Let

$$F_i(q_i) = \left\{ x \left| \frac{r_i}{q_i} \sum_{k=0}^{q_i - 1} T_{r_i/q_i}{}^k (f - g)(x) > 0 \right. \right\} \cap E_i'.$$

Then $\mu(E_i' - F_i(q_i)) < \delta(\epsilon_i)$ for all $q_i \geqq Q_i$. Now find a rational number $r > 0$ such that $r_i = r q_i$, $i = 1, \ldots, N$, and $q_i \geqq Q_i$. It is then clear that

$$\sup_{0 \leqq k \leqq K} \sum_{n=0}^{k} T_r{}^n (f - g) > 0 \quad \text{a.e. on } F = \bigcup_{i=1}^{N} F_i(q_i),$$

where $Kr < t_0$.

To complete the proof we will now recall a result from the discrete case.

Let T be a positive linear contraction on L_1. For any $f \in L_1^+$ and for any measurable set F define the following sequences $\{f_0, f_1, \ldots\}$, $\{h_0, h_1, \ldots\}$ of L_1^+ functions:

$$\begin{aligned} f_0 &= \chi_F f, & h_0 &= \chi_{F^c} f, & \\ f_{n+1} &= \chi_F T h_n, & h_{n+1} &= \chi_{F^c} T h_n, & n \geqq 0. \end{aligned}$$

An induction argument shows that

$$(T^{\chi_F})^n f = f_0 + f_1 + \ldots + f_n + h_n \quad \text{for all } n \geqq 0.$$

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Now if g is another L_1^+ function such that

$$\sup_{0 \leqq k \leqq K} \sum_{n=0}^{k} T^n(f - g) > 0 \quad \text{a.e. on } F,$$

for some integer $K \geqq 0$, then one can prove (cf. [**1**; **2**]) that

$$\int \sum_{k=0}^{K} f_k \, d\mu = \int_F (T^{\chi_F})^K f \, d\mu \geqq \int_F g \, d\mu.$$

Applying this result to our case with $T = T_r$ we then obtain:

$$\int_F (T_r^{\chi_F})^K f \, d\mu \geqq \int_F g \, d\mu,$$

which implies that

$$\varphi_E{}^{t_0} f \geqq \varphi_F{}^{t_0} f \geqq \int_F g \, d\mu \geqq \int_E g \, d\mu - 3\epsilon,$$

and this completes the proof.

LEMMA 3.2. *If* $g' >^{\delta} g$, $g \in L_1^+$, *then* $S_0{}^t g' \leqq S_0{}^{t+\delta} g$ *a.e., for all* $t \geqq 0$.

The proof follows from a simple induction argument on n, where

$$g' = T_{t_n}{}^{\alpha_n} \ldots T_{t_1}{}^{\alpha_1} g.$$

LEMMA 3.3. *Let* $\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g) > 0$ *a.e. on* $E \in \mathscr{F}$, $\mu(E) < \infty$. *Then given* $\epsilon > 0$, *there exists* $F \subset E$, $\mu(E - F) < \epsilon$, *and a number* $\delta_0 > 0$ *such that* $g <^{\delta} g'$ *and* $\delta < \delta_0$ *imply that* $\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g') > 0$ *a.e. on* F.

Proof. As in the proof of Lemma 3.1, consider $E_i, \alpha_i, E_i', i \geqq 1$, such that $\mu(E_i - E_i') < \epsilon_i$, $\sum_{i=1}^{\infty} \epsilon_i < \epsilon$. If $g <^{\delta} g'$, then

$$S_0{}^{r_i}(f - g') \geqq S_0{}^{r_i}(f - g) - S_{r_i}{}^{r_i+\delta} g.$$

But $S_{r_i}{}^{r_i+\delta} g \downarrow 0$ a.e. as $\delta \downarrow 0$. Hence, find $\delta_i > 0$ such that $S_{r_i}{}^{r_i+\delta_i} g < \alpha_i$ on $F_i \subset E_i'$ with $\mu(E_i' - F_i) < \epsilon_i$. Choosing N large enough so that

$$\mu\left(E - \bigcup_{i=1}^{N} E_i\right) < \epsilon$$

and letting $F = \bigcup_{i=1}^{\infty} F_i$, $\delta_0 = \min(\delta_1, \ldots, \delta_N)$, we then have $\mu(E - F) < 3\epsilon$ and $S_0{}^{r_i}(f - g') > 0$ a.e. on F_i, if $g' >^{\delta} g$, $\delta < \delta_0$. Therefore,

$$\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g') > 0$$

a.e. on F, whenever $g' >^{\delta} g$ and $\delta < \delta_0$.

LEMMA 3.4. *Let* $\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g) > 0$ *a.e. on* E, $\mu(E) < \infty$. *Then given* $\epsilon > 0$, *there exists* $F \subset E$, $\mu(E - F) < \epsilon$, *such that* $\varphi_F{}^{t_0} f \geqq \varphi_F g$.

The proof follows directly from Lemmas 3.1 and 3.3.

LEMMA 3.5. *Let* $\sup_{0 \leqq t \leqq t_0} S_0{}^t(f - g) > 0$ *a.e. on* E *for every* $t_0 > 0$, $\mu(E) < \infty$. *Then, given* $\epsilon > 0$, *there exists* $F \subset E$, $\mu(E - F) < \epsilon$, *such that* $\varphi_F f \geqq \varphi_F g$.

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Proof. Let $t_n \downarrow 0$, $\epsilon_n > 0$, $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. For every n, choose $F_n \subset E$ and $\delta_n > 0$ such that $\mu(E - F_n) < \epsilon_n$ and $\sup_{0 \leqq t \leqq t_n} S_0^t(f - g') > 0$ a.e. on F_n, whenever $g' >^{\delta} g$ and $\delta < \delta_n$. Let $F = \bigcap_{n=1}^{\infty} F_n$; hence $\mu(E - F) < \epsilon$. On F, $\sup_{0 \leqq t \leqq t_n} S_0^t(f - g') > 0$ a.e. whenever $g' >^{\delta} g$, $\delta < \delta_n$. Choose t_n such that

$$\varphi_F^{t_n} f \leqq \varphi_F f + \epsilon', \qquad \epsilon' > 0.$$

Then $\varphi_F f + \epsilon' \geqq \varphi_F g$ for all $\epsilon' > 0$.

Proof of Theorem 3.1. The ratio $S_0^t f / S_0^t g$ is defined a.e. on K, for all $t > 0$, because of Lemma 2.3. We may assume that $f \in L_1^+$. If the limit of this ratio fails to exist as $t \downarrow 0$ on a set of positive measure, then there exist two real numbers α, β, $0 < \alpha < \beta$, and a set $E \subset K$, $0 < \mu(E) < \infty$, such that

$$\liminf_{t \downarrow 0} \frac{S_0^t f}{S_0^t g} < \alpha < \beta < \limsup_{t \downarrow 0} \frac{S_0^t f}{S_0^t g} \quad \text{a.e. on } E.$$

Hence,

$$\sup_{0 \leqq t \leqq t_0} S_0^t(f - \beta g) > 0 \quad \text{and} \quad \sup_{0 \leqq t \leqq t_0} S_0^t(\alpha g - f) > 0 \quad \text{a.e. on } E,$$

for all $t_0 > 0$. Choose $t_n \downarrow 0$, $\epsilon_n > 0$, $\sum_{n=1}^{\infty} \epsilon_n < \frac{1}{2}\mu(E)$ and $F_n \subset E$, $\tilde{F}_n \subset E$, $\delta_n > 0$, $\tilde{\delta}_n > 0$, $n \geqq 1$, such that $\mu(E - F_n) < \epsilon_n$, $\mu(E - \tilde{F}_n) < \epsilon_n$,

$$\sup_{0 \leqq t \leqq t_n} S_0^t(f - \beta g') > 0$$

a.e. on F_n, for all $g' >^{\delta} g$ with $\delta < \delta_n$ and $\sup_{0 \leqq t \leqq t_n} S_0^t(\alpha g - f') > 0$ a.e. on $\tilde{F}_n$, whenever $f' >^{\delta} f$, $\delta < \tilde{\delta}_n$. Let $F = \bigcap_{n=1}^{\infty}(F_n \cap \tilde{F}_n)$. Then $\mu(F) > 0$ and $\varphi_F f \geqq \beta \varphi_F g$, $\alpha \varphi_F g \geqq \varphi_F f$. This is a contradiction, since $\varphi_F g \geqq \int_F g \, d\mu > 0$ and $\alpha < \beta$.

4. The initial continuity of T_t. In [3], Krengel proved that if T_t, $t \geqq 0$, is a semigroup of positive linear contractions on L_1, strongly continuous on $[0, \infty)$, then for all $f \in L_1$, $T_0 f = \lim_{t \downarrow 0} (1/t) S_0^t f$ a.e. and also observed that, in most cases a strongly continuous semigroup T_t, $t > 0$, on $(0, \infty)$ can be completed to a strongly continuous semigroup T_t, $t \geqq 0$, on $[0, \infty)$ by a suitable choice of T_0, which, in view of his result and our Theorem 3.2, must be defined as $T_0 f = \lim_{t \downarrow 0} (1/t) S_0^t f$. The following example shows that, however, the resulting semigroup T_t, $t \geqq 0$, in general is not continuous at $t = 0$.

Example 4.1. Let $X = R \cup \{P\}$, where $R = (-\infty, \infty)$ and $P \notin R$ is a single point. Let μ be the measure on X, whose restriction to R is the Lebesgue measure, and $\mu(\{P\}) = 1$. For $f \in L_1(\mu)$ and $t > 0$, define

$$(T_t f)(x) = \begin{cases} f(P) \dfrac{1}{(\pi t)^{1/2}} e^{-x^2/t} + \displaystyle\int_R \frac{1}{(\pi t)^{1/2}} e^{-(x-y)^2/t} f(y) \, dy, & \text{for } x \in R, \\ 0, & \text{for } x = P. \end{cases}$$

It is clear that, if $f = \chi_{\{P\}}$, then $T_0 f = \lim_{t \downarrow 0} (1/t) S_0^t f = 0$ a.e. on X, but $\|T_t f\| = 1$ for all $t > 0$.

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We may, however, give a sufficient condition for the possibility of completing T_t, $t > 0$, to a strongly continuous semigroup on $[0, \infty)$.

THEOREM 4.1. *If* $\mu(D) = 0$ *and if* $T_0 f = \lim_{t\downarrow 0}(1/t)S_0{}^t f$ *a.e.*, $f \in L_1$, *then* T_t, $t \geqq 0$, *is a strongly continuous semigroup on* $[0, \infty)$.

Proof. Clearly, $T_0: L_1 \rightarrow L_1$ is a positive linear contraction. Also, if $g \in L_1$ and $g > 0$ a.e., then $h = S_0{}^1 g > 0$ a.e. and $Th = h$.

Note that the existence of such an invariant function h implies that $||Tf|| = ||f||$ for any $f \in L_1{}^+$. In fact, first assume that $f \in L_1{}^+$ and $f \leqq h$ a.e. Then $h = f + l$ for some $l \in L_1{}^+$. Hence $||h|| = ||f|| + ||l||$ and $||Th|| = ||Tf|| + ||Tl||$. Therefore $||f|| + ||l|| = ||Tf|| + ||Tl||$, or $||f|| - ||Tf|| = ||Tl|| - ||l||$. But $||f|| - ||Tf|| \geqq 0$ and $||Tl|| - ||l|| \leqq 0$. Hence $||f|| = ||Tf||$.

Now, if we have an arbitrary $f \in L_1{}^+$, let $\epsilon > 0$ be a given number and choose a real number r so that $rh \geqq f$ a.e. except on a set G with $\int_G f\, d\mu < \epsilon$. Let $f_1 = \chi_{G^c} f$ and $f_2 = \chi_G f$. Then, from the preceding paragraph, $||f_1|| = ||Tf_1||$, since $f_1 \leqq rh$ a.e. and $Trh = rh$. Hence,

$$||Tf|| = ||Tf_1 + Tf_2|| = ||Tf_1|| + ||Tf_2|| \geqq ||Tf_1|| = ||f_1|| \geqq ||f|| - \epsilon.$$

This shows that $||Tf|| = ||f||$.

We now return to the main proof. Since $||(1/t)S_0{}^t f|| \leqq ||f||$, $t > 0$, we then have $\lim_{t\downarrow 0}(1/t)S_0{}^t f = T_0 f$, in the norm topology of L_1. Hence for every $\tau > 0, f \in L_1$,

$$T_\tau T_0 f = T_\tau \lim_{t\downarrow 0} \frac{1}{t} S_0{}^t f = \lim_{t\downarrow 0} \frac{1}{t} S_0{}^t T_\tau f = T_0 T_\tau f = \lim_{t\downarrow 0} \frac{1}{t} S_\tau{}^{\tau+t} f = T_\tau f,$$

where all the limits are in the norm topology of L_1. Now let $f \in L_1$ and $\epsilon > 0$ be given and choose $t > 0$ small enough so that $||T_0 f - (1/t)S_0{}^t f|| < \epsilon$. Hence, for all $\tau > 0$,

$$\begin{aligned} ||T_\tau f - T_0 f|| &\leqq \left\| T_\tau f - \frac{1}{t} S_0{}^t f \right\| + \epsilon = \left\| T_\tau T_0 f - \frac{1}{t} S_0{}^t f \right\| + \epsilon \\ &\leqq \left\| T_\tau \frac{1}{t} S_0{}^t f - \frac{1}{t} S_0{}^t f \right\| + 2\epsilon \leqq \frac{2\tau}{t} ||f|| + 2\epsilon. \end{aligned}$$

This proves that $\lim_{t\downarrow 0}||T_\tau f - T_0 f|| = 0$. Also,

$$T_0 T_0 f = \lim_{t\downarrow 0} T_0 \frac{1}{t} S_0{}^t f = \lim_{t\downarrow 0} \frac{1}{t} S_0{}^t f = T_0 f,$$

where, again, all the limits are in the norm topology of L_1. This completes the proof.

We may notice that the conclusion of Theorem 4.1 is true under the following weaker condition. There exists a $g \in L_1{}^+$, $g > 0$ a.e. on D, and an $f \in L_1{}^+$ such that $T_t g \leqq f$ for all t, $0 < t \leqq t_0$, with some $t_0 > 0$. The proof is a modification of the proof of Theorem 4.1.

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