

## ON GROWTH FUNCTIONS OF COXETER GROUPS

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**Abstract** Let  $(W, S)$  be a Coxeter system of rank  $n$ , and let  $p_{(W, S)}(t)$  be its growth function. It is known that  $p_{(W, S)}(q^{-1}) < \infty$  holds for all  $n \leq q \in \mathbb{N}$ . In this paper, we will show that this still holds for  $q = n - 1$ , if  $(W, S)$  is 2-spherical. Moreover, we will prove that  $p_{(W, S)}(q^{-1}) = \infty$  holds for  $q = n - 2$ , if the Coxeter diagram of  $(W, S)$  is the complete graph. These two results provide a complete characterization of the finiteness of the growth function in the case of 2-spherical Coxeter systems with a complete Coxeter diagram.

**Keywords:** Coxeter groups; growth function; Poincaré series

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### 1. Introduction

One of the most central results in the theory of lattices is Margulis' Normal Subgroup Theorem for irreducible lattices in connected semi-simple Lie groups of real rank  $\geq 2$  with a finite centre and no non-trivial compact factor [16]. Among all the recent generalizations, let us mention that Bader and Shalom proved a version of the Normal Subgroup Theorem for irreducible cocompact lattices in a product of two locally compact, non-discrete, compactly generated groups [3]. Based on earlier results in [18], Caprace and Rémy applied the Normal Subgroup Theorem to show simplicity for Kac–Moody groups over finite fields of irreducible, non-spherical and non-affine types that are twin building lattices (cf. [10, Theorems 18, 19, 20]). Moreover, it can be used to prove virtual simplicity of certain twin tree lattices with non-trivial commutation relations (cf. [11]).

In [17] and [12], Rémy, and independently Carbone and Garland, proved that certain groups acting on (twin) buildings are lattices. To be more precise: Let  $(W, S)$  be a Coxeter system with  $|S| < \infty$ , and let  $\Phi := \Phi(W, S)$  be its associated set of roots (viewed as half-spaces). Let  $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$  be an RGD system of type  $(W, S)$ , i.e. a group  $G$  together

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with a family  $(U_\alpha)_{\alpha \in \Phi}$  of subgroups (which we call *root groups*) indexed by the set of roots  $\Phi$  satisfying some combinatorial axioms (for the precise definition, we refer to [1, Ch. 7,8]). Then, there exists a *twin building*  $\Delta = (\Delta_+, \Delta_-, \delta_*)$  such that  $G$  acts on  $\Delta$ . It turns out that under some conditions,  $G^\dagger := \langle U_\alpha \mid \alpha \in \Phi \rangle \leq \text{Aut}(\Delta_+) \times \text{Aut}(\Delta_-)$  and  $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+ \rangle \leq \text{Aut}(\Delta_-)$  are lattices (cf. [17], [12]) – and in this case  $G^\dagger$  is an example of a twin building lattice. Sufficient conditions are that every root group is finite,  $W$  is infinite and for  $q_{\min} := \min\{|U_\alpha| \mid \alpha \in \Phi\}$ , one has  $p_{(W,S)}\left(\frac{1}{q_{\min}}\right) < \infty$ , where  $p_{(W,S)}(t)$  denotes the *growth function* of  $(W, S)$ . Some authors call  $p_{(W,S)}(t)$  the (*spherical*) *growth series* (cf. [13, Chapter 17] or [14, Chapter VI]) or the *Poincaré series* of  $(W, S)$  (cf. [6, Chapter 7.1]). It is clear that  $p_{(W,S)}\left(\frac{1}{q_{\min}}\right) < \infty$  holds if  $|S| \leq q_{\min}$ . It is particularly unsatisfying that the criterion  $|S| \leq q_{\min}$  does not apply to Coxeter systems of rank  $n \geq 3$  and  $q_{\min} = 2$ . However, there are examples of Coxeter systems  $(W, S)$  of rank  $n \geq 3$  with  $p_{(W,S)}\left(\frac{1}{2}\right) < \infty$ . Note that the growth function  $p_{(W,S)}(t)$  applied to  $q^{-1}$  with  $q \in \mathbb{N}$  and  $q \geq 2$  is finite for spherical and affine Coxeter systems (cf. [7, Ch. VI, Exercises § 4, 10]).

Suppose  $(W, S)$  is of type  $(4, 4, 4)$ , that is,  $|S| = 3$  and the order of  $st$  in  $W$  equals 4 for all  $s \neq t \in S$ . In [5] we constructed uncountably many new examples of RGD systems of type  $(4, 4, 4)$  in which every root group has cardinality 2. As the criterion  $|S| \leq q_{\min}$  does not apply to such RGD systems, we first asked the question whether  $p_{(W,S)}\left(\frac{1}{2}\right) < \infty$  holds. It turns out that this is indeed the case (cf. Theorem A).

## Main results

Let  $(W, S)$  be a Coxeter system and denote by  $m_{st}$  the order of  $st$  in  $W$ . The Coxeter system is called *2-spherical* if  $m_{st} < \infty$  for all  $s \neq t \in S$ . The *rank* of  $(W, S)$  is given by the cardinality of  $S$ . Throughout this paper we assume that all Coxeter systems under consideration are of finite rank. We prove the following (cf. Theorem 5.3):

**Theorem A.** *Let  $(W, S)$  be a 2-spherical Coxeter system of rank  $n$ . Then  $p_{(W,S)}\left(\frac{1}{n-1}\right) < \infty$ .*

**Remark 1.** After completion of this project, I was informed by Corentin Bodart that a more general version of Theorem A can be deduced from [2, Theorem 1] and we refer to Remark 3 at the end of the introduction for more details. Our methods of the proof are very different, and most of the results proved in the present paper are also used to prove Theorem C below. Our proofs are Coxeter group theoretic, while the proofs in [2] are for non-elementary word hyperbolic groups.

In view of the examples constructed in [5], Theorem A produces many new examples of lattices in (locally compact) automorphism groups of buildings and in a product of two automorphism groups of buildings. Combining Theorem A with [17, Théorème 1], we obtain that almost all RGD systems of 2-spherical type and rank 3 are twin building lattices:

**Corollary B.** *Let  $(W, S)$  be a Coxeter system, and let  $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$  be an RGD system of type  $(W, S)$ . Assume that the following are satisfied:*

- $(W, S)$  is 2-spherical of rank 3 and  $W$  is infinite.
- $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ ,  $|Z(G)| < \infty$  and  $|U_\alpha| < \infty$  for all  $\alpha \in \Phi$ .

Then,  $\mathcal{D}$  is a twin building lattice.

**Corollary B. (Kac–Moody version).** *Let  $(W, S)$  be a 2-spherical Coxeter system of rank 3 such that  $W$  is infinite, and let  $\mathbf{G}$  be the Kac–Moody group (in the sense of [21]) of type  $(W, S)$ . Then  $\mathbf{G}(\mathbb{F}_q)$  is a twin building lattice, where  $\mathbb{F}_q$  denotes the finite field with  $q$  elements.*

Now the question is whether the finiteness still holds for some  $q < n - 1$ . It turns out that in the class of Coxeter systems with  $m_{st} \geq 3$  for all  $s \neq t \in S$  this will not happen (cf. Theorem 5.5):

**Theorem C.** *Let  $(W, S)$  be a Coxeter system of rank  $n \geq 3$  such that  $m_{st} \geq 3$  for all  $s \neq t \in S$ . Then,  $p_{(W, S)}\left(\frac{1}{n-2}\right) = \infty$ .*

Suppose that the Coxeter diagram is 2-spherical, but the Coxeter diagram is not the complete graph. If the number of non-edges in the Coxeter diagram compared to the number of edges is *large*, then it is still possible that  $p_{(W, S)}\left(\frac{1}{n-2}\right) < \infty$  holds (cf. [19]). We also remark that Theorem C can be used to exclude certain subdiagrams for twin building lattices, as parabolic subgroups of twin building lattices are again twin building lattices:

**Corollary D.** *Let  $(W, S)$  be a Coxeter system, let  $\mathcal{D}$  be an RGD system of type  $(W, S)$  with finite root groups and let  $q_{\min} := \min\{|U_\alpha| \mid \alpha \in \Phi\}$ . If  $\mathcal{D}$  is a twin building lattice, then there does not exist a subdiagram of  $(W, S)$  with at least  $q_{\min} + 2$  vertices, whose underlying Coxeter diagram is the complete graph.*

Several remarks of our main results are in order.

**Remark 2.** The proofs of Theorem A and Theorem C make essential use of a result of Terragni [20, Theorem A]. We recall this result in Subsection 5.1.

**Remark 3.** A more general version of Theorems A can be deduced from [2, Theorem 1]: Let  $(W, S)$  and  $(W', S')$  be two Coxeter systems of rank  $n \geq 3$ . Suppose that  $(W', S')$  is of *universal* type, i.e.  $m_{st} = \infty$  for all  $s \neq t \in S$ . Note that  $W'$  is word-hyperbolic (cf. [13, Corollary 12.6.3]) and non-elementary (in the sense of [2]; cf. [13, Theorem 8.6.1, 8.7.3]). Suppose that  $(W, S)$  is not of universal type. This means that  $m_{st} < \infty$  for some  $s \neq t \in S$ . Let  $\pi : W' \rightarrow W$  be a canonical homomorphism which induces a bijection between  $S'$  and  $S$ . Then  $N := \ker(\pi)$  is a normal subgroup which is infinite. We now use the notation from [2]. One can show  $\lambda(W', S') = n - 1$ , and by [2, Theorem 1] we have  $\lambda(W, S) < \lambda(W', S') = n - 1$ . We deduce from [14, Chapter VI.C, Observation 50] that  $p_{(W, S)}\left(\frac{1}{n-1}\right) < \infty$ . This implies that we can replace in Theorem A 2-spherical by non-universal.

## Overview

In § 2, we fix notation and recall some basic results. In § 2.2, we define two subsets  $C_i$  and  $D_i$  of the Coxeter group  $W$ , which play a central role in this paper. In § 3, we recall the definition of reflection and combinatorial triangles and prove some results about them. In § 4, we establish some (in-)equalities concerning the cardinalities  $|C_i|$  and  $|D_i|$ . In § 5, we recall a result due to Terragni and prove our main results.

## 2. Preliminaries

This section is devoted to fixing notation. In § 2.1 which is based on [20], we recall the notion of growth functions in finitely generated groups. In § 2.2 and § 2.3, we recall some basic definitions about Coxeter systems. Moreover, we introduce two sets  $C_i$  and  $D_i$  which play a central role in this paper. In § 2.4, we recall some basic results about roots and walls in Coxeter systems. § 2.2, § 2.3 and § 2.4 are based on [1, § 5].

### 2.1. Growth of finitely generated groups

Let  $G$  be a finitely generated group, and let  $X = X^{-1} \subseteq G \setminus \{1\}$  be a finite, symmetric set of generators. The *length* of  $g \in G$  with respect to  $X$  is the minimal  $n$  such that  $g = x_1 \cdots x_n$  with  $x_i \in X$ ; the *length function* will be denoted by  $\ell_{(G,X)} : G \rightarrow \mathbb{N}$ . For  $n \in \mathbb{N}$ , the *sphere* in  $\text{Cay}(G, X)$  centred around  $1_G$  with radius  $n$  will be denoted by

$$C_n^{(G,X)} := \{g \in G \mid \ell_{(G,X)}(g) = n\}.$$

The cardinalities are defined as  $c_n^{(G,X)} := |C_n^{(G,X)}|$ . The *growth function* of  $(G, X)$  is given by

$$p_{(G,X)}(t) := \sum_{n \geq 0} c_n^{(G,X)} t^n \in \mathbb{Z}[[t]].$$

### 2.2. Coxeter systems

Let  $W$  be a group, and let  $S \subseteq W$  be a generating set of elements of order 2. For  $s, t \in S$ , we denote the order of  $st$  in  $W$  by  $m_{st}$ . Then, the pair  $(W, S)$  is called *Coxeter system* if the group  $W$  admits the presentation

$$W \cong \langle S \mid (st)^{m_{st}} = 1 \rangle,$$

where there is one relation for each pair  $s, t$  (possibly  $s = t$ ) with  $m_{st} < \infty$ . Let  $(W, S)$  be a Coxeter system, and let  $\ell := \ell_{(W,S)}$  be the corresponding length function. The *Coxeter diagram* corresponding to  $(W, S)$  is the labelled graph  $(S, E(S))$ , where  $E(S) = \{\{s, t\} \mid m_{st} > 2\}$  and where each edge  $\{s, t\}$  is labelled by  $m_{st}$  for all  $s, t \in S$ . The *rank* of the Coxeter system is the cardinality of the set  $S$ . Recall from the introduction that in this paper all Coxeter systems under consideration are assumed to be of finite rank.

It is well-known that for each  $J \subseteq S$ , the pair  $(\langle J \rangle, J)$  is a Coxeter system (cf. [7, Ch. IV, §1 Theorem 2]). A subset  $J \subseteq S$  is called *spherical* if  $\langle J \rangle$  is finite. The Coxeter system is called *2-spherical* if  $\langle J \rangle$  is finite for all  $J \subseteq S$  containing at most 2 elements (i.e.  $m_{st} < \infty$  for all  $s, t \in S$ ). Given a spherical subset  $J$  of  $S$ , there exists a unique element of maximal length in  $\langle J \rangle$ , which we denote by  $r_J$  (cf. [1, Corollary 2.19]).

For  $i \in \mathbb{N}$ , we define

- $C_i := C_i^{(W, S)} = \{w \in W \mid \ell(w) = i\}$  and  $c_i := |C_i| = c_i^{(W, S)}$ ;
- $D_i := \{w \in C_i \mid \exists! s \in S : \ell(ws) < \ell(w)\}$  and  $d_i := |D_i|$ .

The set  $C_i$  consists of all elements  $w \in W$  of length  $i$ . The set  $D_i$  consists of all elements  $w \in W$  of length  $i$  whose right descent set contains a single element of  $S$ .

### 2.3. The chamber system $\Sigma(W, S)$

Let  $(W, S)$  be a Coxeter system. Defining  $w \sim_s w'$  if and only if  $w^{-1}w' \in \langle s \rangle$ , we obtain a chamber system (for the definition of a chamber system, see [1, Definition 5.21]) with chamber set  $W$  and equivalence relations  $\sim_s$  for  $s \in S$ , which we denote by  $\Sigma(W, S)$ . We call two chambers  $w, w'$  *s-adjacent* if  $w \sim_s w'$  and *adjacent* if they are *s-adjacent* for some  $s \in S$ . A *gallery of length  $n$*  from  $w_0$  to  $w_n$  is a sequence  $(w_0, \dots, w_n)$  of chambers, where  $w_i$  and  $w_{i+1}$  are adjacent for each  $0 \leq i < n$ . A gallery  $(w_0, \dots, w_n)$  is called *minimal* if there exists no gallery from  $w_0$  to  $w_n$  of length  $k < n$ , and we denote the length of a minimal gallery from  $w_0$  to  $w_n$  by  $\ell(w_0, w_n)$ . For  $J \subseteq S$ , we define the *J-residue* of a chamber  $c \in W$  to be the set  $R_J(c) := c\langle J \rangle$ . A *residue*  $R$  is a *J-residue* for some  $J \subseteq S$ ; we call  $J$  the *type* of  $R$ , and the cardinality of  $J$  is called the *rank* of  $R$ . A residue is called *spherical* if its type is a spherical subset of  $S$ . Let  $R$  be a spherical *J-residue*. Two chambers  $x, y \in R$  are called *opposite in  $R$*  if  $x^{-1}y = r_J$ . Two residues  $P, Q \subseteq R$  are called *opposite in  $R$*  if for each  $p \in P$  there exists  $q \in Q$  such that  $p, q$  are opposite in  $R$ . A *panel* is a residue of rank 1. It is a fact that for every chamber  $x \in W$  and every residue  $R$ , there exists a unique chamber  $z \in R$  such that  $\ell(x, y) = \ell(x, z) + \ell(z, y)$  holds for each chamber  $y \in R$ . The chamber  $z$  is called the *projection* or the *gate* of  $x$  onto  $R$  and is denoted by  $z = \text{proj}_R x$ .

A subset  $\Sigma \subseteq W$  is called *convex* if for any two chambers  $c, d \in \Sigma$  and any minimal gallery  $(c_0 = c, \dots, c_k = d)$ , we have  $c_i \in \Sigma$  for all  $0 \leq i \leq k$ . Note that residues are convex by [1, Example 5.44(b)].

For two residues  $R$  and  $T$ , we define  $\text{proj}_T R := \{\text{proj}_T r \mid r \in R\}$ . By [1, Lemma 5.36(2)],  $\text{proj}_T R$  is a residue contained in  $T$ . The residues  $R$  and  $T$  are called *parallel* if  $\text{proj}_T R = T$  and  $\text{proj}_R T = R$ .

### 2.4. Roots and walls

Let  $(W, S)$  be a Coxeter system. A *reflection* is an element of  $W$  that is conjugate to an element of  $S$ . For  $s \in S$  we let  $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$  be the *simple root* corresponding to  $s$ . A *root* is a subset  $\alpha \subseteq W$  such that  $\alpha = v\alpha_s$  for some  $v \in W$  and  $s \in S$ . We denote the set of all roots by  $\Phi := \Phi(W, S)$ . The set  $\Phi_+ := \{\alpha \in \Phi \mid 1_W \in \alpha\}$  is

the set of all *positive roots*, and  $\Phi_- := \{\alpha \in \Phi \mid 1_W \notin \alpha\}$  is the set of all *negative roots*. For each root  $\alpha \in \Phi$ , we denote the *opposite root* by  $-\alpha$  and we denote the unique reflection which interchanges these two roots by  $r_\alpha$ . For  $\alpha \in \Phi$ , we denote by  $\partial\alpha$  (respectively,  $\partial^2\alpha$ ) the set of all panels (respectively, spherical residues of rank 2) stabilized by  $r_\alpha$ . Furthermore, we define  $\mathcal{C}(\partial\alpha) := \bigcup_{P \in \partial\alpha} P$  and  $\mathcal{C}(\partial^2\alpha) := \bigcup_{R \in \partial^2\alpha} R$ .

The set  $\partial\alpha$  is called the *wall* associated with  $\alpha$ . Let  $G = (c_0, \dots, c_k)$  be a gallery with  $c_{i-1} \neq c_i$  for each  $1 \leq i \leq k$ . We say that  $G$  *crosses the wall*  $\partial\alpha$  if there exists  $1 \leq i \leq k$  such that  $\{c_{i-1}, c_i\} \in \partial\alpha$ . It is a basic fact that a minimal gallery crosses a wall at most once (cf. [1, Lemma 3.69]). Moreover, a gallery which crosses each wall at most once is already minimal.

A pair  $\{\alpha, \beta\} \subseteq \Phi$  of roots is called *prenilpotent*, if  $\alpha \cap \beta \neq \emptyset \neq (-\alpha) \cap (-\beta)$ . For a prenilpotent pair  $\{\alpha, \beta\}$  of roots, we will write  $[\alpha, \beta] := \{\gamma \in \Phi \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq (-\gamma)\}$  and  $(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}$ . We note that roots are convex (cf. [1, Lemma 3.44]).

Let  $(c_0, \dots, c_k)$  and  $(d_0 = c_0, \dots, d_k = c_k)$  be two minimal galleries from  $c_0$  to  $c_k$ , and let  $\alpha \in \Phi$ . Then,  $\partial\alpha$  is crossed by the minimal gallery  $(c_0, \dots, c_k)$  if and only if it is crossed by the minimal gallery  $(d_0, \dots, d_k)$ .

**Lemma 2.1.** *Let  $R$  be a spherical residue of  $\Sigma(W, S)$  of rank 2, and let  $\alpha \in \Phi$ . Then, exactly one of the following holds:*

- (a)  $R \subseteq \alpha$ ;
- (b)  $R \subseteq (-\alpha)$ ;
- (c)  $R \in \partial^2\alpha$ .

**Proof.** It is clear that the three cases are exclusive. Suppose that  $R \not\subseteq \alpha$  and  $R \not\subseteq (-\alpha)$ . Then, there exist  $c \in R \cap (-\alpha)$  and  $d \in R \cap \alpha$ . Let  $(c_0 = c, \dots, c_k = d)$  be a minimal gallery. As residues are convex, we have  $c_i \in R$  for each  $0 \leq i \leq k$ . As  $c \in (-\alpha), d \in \alpha$ , there exists  $1 \leq i \leq k$ , with  $c_{i-1} \in (-\alpha), c_i \in \alpha$ . In particular,  $\{c_{i-1}, c_i\} \in \partial\alpha$  and hence  $R \in \partial^2\alpha$ .  $\square$

**Lemma 2.2.** *Let  $R, T$  be two spherical residues of  $\Sigma(W, S)$ . Then, the following are equivalent:*

- (i)  $R, T$  are parallel;
- (ii) a reflection of  $\Sigma(W, S)$  stabilizes  $R$  if and only if it stabilizes  $T$ ;
- (iii) there exist two sequences  $R_0 = R, \dots, R_n = T$  and  $T_1, \dots, T_n$  of residues of spherical type such that for each  $1 \leq i \leq n$ , the rank of  $T_i$  is equal to  $1 + \text{rank}(R)$ , the residues  $R_{i-1}, R_i$  are contained and opposite in  $T_i$  and moreover, we have  $\text{proj}_{T_i} R = R_{i-1}$  and  $\text{proj}_{T_i} T = R_i$ .

**Proof.** This is [9, Proposition 2.7].  $\square$

**Lemma 2.3.** *Let  $\alpha \in \Phi$  be a root, and let  $x, y \in \alpha \cap \mathcal{C}(\partial\alpha)$ . Then, there exists a minimal gallery  $(c_0 = x, \dots, c_k = y)$  such that  $c_i \in \mathcal{C}(\partial^2\alpha)$  for each  $0 \leq i \leq k$ . Moreover, for each  $1 \leq i \leq k$ , there exists  $L_i \in \partial^2\alpha$  with  $\{c_{i-1}, c_i\} \subseteq L_i$ .*

**Proof.** This is a consequence of [8, Lemma 2.3] and its proof.  $\square$

**Lemma 2.4.** Let  $\alpha, \beta \in \Phi, \alpha \neq \pm\beta$  be two roots, and let  $R, T \in \partial^2\alpha \cap \partial^2\beta$ .

- (a) The residues  $R$  and  $T$  are parallel.
- (b) If  $|\langle J \rangle| = \infty$  holds for all  $J \subseteq S$  containing three elements, then  $R = T$ .

**Proof.** As  $R, T \in \partial^2\alpha \cap \partial^2\beta$ , there exist panels  $P_1, Q_1 \in \partial\alpha$  and  $P_2, Q_2 \in \partial\beta$  such that  $P_1, P_2 \subseteq R$  and  $Q_1, Q_2 \subseteq T$  (as in the proof of Lemma 2.1). By Lemma 2.2, the panels  $P_i, Q_i$  are parallel for both  $i \in \{1, 2\}$ . Now [15, Lemma 17] yields that  $P_i, \text{proj}_T P_i$  are parallel, and hence  $\text{proj}_T P_1 \in \partial\alpha, \text{proj}_T P_2 \in \partial\beta$  by Lemma 2.2. As  $\alpha \neq \pm\beta$ , we deduce  $\text{proj}_T P_1 \neq \text{proj}_T P_2$ , and hence  $\text{proj}_T R$  contains the two different panels  $\text{proj}_T P_1$  and  $\text{proj}_T P_2$ . In particular,  $\text{proj}_T R$  is not a panel. Since  $\text{proj}_T R$  is a residue contained in  $T$ , we deduce  $\text{proj}_T R = T$ . Using similar arguments, we found that  $\text{proj}_R T = R$  and  $R, T$  are parallel. This proves (a). Moreover, Lemma 2.2 yields  $R = T$ , as there are no spherical residues of rank 3 by assumption. This finishes the proof.  $\square$

### 3. Reflection and combinatorial triangles in $\Sigma(W, S)$

Reflection triangles and combinatorial triangles were introduced in [8]. A *reflection triangle* is a set  $\mathcal{R}$  of three reflections such that the order of  $tt'$  is finite for all  $t, t' \in \mathcal{R}$ , and  $\bigcap_{t \in \mathcal{R}} \partial^2\beta_t = \emptyset$ , where  $\beta_t$  is one of the two roots associated with the reflection  $t$ . Note that  $\partial^2\beta_t = \partial^2(-\beta_t)$ . A set of three roots  $\mathcal{T}$  is called a *combinatorial triangle* (or simply *triangle*) if the following holds:

- (CT1) The set  $\{r_\alpha \mid \alpha \in \mathcal{T}\}$  is a reflection triangle.
- (CT2) For each  $\alpha \in \mathcal{T}$ , there exists  $\sigma \in \partial^2\beta \cap \partial^2\gamma$  such that  $\sigma \subseteq \alpha$ , where  $\{\beta, \gamma\} = \mathcal{T} \setminus \{\alpha\}$ .

**Lemma 3.1.** Suppose that  $(W, S)$  is 2-spherical and the Coxeter diagram is the complete graph. If  $\mathcal{T}$  is a triangle, then  $(-\alpha, \beta) = \emptyset$  holds for all  $\alpha \neq \beta \in \mathcal{T}$ .

**Proof.** This is [4, Proposition 2.3].  $\square$

**Proposition 3.2.** Assume that  $(W, S)$  is 2-spherical and the Coxeter diagram is the complete graph. Let  $R \neq T$  be two residues of rank 2 such that  $P := R \cap T$  is a panel. If  $\ell(1_W, \text{proj}_R 1_W) < \ell(1_W, \text{proj}_T 1_W)$ , then  $\text{proj}_T 1_W = \text{proj}_P 1_W$ .

**Proof.** We let  $\alpha \in \Phi_+$  be the root with  $P \in \partial\alpha$ . Let  $(c_0 = 1_W, \dots, c_{k'} = \text{proj}_P c_0)$  be a minimal gallery with  $c_k = \text{proj}_R c_0$  for some  $0 \leq k \leq k'$  and  $c_k, \dots, c_{k'} \in R$ .

We assume that  $\text{proj}_T c_0 \neq \text{proj}_P c_0$  holds. Then, we have  $k' > \ell(1_W, \text{proj}_T 1_W) > \ell(1_W, \text{proj}_R 1_W) = k$ . Let  $(d_0 = 1_W, \dots, d_{m'} = \text{proj}_P d_0)$  be a minimal gallery with  $d_m = \text{proj}_T c_0$  for some  $0 \leq m \leq m'$  and  $d_m, \dots, d_{m'} \in T$ . We let  $\beta \in \Phi_+$  be the root with  $\{d_m, d_{m+1}\} \in \partial\beta$ , and we let  $\gamma \in \Phi_+$  be the root with  $\{c_k, c_{k+1}\} \in \partial\gamma$ . We will show that  $\{\alpha, -\beta, -\gamma\}$  is a triangle. Thus, we first show that  $\{r_\alpha, r_\beta, r_\gamma\}$  is a reflection triangle. We have  $T \in \partial^2\alpha \cap \partial^2\beta$ , and, as a minimal gallery crosses a wall at most once, we deduce  $\alpha \neq \beta$ . Note that the wall  $\partial\beta$  is crossed by the minimal gallery  $(c_0, \dots, c_{k'})$ .



Since  $\partial^2\alpha \ni R \neq T \in \partial^2\alpha \cap \partial^2\beta$  and  $\alpha \neq \pm\beta$ , Lemma 2.4(b) implies  $R \notin \partial^2\beta$ , and hence  $\partial\beta$  is crossed by  $(c_0, \dots, c_k)$ . As  $k < k'$ , we have  $\text{proj}_R 1_W \neq \text{proj}_P 1_W$  and hence  $\alpha \neq \gamma$ . As  $\alpha, \gamma \in \Phi_+$ , we have  $\alpha \neq \pm\gamma$ .

Assume that  $o(r_\beta r_\gamma) = \infty$ . We deduce  $\beta \subseteq \gamma$ . But  $\partial\gamma$  has to be crossed by the gallery  $(d_0, \dots, d_{m'})$ . Since  $\partial^2\alpha \ni T \neq R \in \partial^2\alpha \cap \partial^2\gamma$  and  $\alpha \neq \pm\gamma$ , we have  $T \notin \partial\gamma^2$  by Lemma 2.4(b) as before. This implies that  $(d_0, \dots, d_m)$  crosses the wall  $\partial\beta$  and hence  $\gamma \subseteq \beta$ . This yields a contradiction, and we have  $o(r_\beta r_\gamma) < \infty$ .

As  $R \in \partial^2\alpha \cap \partial^2\gamma$ , Lemma 2.4(b) implies  $\partial^2\alpha \cap \partial^2\gamma = \{R\}$ . As  $R \notin \partial^2\beta$ , we deduce  $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma = \emptyset$ , and hence  $\{r_\alpha, r_\beta, r_\gamma\}$  is a reflection triangle.

Now we have to verify (CT2). As  $\partial^2\gamma \not\ni T \in \partial^2\alpha \cap \partial^2\beta$  and  $P \subseteq T \cap (-\gamma)$ , we have  $T \subseteq (-\gamma)$  by Lemma 2.1. As  $\partial^2\beta \not\ni R \in \partial^2\alpha \cap \partial^2\gamma$  and  $P \subseteq R \cap (-\beta)$ , we have  $R \subseteq (-\beta)$ . Let  $1 \leq i \leq k$  be such that  $\{c_{i-1}, c_i\} \in \partial\beta$ . Note that  $\{d_m, d_{m+1}\} \in \partial\beta$ ,  $d_{m+1} \in (-\beta) \cap T \subseteq (-\gamma)$  and  $c_i \in (-\beta) \cap \gamma$ . By Lemma 2.3 there exists a minimal gallery  $(e_0 = d_{m+1}, \dots, e_z = c_i)$  such that  $e_j \in \mathcal{C}(\partial^2\beta)$ . As  $d_{m+1} \in (-\gamma)$  and  $c_i \in \gamma$ , there exists  $1 \leq p \leq z$  such that  $e_{p-1} \in (-\gamma)$  and  $e_p \in \gamma$ . Again, by Lemma 2.3, there exists  $L \in \partial^2\beta$  such that  $\{e_{p-1}, e_p\} \subseteq L$ , and hence  $L \in \partial^2\beta \cap \partial^2\gamma$ . As roots are convex and  $e_0 = d_{m+1}, e_z = c_i \in \alpha$ , we have  $e_p \in L \cap \alpha$ . As  $\{r_\alpha, r_\beta, r_\gamma\}$  is a reflection triangle (and hence  $L \notin \partial^2\alpha$ ), we obtain  $L \subseteq \alpha$  by Lemma 2.1. This implies that  $\{\alpha, -\beta, -\gamma\}$  is a triangle, and hence  $(\alpha, \gamma) = \emptyset$  holds by Lemma 3.1. In particular,  $k+1 = k'$  and  $\ell(1_W, \text{proj}_R 1_W) = \ell(1_W, \text{proj}_P 1_W) - 1 \geq \ell(1_W, \text{proj}_T 1_W)$ . This is a contradiction to the assumption, and we conclude  $\text{proj}_T 1_W = \text{proj}_P 1_W$ .  $\square$

**Corollary 3.3.** *Assume that  $(W, S)$  is 2-spherical and that the underlying Coxeter diagram is the complete graph. Suppose  $w \in W$  and  $s \neq t \in S$  with  $\ell(ws) = \ell(w) + 1 = \ell(wt)$  and suppose  $w' \in \langle s, t \rangle$  with  $\ell(w') \geq 2$ . Then we have  $\ell(ww'r) = \ell(w) + \ell(w') + 1$  for each  $r \in S \setminus \{s, t\}$ .*

**Proof.** Suppose  $r \in S \setminus \{s, t\}$ , and assume that  $\ell(ww'r) = \ell(ww') - 1$  holds for some  $w' \in \langle s, t \rangle$  with  $\ell(w') \geq 2$ . Suppose  $w'$  starts with  $s$ , i.e.  $w' = sw''$  for some  $w'' \in \langle s, t \rangle$  with  $\ell(w'') = \ell(w') - 1$ . As  $\ell(ww'r) = \ell(ww') - 1$ , one easily sees that  $\ell(wstr) = \ell(wst) - 1$  and  $\ell(wsr) = \ell(ws) - 1$  hold, too. We define  $R := R_{\{r, t\}}(ws)$ ,  $T := R_{\{s, t\}}(w)$  and  $P := R \cap T = \mathcal{P}_t(ws)$ . Clearly,  $\text{proj}_T 1_W \neq \text{proj}_P 1_W$ . As  $m_{rt} \geq 3$ , we deduce  $\ell(1_W, \text{proj}_R 1_W) < \ell(1_W, \text{proj}_T 1_W)$  and Proposition 3.2 yields a contradiction.  $\square$

**Lemma 3.4.** *Assume that  $(W, S)$  is 2-spherical and that  $m_{st} \geq 4$  holds for all  $s \neq t \in S$ . Suppose  $w \in W$  and  $s \neq t \in S$  with  $\ell(ws) = \ell(w) + 1 = \ell(wt)$ . Then we have  $\ell(w) + 2 \in \{\ell(wsr), \ell(wtr)\}$  for all  $r \in S \setminus \{s, t\}$ .*

**Proof.** Assume that  $\ell(wsr) = \ell(w) = \ell(wtr)$ . Then,  $\ell(wr) = \ell(w) - 1$  and  $\ell(wrs) = \ell(w) - 2 = \ell(wrt)$ . Let  $R$  be the  $\{r, s\}$ -residue containing  $w$ . As  $m_{rs} \geq 4$ , we deduce  $\ell(wrsr) = \ell(wrs) - 1$ . Let  $w' \in \langle s, t \rangle$  be such that  $wr = (\text{proj}_R 1_W)w'$ . Then,  $\ell(w') \geq 2$  and the previous corollary implies  $\ell(wrt) = \ell(wr) + 1$ , which is a contradiction. This finishes the proof.  $\square$

**Remark 3.5.** Note that Lemma 3.4 is false without the assumption  $m_{st} \geq 4$ . To see this, one can consider the Coxeter system of type  $\tilde{A}_2$ .



#### 4. Some (in-)equalities

To show the two main results (Theorem 5.3 and 5.5), we will apply the ratio test. In order to do so, we need a few inequalities, which we establish in this and the next section. We recall that for  $i \in \mathbb{N}$  we have

- $C_i := \{w \in W \mid \ell(w) = i\}$  and  $c_i := |C_i|$ ;
- $D_i := \{w \in C_i \mid \exists! s \in S : \ell(ws) < \ell(w)\}$  and  $d_i := |D_i|$ ;

**Lemma 4.1.** *Suppose that the Coxeter diagram of  $(W, S)$  is the complete graph. Then, for each  $w \in W \setminus \{1_W\}$ , there is either a unique element  $s_w \in S$  with  $\ell(ws_w) = \ell(w) - 1$ , or else there are exactly two elements  $s_w \neq t_w \in S$  with  $\ell(ws_w) = \ell(w) - 1 = \ell(wt_w)$ .*

**Proof.** Let  $J \subseteq S$  with  $\ell(wj) < \ell(w)$  for each  $j \in J$ . Then [1, Corollary 2.18] implies that  $J$  is spherical. As the underlying Coxeter diagram is the complete graph, it follows that each subset of  $S$  containing at least three elements is non-spherical. This finishes the proof.  $\square$

**Convention 4.2.** *In this section, we assume that  $(W, S)$  is of rank  $n \geq 3$  and that there exists  $m \geq 3$  such that  $m_{st} = m$  holds for all  $s \neq t \in S$ . Moreover, we let  $i > m$ .*

**Lemma 4.3.**  $c_i - d_i = \binom{n-2}{2} c_{i-m} + (n-2) d_{i-m}.$

**Proof.** Let  $v \in C_i \setminus D_i$  be an element. By Lemma 4.1, there exist exactly two elements  $s \neq t \in S$  with  $\ell(vs) = \ell(v) - 1 = \ell(vt)$ . We define  $R_v := R_{\{s,t\}}(v)$ . Then, we consider the mapping

$$f : C_i \setminus D_i \rightarrow C_{i-m}, v \mapsto \text{proj}_{R_v} 1_W$$

Note that  $C_{i-m} = D_{i-m} \cup C_{i-m} \setminus D_{i-m}$ . If  $w \in C_{i-m} \setminus D_{i-m}$ , Lemma 4.1 implies that there are exactly two elements in  $S$ , say  $s_w \neq t_w \in S$ , which decreases the length of  $w$  (as  $i > m$ ). Any other element  $r \in S \setminus \{s_w, t_w\}$  increases the length of  $w$ . For  $n > 3$  and  $r_1 \neq r_2 \in S \setminus \{s_w, t_w\}$ , we have  $f(wr_{\{r_1, r_2\}}) = w$ . For  $n = 3$ , we have  $f^{-1}(w) = \emptyset$ .

In both cases,  $w$  has  $\binom{n-2}{2}$  many preimages. If  $w \in D_{i-m}$  is, there exists a unique

$s_w \in S$  which decreases the length of  $w$  and (similarly as before)  $w$  has  $\binom{n-1}{2}$  many

preimages. Note that  $\binom{n-1}{2} - \binom{n-2}{2} = n - 2$ . We conclude

$$\begin{aligned}
c_i - d_i &= |C_i \setminus D_i| = \sum_{w \in C_{i-m}} |f^{-1}(w)| \\
&= \sum_{w \in C_{i-m} \setminus D_{i-m}} |f^{-1}(w)| + \sum_{w \in D_{i-m}} |f^{-1}(w)| \\
&= \binom{n-2}{2} (c_{i-m} - d_{i-m}) + \binom{n-1}{2} d_{i-m} \\
&= \binom{n-2}{2} c_{i-m} + (n-2) d_{i-m}.
\end{aligned}$$

□

**Lemma 4.4.**  $2c_{i+1} - d_{i+1} = (n-2)c_i + d_i$ .

**Proof.** We put  $M_i := \{(w, s) \in C_i \times S \mid ws \in C_{i+1}\}$ . We prove the claim by showing that both sides of the equation are equal to  $|M_i|$ .

(a)  $2c_{i+1} - d_{i+1} = |M_i|$ : We consider the mapping

$$\pi : M_i \rightarrow C_{i+1}, (w, s) \mapsto ws.$$

Clearly,  $\pi$  is surjective. We define

$$C_{i+1}^1 := \{w \in C_{i+1} \mid |\pi^{-1}(w)| = 1\} \quad \text{and} \quad C_{i+1}^{>1} := \{w \in C_{i+1} \mid |\pi^{-1}(w)| > 1\}.$$

We show that  $C_{i+1}^{>1} = C_{i+1} \setminus D_{i+1}$ . Let  $\bar{w} \in C_{i+1}^{>1}$  be an element. Then, there exist  $(w, s) \neq (w', s') \in \pi^{-1}(\bar{w})$ . It follows that  $s \neq s'$ , and hence  $\bar{w} \in C_{i+1} \setminus D_{i+1}$ . Now, let  $w \in C_{i+1} \setminus D_{i+1}$ . By Lemma 4.1, there exist exactly two elements  $s_w \neq t_w \in S$ , which decreases the length of  $w$ . This implies  $(ws_w, s_w) \neq (wt_w, t_w) \in \pi^{-1}(w)$ . As  $|\langle J \rangle| = \infty$  for all  $J \subseteq S$  containing three elements, we deduce for every  $1 \neq w \in W$  that

$$|\pi^{-1}(w)| \in \{1, 2\}.$$

We infer  $C_{i+1}^1 = C_{i+1} \setminus C_{i+1}^{>1} = C_{i+1} \setminus (C_{i+1} \setminus D_{i+1}) = D_{i+1}$  and compute

$$\begin{aligned}
|M_i| &= \sum_{w \in C_{i+1}} |\pi^{-1}(w)| = \sum_{w \in D_{i+1}} |\pi^{-1}(w)| + \sum_{w \in C_{i+1} \setminus D_{i+1}} |\pi^{-1}(w)| \\
&= d_{i+1} + 2(c_{i+1} - d_{i+1}) \\
&= 2c_{i+1} - d_{i+1}.
\end{aligned}$$

(b)  $(n-2)c_i + d_i = |M_i|$ : For a subset  $T \subseteq C_i$ , we define

$$M_{i,T} := \{(w, s) \in M_i \mid w \in T\}.$$

For  $w \in D_i$  there are exactly  $n-1$  elements which increase the length of  $w$ . Thus, we have  $|M_{i,D_i}| = (n-1)d_i$ . For  $w \in C_i \setminus D_i$ , there are exactly  $n-2$  elements in  $S$ , which increases the length of  $w$  (cf. Lemma 4.1). Thus, we have  $|M_{i,C_i \setminus D_i}| = (n-2)(c_i - d_i)$ . We conclude

$$|M_i| = |M_{i,C_i \setminus D_i}| + |M_{i,D_i}| = (n-2)(c_i - d_i) + (n-1)d_i = (n-2)c_i + d_i.$$

□

**Lemma 4.5.**  $c_{i+1} \leq (n-1)c_i - (n-2)d_{i-m+1} \leq (n-1)c_i$ .

**Proof.** The last inequality is obvious. Using Lemma 4.3 and 4.4, we deduce the following:

$$c_{i+1} + (n-2)d_{i-m+1} \leq 2c_{i+1} - d_{i+1} = (n-2)c_i + d_i \leq (n-1)c_i.$$

□

**Lemma 4.6.** Suppose  $m > 3$ . Then, the following holds:

- (a)  $(n-2)c_i \leq c_{i+1}$ ;
- (b)  $(n-2)d_i \leq d_{i+1}$ ;

**Proof.** We define  $N_i := \{(w, s) \in C_i \times S \mid ws \in D_{i+1}\}$ . Then,  $N_i \rightarrow D_{i+1}, (w, s) \mapsto ws$  is a bijection, and hence  $|N_i| = d_{i+1}$ . As in the proof of Lemma 4.4, we define for a subset  $T \subseteq C_i$ :

$$N_{i,T} := \{(w, s) \in N_i \mid w \in T\}.$$

We see that  $c_{i+1} \geq d_{i+1} = |N_i| = |N_{i,D_i}| + |N_{i,C_i \setminus D_i}|$ . Let  $w \in C_i$ . We now count pairs  $(w, s) \in N_i$ . We distinguish the following two cases:

- (i)  $w \in D_i$ : Let  $s_w \in S$  be the unique element with  $\ell(ws_w) < \ell(w)$ . Let  $t \in S \setminus \{s_w\}$ . Then,  $wt \in C_{i+1}$ . Suppose  $wt \notin D_{i+1}$ . Then, there exists  $t \neq r \in S$  with  $\ell(wtr) < \ell(wt)$ . This implies  $\ell(wr) < \ell(w)$ , and the uniqueness of  $s_w$  yields  $r = s_w$ . Now, let  $r \in S \setminus \{s_w, t\}$ . Then  $wr \in C_{i+1}$ . Again, if  $wr \notin D_{i+1}$ , then  $s_w$  would decrease the length of  $w$ . But this is a contradiction to Lemma 3.4. This implies  $(w, r) \in N_{i,D_i}$  for all  $r \in S \setminus \{s_w, t\}$ . This shows (b).
- (ii)  $w \in C_i \setminus D_i$ : Let  $s_w \neq t_w \in S$  be the two elements with  $\ell(ws_w) = \ell(wt_w) < \ell(w)$ . Now let  $r \in S \setminus \{s_w, t_w\}$ . Then,  $wr \in C_{i+1}$ . We assume by contradiction that  $wr \notin D_{i+1}$ . Then, there would exist  $u \in S \setminus \{r\}$  with  $\ell(wru) = \ell(w)$ , and hence  $\ell(wu) < \ell(w)$ . As  $s_w$  and  $t_w$  are the only two elements in  $S$  with the property that they decrease the length of  $w$ , we obtain  $u \in \{s_w, t_w\}$ . But then, we obtain a contradiction to Corollary 3.3. We conclude  $(w, r) \in N_{i,C_i \setminus D_i}$ .

We infer  $c_{i+1} \geq |N_{i,D_i}| + |N_{i,C_i \setminus D_i}| \geq (n-2)d_i + (n-2)(c_i - d_i) = (n-2)c_i$ .  $\square$

## 5. Main results

In this section, we prove our main results. In § 5.1, we recall a reduction result due to Terragni. In § 5.2, we use the reduction result to prove convergence of  $p_{(W,S)}\left(\frac{1}{n-1}\right)$ , where  $(W, S)$  is 2-spherical of rank  $n \geq 3$ . In § 5.3, we use the reduction result to prove divergence of  $p_{(W,S)}\left(\frac{1}{n-2}\right)$ , where  $(W, S)$  is of rank  $n \geq 4$  and the underlying Coxeter diagram is the complete graph.

### 5.1. Reduction step

Let  $(W, S)$  and  $(W', S')$  be two Coxeter systems. Following [20], we define  $(W, S) \preceq (W', S')$  if there exists an injective map  $\varphi : S \rightarrow S'$  satisfying  $m_{st} \leq m'_{\varphi(s)\varphi(t)}$  for all  $s, t \in S$ .

**Theorem 5.1.** *Let  $(W, S)$  and  $(W', S')$  be two Coxeter systems, and let  $c_n := c_n^{(W,S)}$  and  $c'_n := c_n^{(W',S')}$ . If  $(W, S) \preceq (W', S')$ , then  $c_n \leq c'_n$ .*

**Proof.** This is [20, Theorem A].  $\square$

### 5.2. Convergence

**Lemma 5.2.** *Let  $(W, S)$  be of rank  $n \geq 3$ , and assume that there exists  $m \geq 4$  such that  $m_{st} = m$  holds for all  $s \neq t \in S$ . Then, there exists  $k \in \mathbb{R}$  such that  $\frac{d_i}{c_i} \geq k > 0$  holds for all  $i > m$ .*

**Proof.** Using Lemma 4.3 and 4.6, we compute

$$\begin{aligned} 1 = \frac{c_i - d_i + d_i}{c_i} &= \frac{1}{c_i} \left( \binom{n-2}{2} c_{i-m} + (n-2)d_{i-m} + d_i \right) \\ &\leq \frac{1}{c_i} \left( \binom{n-2}{2} \frac{1}{(n-2)^m} c_i + \left( \frac{1}{(n-2)^{m-1}} + 1 \right) d_i \right) \\ &= \frac{1}{c_i} \left( \frac{(n-3)}{2(n-2)^{m-1}} c_i + \left( \frac{1}{(n-2)^{m-1}} + 1 \right) d_i \right) \\ &\leq \frac{1}{2(n-2)^{m-2}} + \left( \frac{1}{(n-2)^{m-1}} + 1 \right) \frac{d_i}{c_i}. \end{aligned}$$

We put

$$k := \left( 1 - \frac{1}{2(n-2)^{m-2}} \right) \cdot \left( \frac{1}{(n-2)^{m-1}} + 1 \right)^{-1}.$$

As  $n \geq 3$  and  $m \geq 4$ , we have  $k > 0$ . This proves the claim.  $\square$

**Theorem 5.3.** *Let  $(W, S)$  be 2-spherical and of rank  $n \geq 3$ . Then,  $p_{(W, S)}\left(\frac{1}{n-1}\right) < \infty$ .*

**Proof.** Let  $m := \max\{4, m_{st} \mid s, t \in S\}$ , and let  $(W', S')$  be the Coxeter system of rank  $n$  with  $m'_{st} = m$  for all  $s \neq t \in S'$ . Using Theorem 5.1, it suffices to show that

$$p_{(W', S')}\left(\frac{1}{n-1}\right) < \infty.$$

By Lemma 5.2, there exists  $k \in \mathbb{R}$  such that  $\frac{d_i}{c_i} \geq k > 0$  holds for all  $i > m$ . We apply the ratio test. We use Lemma 4.5 and compute for  $i > 2m - 1$  and  $t = \frac{1}{n-1}$ :

$$\frac{c_{i+1}t^{i+1}}{c_i t^i} \leq \frac{(n-1)c_i - (n-2)d_{i-m+1}}{(n-1)c_i} \leq 1 - \frac{(n-2)d_{i-m+1}}{(n-1)^m c_{i-m+1}} \leq 1 - \frac{n-2}{(n-1)^m} k < 1.$$

$\square$

### 5.3. Divergence

In this subsection, we prove that the new lower bound  $\frac{1}{n-1}$  for the finiteness of the growth function is optimal for the class of 2-spherical Coxeter systems with a complete Coxeter diagram.

**Lemma 5.4.** *Let  $(W, S)$  be 2-spherical and of rank  $n \geq 4$ , and assume that the underlying Coxeter diagram is the complete graph. Then  $(n-2)c_i \leq d_i + d_{i+1}$ .*

**Proof.** For  $i = 0$ , we have  $c_0 = 1$ ,  $d_0 = 0$  and  $d_1 = n$ , and the claim follows. Thus, we can assume  $i > 0$ . As in Lemma 4.6, we define  $N_i := \{(w, s) \in C_i \times S \mid ws \in D_{i+1}\}$  as well as  $N_{i,T} := \{(w, s) \in N_i \mid w \in T\}$  for  $T \subseteq C_i$ . We consider the mapping

$$\pi : N_i \rightarrow D_{i+1}, (w, s) \mapsto ws.$$

As before,  $\pi$  is a bijection and we have  $|N_i| = d_{i+1}$ . Moreover, we have  $N_i = N_{i,D_i} \cup N_{i,C_i \setminus D_i}$  and this union is disjoint. We now count pairs  $(w, s)$  in  $N_i$ .

We fix  $w \in D_i$ , and we let  $s_w \in S$  be the unique element with  $\ell(ws_w) = \ell(w) - 1$ . Assume that there are  $r, s, t \in S \setminus \{s_w\}$  pairwise distinct with  $wr, ws, wt \in C_{i+1} \setminus D_{i+1}$ . Similarly, as in Lemma 4.6(b), we deduce  $\ell(wzs_w) = \ell(w)$  for each  $z \in \{r, s, t\}$ . As  $m_{pq} \geq 3$  holds for all  $p \neq q \in S$ , we infer  $\ell(ws_w z) = \ell(ws_w) - 1$ . As  $\{r, s, t\}$  is not spherical, this is a contradiction and we have for a fixed  $w \in D_i$  at least  $n-3$  tuples  $(w, s)$  in  $N_i$ .

We fix  $w \in C_i \setminus D_i$ , and we let  $s_w \neq t_w \in S$  be the two elements with  $\ell(ws_w) = \ell(w) - 1 = \ell(wt_w)$ . Assume that there is  $s \in S \setminus \{s_w, t_w\}$  with  $ws \in C_{i+1} \setminus D_{i+1}$ . Then,  $\ell(w) \in \{\ell(wss_w), \ell(wst_w)\}$ . W.l.o.g. we assume  $\ell(wss_w) = \ell(w)$ . But then Corollary 3.3 implies  $\ell(wt_w) = \ell(w) + 1$ , which is a contradiction. Thus, we have for a fixed  $w \in C_i \setminus D_i$

exactly  $n - 2$  tuples  $(w, s)$  in  $N_i$  (cf. Lemma 4.1). This implies that  $(n - 2)c_i - d_i = (n - 3)d_i + (n - 2)(c_i - d_i) \leq d_{i+1}$ .  $\square$

**Theorem 5.5.** *Let  $(W, S)$  be of rank  $n \geq 4$ , and assume that the underlying Coxeter diagram is the complete graph. Then,  $p_{(W, S)}\left(\frac{1}{n-2}\right) = \infty$ .*

**Proof.** Let  $(W', S')$  be the Coxeter system of rank  $n$  with  $m'_{st} = 3$  for all  $s \neq t \in S'$ . Using Theorem 5.1, it suffices to show that

$$p_{(W', S')}\left(\frac{1}{n-2}\right) = \infty.$$

As before, we apply the ratio test. Using Lemmas 4.4 and 5.4, we deduce the following for  $i > m = 3$  and  $t = \frac{1}{n-2}$ :

$$\frac{c_{i+1}t^{i+1}}{c_it^i} = \frac{(n-2)c_i + d_i + d_{i+1}}{2(n-2)c_i} = \frac{1}{2} + \frac{d_i + d_{i+1}}{2(n-2)c_i} \geq \frac{1}{2} + \frac{1}{2} = 1.$$

$\square$

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