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SHORT NOTE

STEADY PROFILE OF A FINITE-AMPLITUDE KINEMATIC WAVE ON A GLACIER*

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ABSTRACT. The steady profile of a finite-amplitude kinematic wave on a glacier is calculated with the assumption that the wave velocity varies linearly with h_1, the departure of the ice thickness from the datum state. If the flow of ice is due mainly to sliding of the glacier on its bed, the width of the calculated steady profile is several hundred times the datum state ice thickness. The width of an observed kinematic wave front on Nisqually Glacier, Mt. Rainier, Washington is at least an order of magnitude smaller than the calculated steady profile. This indicates that the observed steepening of the wave may be due to effects other than variation of wave velocity with ice thickness.

RÉSUMÉ. *Profil permanent d'une onde cinématique d'amplitude finie sur un glacier.* Le profil permanent d'une onde cinématique d'amplitude finie sur un glacier est calculé suivant l'hypothèse que la vitesse de l'onde varie linéairement avec h_1, l'épaisseur au départ de la glace à la date initiale. Si l'ecoulement de la glace est dû principalement au glissement sur son lit, l'étendue du profil permanent calculé est plusieurs centaines de fois égale à l'épaisseur initiale de la glace. L'étendue du front d'une onde cinématique observée sur le Nisqually Glacier, Mt. Rainier, Washington, est finalement d'un ordre de grandeur plus petite que celle du profil permanent calculé. Cela indique que l'augmentation de pente observée de l'onde peut être due à d'autres effets qu'a celui de la variation de la vitesse de l'onde avec l'épaisseur de la glace.

ZUSAMMENFASSUNG. *Stationäres Profil einer kinematischen Welle mit begrenzter Amplitude auf einem Gletscher.* Unter der Annahme, dass die Wellengeschwindigkeit sich linear mit h_1, dem Ausgangswert der Eisdicke, ändert, wird das stationäre Profil einer kinematischen Welle mit begrenzter Amplitude auf einem Gletscher berechnet. Wenn die Eisbewegung hauptsächlich aus dem Gleiten am Untergrund besteht, beträgt die Längsausdehnung des berechneten stationären Profiles das Mehrhundertfache der Ausgangseisdicke. Die Längsausdehnung einer am Nisqually-Gletscher, Mt. Rainier, Washington, beobachteten Wellenfront ist wenigstens eine Grössenordnung kleiner als das berechnete stationäre Profil. Dies weist darauf hin, dass die beobachtete Versteilung der Welle durch andere Einflüsse als die Änderung der Wellengeschwindigkeit mit der Eisdicke verursacht sein mag.

Previous theoretical discussions (Nye, 1960; Weertman, 1958) of kinematic wave propagation on glaciers generally ignore a quantitative description of non-linear effects; only small perturbations from the datum state are considered. There are several reasons for this neglect. In the first place, non-linear partial differential equations are difficult to solve analytically, and secondly the linear case has not yet been completely investigated. A particular example of a non-linear effect is the variation of wave velocity with ice thickness and the idea that discontinuities or shock waves may develop in kinematic waves on glaciers (Nye, 1958; Meier and Johnson, 1962). However, because of the dependence of the flow q on the slope of the glacier surface, a diffusive term is introduced into the wave equation and this limits the steepness of wave fronts that may develop (Nye, 1960). Here q is defined as the volume of ice, for unit glacier width, passing a point in unit time. The diffusive term has the effect of spreading out the kinematic wave front so that discontinuities cannot develop. The competition between the increase of wave speed with increasing ice thickness and the diffusive effect may result in a steady profile of the kinematic wave, if the wave is allowed to travel far enough. By the word "steady profile" I mean that if an observer were traveling with the wave velocity in the direction of wave propagation, he would see no change in the wave shape. The idea of a steady profile is very well known in the field of shock wave propagation in gases and solids (Rayleigh, 1910; Band and Duvall, 1961). For example, in gases the wave speed increases with pressure and therefore compressive waves tend to form discontinuities. However, because of thermal conduction and viscosity, actual discontinuities cannot form and steady profiles may be achieved.

As in the case of shock waves in solids and gases, it is possible to find a mathematical expression for the steady profile of a kinematic wave on a glacier. Let $h_1(x, t)$ be the departure of ice thickness from some datum state $h_0(x)$. Here x is the position coordinate in the direction of flow and t is the time. We shall

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suppose the datum state in the region of interest to be $h_0(x) = h_0$, a constant. That is, we are looking at the propagation of waves in a region where the glacier thickness is nearly uniform. We shall ignore the loss or accumulation of ice at the glacier surface and assume the relationship between the flow, ice thickness, and surface slope to be independent of position. Then, following a method similar to that of Nye (1960), we obtain the following differential equation for $h_1(x, t)$:

$$\partial h_1/\partial t + (C_0 + B_0 h_1)\, \partial h_1/\partial x = D_0\, \partial^2 h_1/\partial x^2. \tag{1}$$

The difference between Equation (1) and the equation obtained by Nye, besides the additional restrictions I have already mentioned, is the non-constant wave velocity. In Equation (1) the wave velocity varies linearly with h_1. The constants C_0, B_0, and D_0 are defined as follows: $C_0 = (\partial q/\partial h)_0$, $B_0 = (\partial^2 q/\partial h^2)_0$, and $D_0 = (\partial q/\partial \alpha)_0$, where α is the surface slope and the subscript zero means the partial derivatives are to be evaluated at the datum state. Because of the variation of wave velocity with h_1, a steady profile may form. To find an expression for the steady profile let us imagine a wave of amplitude H propagating into an undisturbed region of the glacier where the ice thickness is $h_0 - \frac{1}{2}H$. The ice thickness behind the wave will then be $h_0 + \frac{1}{2}H$. If the wave has achieved a steady profile, the velocity with which the steady profile propagates is C_0. This is easily verified by consideration of mass conservation. Since we have assumed the profile to be steady in time, we look for a solution of Equation (1) in the form $h_1 = h_1(x - C_0 t)$. Such a solution which satisfies the necessary requirements is easily found:

$$h_1(x - C_0 t) = (\tfrac{1}{2}H) \tanh[-B_0 H(x - C_0 t)/4D_0]. \tag{2}$$

From Equation (2) it is seen that h_1 asymptotically approaches $-\frac{1}{2}H$ and $+\frac{1}{2}H$ at points ahead of and behind the wave, respectively. We can estimate the width of the wave in the following way: Define the width W as the distance between the points at which $h_1 = 0.9(\frac{1}{2}H)$ and $h_1 = -0.9(\frac{1}{2}H)$. Then

$$W = (8D_0/B_0 H) \tanh^{-1}(0.9) \approx (12D_0/B_0 H). \tag{3}$$

For calculational purposes we shall suppose that the motion of the glacier is due entirely to slipping on its bed so the relationship between q, h, and α has the form (Nye, 1960)

$$q = \theta h^{m+1} \alpha^m \tag{4}$$

where θ and m are constants ($m \approx 2$). Here we have made the approximation that $\sin\alpha \approx \alpha$, $\alpha \ll 1$. From Equations (3) and (4) and the definition of the constants B_0 and D_0 we find

$$W \approx [12h_0/(m+1) H\alpha_0]\, h_0. \tag{5}$$

Hence, if we take $H/h_0 = 0.10$ and $\alpha_0 = 0.10$, we find $W \approx 400h_0$; the steady profile of the wave is very much larger than the datum-state ice thickness.

The conclusion we may draw from this result is that, although the non-constant wave velocity may in principle cause the kinematic wave front to steepen, the diffusive term in the wave equation, Equation (1), has such a strong influence that the steady profile, toward which any wave-form tends, is very wide. It has been previously reported (Meier and Johnson, 1962) that a kinematic wave on Nisqually Glacier was developing a shock front. The width of the wave front was more than an order of magnitude smaller than the calculated width of the steady profile, Equation (5). This suggests that the observed steepening of the wave front on Nisqually Glacier may be due to effects other than the variation of wave velocity with ice thickness.

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