THE FREE CENTRE-BY-METABELIAN GROUPS

Dedicated to the memory of Hanna Neumann

CHANDER KANTA GUPTA

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1. Introduction

Let $G_n = F_n/[F_n'', F_n]$ be the free centre-by-metabelian group of rank n. In this paper, our main result is the following

THEOREM. *For $n \geqq 4$, G_n has a finite elementary abelian subgroup H_n of rank $\binom{n}{4}$. More precisely, H_n is a minimal fully invariant subgroup contained in the centre of G_n and G_n/H_n is isomorphic to a group of 3×3 matrices over a finitely generated integral domain of characteristic zero.*

The presence of elements of order 2 in G_n ($n \geqq 4$) contradicts most of the results in sections 7 and 8 of Hurley [4]. It also contradicts an earlier claim of Ward [5] that $F_n''/[F_n'', F_n]$ is free abelian.

An error in Hurley's power series representation of $F/[F'',F]$ was first noted by Narain Gupta and Frank Levin whom I thank for communicating the information. The contents of this paper arose in an attempt to revive Hurley's results.

2. Preliminaries

The following commutator identities will be used without reference. For all d, d_1 in G_n' and $a, a_1, a_2, \cdots, a_r \in G_n$,

(i) $[d, a; d_1] = [d; d_1, a^{-1}]$

(ii) $[d; d_1, a_1, \cdots, a_r] = [d; d_1, a_{1\sigma}, \cdots, a_{r\sigma}]$ where σ is any permutation of $\{1, \cdots, r\}$.

(iii) $[d; a_1, a_2, a_3] = [d; a_1, a_3, a_2]\,[d; a_3, a_2, a_1]$

(iv) $[d, a, b; d]^2 = [d, a; d, a, b]\,[d, b; d, a, b]$

For the proof of (i), (ii) and (iii) see [1]. For the proof of (iv), we note that

$$[d, a, b; d]^{-1} = [d; d, a, b] = [d, a^{-1}, b^{-1}; d] \text{ by (i)}$$
$$= [[d, a, b]^{a^{-1}b^{-1}}; d]$$

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$$= [d,a,b;d][d,a,b,a^{-1};d][d,a,b,b^{-1};d][d,a,b,a^{-1},b^{-1};d]$$

$$= [d,a,b;d][d,a,b;d,a][d,a,b;d,b][d,a,b;d,a,b] \text{ by (i).}$$

Thus, $[d,a,b;d]^{-2} = [d,a,b;d,a]\,[d,a,b;d,b]$.

3. Proof of the theorem

Let $G_n (n \geqq 4)$ be the free centre-by-metabelian group freely generated by $x_1, x_2, \cdots, x_n$. Let H_n be the fully invariant closure in G_n of $w(x_1,x_2,x_3,x_4) = [x_1^{-1},x_2^{-1};\ x_3,x_4]\ [x_1^{-1},\ x_4^{-1};\ x_2,x_3]\ [x_1^{-1},x_3^{-1};\ x_4,x_2]\ [x_4^{-1},x_2^{-1};x_1,x_3]\ [x_2^{-1},x_3^{-1};x_1,x_4][x_3^{-1},x_4^{-1};x_1,x_2]$.

By repeated applications of (i), (ii) and (iii), it is easily shown that $w(x_1,x_2,x_3,x_4x_5)\ w^{-1}(x_1,x_2,x_3,x_4)\ w^{-1}(x_1,x_2,x_3,x_5)$ lies in $[F'',F]$. More generally, $w(u_1,u_2,u_3,u_4) = \Pi^{\alpha(i,j,k,l)}_{1\leqq i<j<k<l\leqq n}\ w(x_i,x_j,x_k,x_l)$, where $\alpha(i,j,k,l) \in Z$ and u_1, u_2, u_3, u_4 are words in G_n. Thus, H_n is a minimal fully invariant subgroup of G_n generated by $\binom{n}{4}$ independent elements $w(x_i,x_j,x_k,x_l)$. Also, it was shown in Gupta [1] that G_n/H_n is isomorphic to a group of 3×3 matrices over a commutative integral domain of characteristic zero. Thus, the proof of our theorem follows from the following two lemmas.

LEMMA 1. $w^2(x_1,x_2,x_3,x_4) = e$.

LEMMA 2. $w(x_1,x_2,x_3,x_4) \neq e$.

PROOF OF LEMMA 1. Except for rearrangements of various factors at various stages, the proof requires straight expansion of $w(x_1,x_2,x_3,x_4)$ and $w^{-2}(x_1,x_2,x_3,x_4)$ using the identities (i)–(iv). For the sake of brevity, we shall use the following notation: $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_1^{-1} = \bar{1}, x_2^{-1} = \bar{2}, x_3^{-1} = \bar{3},\ x_4^{-1} = \bar{4}$, $[x_i,x_j] = (ij)$, $[[x_i,x_j],\ x_k] = (ijk)$, $((ijk),\ (lm)) = (ijk;lm)$ etc.

$$w(1,2,3,4) = (\bar{1}\bar{2};34)\,(\bar{1}\bar{3};42)\,(\bar{1}\bar{4};23)\,(\bar{3}\bar{4};12)\,(\bar{4}\bar{2};13)\,(\bar{2}\bar{3};14) = A_1A_2A_3,$$

where

$$A_1 = (12;34)\,(13;42)\,(14;23)\,(34;12)\,(42;13)\,(23;14) = e,$$

$$A_2 = (12;341)\,(13;421)\,(14;231)\,(34;123)\,(42;134)\,(23;142)\,(12;342)\,(13;423)\,(14;234)\,(34;124)\,(42;132)\,(23;143),$$

and

$$A_3 = (12;3412)\,(13;4213)\,(14;2314)\,(34;1234)\,(42;1342)\,(23;1423).$$

Using the identity (iii) to each factor of A_2 and rearranging using (i) gives

$$A_2 = (12\bar{4};31)\,(124;31) \qquad (13\bar{2};41)\,(132;41)$$
$$(14\bar{3};21)\,(143;21) \qquad (34\bar{2};13)\,(342;13)$$

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$$(42\bar{3};14)\,(423;14)\qquad(23\bar{4};12)\,(234;12)$$

$$(34\bar{2};14)\,(342;14)\qquad(42\bar{3};12)\,(423;12)$$

$$(23\bar{4};13)\,(234;13)\qquad(34\bar{1};32)\,(341;32)$$

$$(42\bar{1};43)\,(421;43)\qquad(23\bar{1};24)\,(231;24)$$

$$= (124\bar{4};31)^{-1}\,(132\bar{2};41)^{-1}\,(143\bar{3};21)^{-1}\,(342\bar{2};13)^{-1}\,(423\bar{3};14)^{-1}\,(234\bar{4};12)^{-1}\,(342\bar{2};14)^{-1}\,(423\bar{3};12)^{-1}\,(234\bar{4};13)^{-1}\,(341\bar{1};32)^{-1}\,(421\bar{1};43)^{-1}\,(231\bar{1};24)^{-1}.$$

Thus, $w^{-1} = A_3^{-1}A_2^{-1} = B_1B_2B_3B_4B_5B_6$, where

$$B_1 = (3412;12)\,(132\bar{2};41)\,(231\bar{1};24)$$

$$B_2 = (4213;13)\,(143\bar{3};21)\,(341\bar{1};32)$$

$$B_3 = (2314;14)\,(124\bar{4};31)\,(421\bar{1};43)$$

$$B_4 = (1423;23)\,(342\bar{2};13)\,(423\bar{3};12)$$

$$B_5 = (1342;42)\,(234\bar{4};12)\,(342\bar{2};14)$$

$$B_6 = (1234;34)\,(423\bar{3};14)\,(234\bar{4};13).$$

Now,

$$\begin{aligned}
B_1 &= (3142;12)(1432;12)(132;412)(231;241) = (312;12\bar{4})(142;12\bar{3})(312;142)\,(231;241)\\
&= (312;12\bar{4})\,(312;124)\,(312;241)\,(124;12\bar{3})\,(241;12\bar{3})\,(231;241)\\
&= (312;12\bar{4})\,(312;124)\,(241;12\bar{3})\,(241;213)^{-1}\,(241;321)^{-1}\,(231;241)\,(1234;12)\\
&= (312;124\bar{4})^{-1}\,(241;123\bar{3})^{-1}\,(1234;12).
\end{aligned}$$

Let $B_1(2\to 3\to 4\to 2)$ denote the product of commutators obtained from B_1 on replacing simultaneously 2 by 3, 3 by 4 and 4 by 2. Then it is easily seen that $B_2 = B_1(2\to 3\to 4\to 2)$, $B_3 = B_2(2\to 3\to 4\to 2)$, $B_6 = B_1(1\to 3\to 1, 2\to 4\to 2)$, $B_5 = B_6(2\to 3\to 4\to 2)$, $B_4 = B_5(2\to 3\to 4\to 2)$. Thus

$$B_1 = (312;124\bar{4})^{-1}\,(241;123\bar{3})^{-1}\,(1234;12)$$

$$B_2 = (413;132\bar{2})^{-1}\,(321;134\bar{4})^{-1}\,(1342;13)$$

$$B_3 = (214;143\bar{3})^{-1}\,(431;142\bar{2})^{-1}\,(1423;14)$$

$$B_4 = (123;234\bar{4})^{-1}\,(342;231\bar{1})^{-1}\,(2314;23)$$

$$B_5 = (142;423\bar{3})^{-1}\,(234;421\bar{1})^{-1}\,(4213;42)$$

$$B_6 = (134;342\bar{2})^{-1}\,(423;341\bar{1})^{-1}\,(3412;34).$$

Now

$w^{-1} = B_1B_2B_3B_4B_5B_6 = C_1C_2$, where

$C_1 = (1234; 12)\ (1342; 13)\ (1423; 14)\ (2314; 23)\ (2413; 24)\ (3412; 34)$

$$
\begin{aligned}
C_2 = {} & (234; 1234)\ (423; 1423)\ (342; 1324)\ (413; 1234)\ (134; 3214) \\
& (341; 4213)\ (214; 1324)\ (142; 4312)\ (421; 2314)\ (312; 1432) \\
& (123; 2413)\ (231; 3412) \\
= {} & (423; 2413)\ (342; 2314)\ (134; 3124)\ (341; 4123)\ (142; 4123) \\
& (421; 2134)\ (123; 2143)\ (231; 3124).
\end{aligned}
$$

Finally, using the commutator identity (iv), we get $w^{-2} = C_1^2C_2^2 = D_1D_2D_3D_4$, where

$D_1 = (423; 2413)\ (243; 2314)\ (342; 2314)\ (342; 3412)$

$D_2 = (134; 3124)\ (341; 4123)\ (134; 1423)\ (341; 3412)$

$D_3 = (142; 4123)\ (421; 2134)\ (142; 1234)\ (241; 2413)$

$D_4 = (123; 2134)\ (231; 3124)\ (231; 2134)\ (132; 1324).$

Further,

$$
\begin{aligned}
D_1 &= (423; 3421)\ (342; 2431) = (423\bar{1}; 342)\ (4231; 342) \\
&= (4231\bar{1}; 342)^{-1} = (3412; 4231) \\
D_2 &= (134; 3412)\ (341; 3142) = (134\bar{2}; 341)\ (1342; 341) \\
&= (1342\bar{2}; 341)^{-1} = (3412; 1342) \\
D_3 &= (142; 4213)\ (421; 4123) = (142\bar{3}; 421)\ (1423; 421) \\
&= (1423\bar{3}; 421)^{-1} = (4213; 1423) \\
D_4 &= (132\bar{4}; 213)\ (213\bar{4}; 312) = (132\bar{4}; 213)\ (1324; 213) \\
&= (1324\bar{4}; 213)^{-1} = (2134; 1324).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
w^{-2} &= D_1D_2D_3D_4 \\
&= (3412; 4231)\ (3412; 1342)\ (4213; 1423)\ (2134; 1324) \\
&= ((3412)\ (4123),\ (4231))\ ((3412)\ (2134),\ (1324)) = (3142;\ 4231) \\
&\quad ((3142),\ (3412)\ (2134)) \\
&= ((3142), (4231)\ (3412)\ (2134)) = ((3142), (3142)) = e.
\end{aligned}
$$

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This completes the proof of Lemma 1.

PROOF OF LEMMA 2. Let G be the free nilpotent of class 6 group freely generated by x_1, x_2, x_3, x_4. Then $\gamma_6(G)$ is a free abelian group freely generated by all basic commutators of weight 6. Let A be the subgroup of G gnerated by all basic commutators of weight 6 *other than* the following eleven commutators:

$$a_1 = (2134; 21),\ a_2 = (2114; 32),\ a_3 = (2123; 41),\ a_4 = (2112; 43),$$

$$a_5 = (4112; 32),\ a_6 = (4123; 21),\ a_7 = (324; 211),\ a_8 = (413; 212),$$

$$a_9 = (412; 213), a_{10} = (214; 213), a_{11} = (411; 3.22).$$

Let B be the subgroup of G generated by $a_1^2, a_1a_2^{-1}, \cdots, a_1a_{11}^{-1}$. Put $K = G/AB$ but retain (without risk of confusion) the same notation as in G, thus $x_1, \cdots, x_4$ generate K and $\gamma_6(K)$ is a cyclic group of order 2 generated by a_1.

Let C be the normal subgroup of K generated by all basic commutators of weight 5 which are of the form $(ijk; lm)$. It can be easily seen that $a_1 \notin C$. Let $H = K/C$. Since H'' is generated by all basic commutators of the type $(ij; kl)$ and $(ijk; lm)$ modulo $\gamma_6(H)$, to show that H is centre-by-metabelian it is sufficient to show that $(ij; kl; m) = e$. But $(ij; kl; m) = (ijm; kl)\ (kj; klm)\ (ijm;\ klm)$. Thus, it is sufficient to show that $(ijm; klm) = e$ in H. There are only two commutators of this type, namely, $(412; 312)$ and $(421; 321)$. The commutator $(412; 312) \in A$ and $(421; 321) = (412; 312)\ (412; 213)\ (214; 312)\ (214; 213) = a_1^2 = e$. Therefore, the group H is a centre-by-metabelian group of class precisely 6 in which $a_1 \neq e$.

REMARK 1. It should be noted that the centre-by-metabelian variety $\mathfrak{C}$ is the first example of a variety defined by a commutator subgroup function (see, Hall [3], p. 422) which is not torsion-free. Let R be a normal subgroup (of index at least 3) of a non-cyclic free group. Since R is also free, by the theorem $R/[R'', R]$- has elements of order 2. Therefore, it follows that there are infinitely many varieties defined by commutator subgroup functions which are not torsion-free.

REMARK 2. It has been noted by Narain Gupta and Frank Levin that Hurley's power series representation is a faithful representation of G_n/H_n for $n \geqq 2$. Consequently, G_n/H_n is residually a finite p-group for all primes p and is residually torsion-free nilpotent (see Hurley [4], p. 290). If $n = 2$ or 3, then $H_n = \{e\}$ so that G_2 and G_3 admit Hurley's representation. The presence of elements of order 2 in G_n $(n \geqq 4)$ shows that G_n is not residually a finite p-group for every prime p. However, it can be deduced with the help of Lemma 2 that G_n $(n \geqq 4)$ is residually a finite 2-group.

REMARK 3. The example in Lemma 2 has been modified from one in the author's thesis [2].

References

[1] C. K. Gupta, 'A faithful matrix representation for certain centre-by-metabelian groups', *J. Austral. Math. Soc.* 10 (1969), 451–464.

[2] C. K. Gupta, *Center-by-metabelian groups* (Ph.D.Thesis), (Austral. Nat. Univ., 1967 Canberra).

[3] P. Hall, 'Finiteness conditions for soluble groups', *Proc. London Math. Soc.* 4 (1954), 419–436.

[4] T. C. Hurley, 'Representations of some relatively free groups in power series rings', *Proc. London Math. Soc.* 24 (1972), 257–294.

[5] M. A. Ward, *A report on Summer Research Institute of the Austral. Math. Soc.* 7 Jan. –14 Feb. 1969.

University of Manitoba
Winnipeg, Manitoba
Canada

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