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# PURITY AND COPURITY IN SYSTEMS OF LINEAR TRANSFORMATIONS

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**1. Introduction.** Consider a system of $N$ linear transformations $A_1, \ldots, A_N: V \rightarrow W$, where $V$ and $W$ are complex vector spaces. Denote it for short by $(V, W)$. A pair of subspaces $X \subset V$, $Y \subset W$ such that $\sum_{j=1}^{N} A_j X \subset Y$ determines a subsystem $(X, Y)$ and a quotient system $(V/X, W/Y)$ (with the induced transformations). The subsystem $(X, Y)$ is of *finite codimension* in $(V, W)$ if and only if $V/X$ and $W/Y$ are finite-dimensional. It is a *direct summand* of $(V, W)$ in case there exist supplementary subspaces $P$ of $X$ in $V$ and $Q$ of $Y$ in $W$ such that $(P, Q)$ is a subsystem.

The main result is that if $(X, Y)$ is of finite codimension in $(V, W)$ and for every subsystem $(U, Z)$ of finite codimension in $(X, Y)$, $(X/U, Y/Z)$ is a direct summand of $(V/U, W/Z)$, then $(X, Y)$ is a direct summand of $(V, W)$.

The proof uses a dual theorem (Aronszajn and Fixman [**2**, Theorem 5.5] in case $N = 2$) and topological systems of $N$ *continuous* linear transformations between topological vector spaces.

**2. Topological $\mathbf{C}^N$-systems.** We begin with an outline of some terminology and facts, referring the reader to [**4**] for further explications. A *topological* $\mathbf{C}^N$*-system* is a pair $(V, W)$ of complex, separated, locally convex topological vector spaces, along with a system operation assigning to every $N$-tuple $e \in \mathbf{C}^N$ and $v \in V$ an element $ev \in W$ such that

(i) for each $e \in \mathbf{C}^N$, the map $v \rightarrow ev$ is a continuous linear transformation of $V$ to $W$, and

(ii) $(\alpha_1 e_1 + \alpha_2 e_2)v = \alpha_1(e_1 v) + \alpha_2(e_2 v)$ for all $v \in V$, $e_1, e_2 \in \mathbf{C}^N$, $\alpha_1, \alpha_2 \in \mathbf{C}$.

Considering the maps $A_j: v \rightarrow e_j v$, where $(e_j)_{j=1}^{N}$ is the canonical basis of $\mathbf{C}^N$, we see that the present concept of topological system is equivalent to that of Section 1.

A *subsystem* $(X, Y)$ of $(V, W)$ is a pair of subspaces $X$ of $V$ and $Y$ of $W$ such that, for $e \in \mathbf{C}^N$ and $x \in X$ we get $ex \in Y$. The pair $(X, Y)$ is itself a topological $\mathbf{C}^N$-system, with the action of $\mathbf{C}^N$ from $X$ to $Y$ induced by its action from $V$ to $W$. A subsystem $(X, Y)$ of $(V, W)$ is said to be *closed* in $(V, W)$ in case $X$ is closed in $V$ and $Y$ is closed in $W$. By convention, *all subsystems considered here are closed.* The *quotient system* $(V/X, W/Y)$ is determined by a subsystem

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$(X, Y)$ of $(V, W)$, by taking $V/X$, $W/Y$ as the usual separated, locally convex quotient spaces. Each $e \in \mathbf{C}^N$ operates on any $v + X \in V/X$ according to $e(v + X) = ev + Y$. We call $(X, Y)$ a *direct summand* of $(V, W)$ whenever there exist continuous projections of $V$ onto $X$ and $W$ onto $Y$ with respective kernels $P$ and $Q$, such that the pair $(P, Q)$ forms a subsystem of $(V, W)$. A topological $\mathbf{C}^N$-system $(V, W)$ is called *finite-dimensional* if $V$ and $W$ are finite-dimensional. A subsystem $(X, Y)$ is of *finite codimension* in $(V, W)$ if the quotient system $(V/X, W/Y)$ is finite-dimensional. In this case $(V, W)$ is called a *finite-dimensional extension of* $(X, Y)$. The subsystem $(X, Y)$ is *pure* in $(V, W)$ whenever it is a direct summand of every finite-dimensional extension of $(X, Y)$ inside $(V, W)$. Dually, $(X, Y)$ is *copure* in $(V, W)$ if the quotient system $(X/U, Y/Z)$ is a direct summand of $(V/U, W/Z)$, for any subsystem $(U, Z)$ of finite codimension in $(X, Y)$. For example, a direct summand $(X, Y)$ of $(V, W)$ is both pure and copure in $(V, W)$.

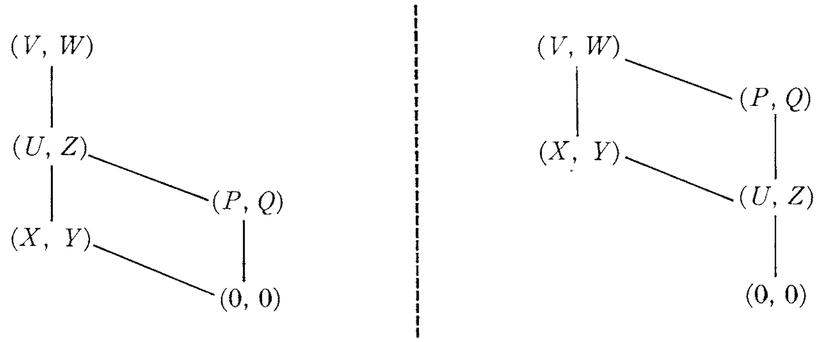
PROPOSITION 1. *Let $(X, Y)$ be a subsystem of a topological $\mathbf{C}^N$-system $(V, W)$. Then $(X, Y)$ is pure in $(V, W)$ if and only if for any finite-dimensional extension $(U, Z)$ of $(X, Y)$ there exists a subsystem $(P, Q)$ of $(U, Z)$ such that*

$$P + X = U, \quad Q + Y = Z \quad \text{and} \quad P \cap X = 0, \quad Q \cap Y = 0.$$

*Dually, $(X, Y)$ is copure in $(V, W)$ if and only if for any subsystem $(U, Z)$ of finite codimension in $(X, Y)$ there exists a subsystem $(P, Q)$ of $(V, W)$ such that*

$$P + X = V, \quad Q + Y = W \quad \text{and} \quad P \cap X = U, \quad Q \cap Y = Z.$$

The above lattice descriptions of purity and copurity, whose proofs are straight forward, can be summed up by the diagrams below.




Pure Copure

For a topological vector space $V$, let $V'$ denote its dual space endowed with the $\sigma(V', V)$ topology (see e.g. [**6**]). The value of a functional $v' \in V'$ at a vector $v \in V$ is written as $\langle v, v' \rangle$. The *dual* of a topological $\mathbf{C}^N$-system $(V, W)$ is $(W', V')$, where the operation of $\mathbf{C}^N$ from $W'$ to $V'$ is given by the transpose

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rule:

$$\langle v, ew' \rangle = \langle ev, w' \rangle \quad \text{for } v \in V, w' \in W', e \in \mathbf{C}^N.$$

To a closed subspace $X$ of a locally convex space $V$ we associate its *polar* $X^0 = \{v' \in V' : \langle x, v' \rangle = 0, \text{ for all } x \in X\}$ inside $V'$. The *polar of a subsystem* $(X, Y)$ of $(V, W)$ is defined as the subsystem $(Y^0, X^0)$ of $(W', V')$. The operation of taking polars is an anti-isomorphism of the lattice of closed subspaces of a locally convex space $V$ onto the lattice of $\sigma(V', V)$-closed subspaces of its dual $V'$. Therefore, there is a lattice anti-isomorphism between the subsystems of a topological $\mathbf{C}^N$-system and the subsystems of its dual, given by taking polars of subsystems. Using these facts and the characterization of pure and copure subsystems given in the former lattice diagrams, one can readily verify the following.

PROPOSITION 2. *A subsystem $(X, Y)$ of a topological $\mathbf{C}^N$-system $(V, W)$ is pure (copure) in $(V, W)$ if and only if its polar $(Y^0, X^0)$ is copure (pure) in $(W', V')$.*

**3. Algebraic $\mathbf{C}^N$-systems.** If, in the definition of a topological $\mathbf{C}^N$-system $(V, W)$, we only require that $V$ and $W$ be ordinary complex vector spaces and place no continuity condition on the system operation, we get what is called an *algebraic $\mathbf{C}^N$-system*. Every algebraic $\mathbf{C}^N$-system can be considered as a topological $\mathbf{C}^N$-system by putting on $V$ and $W$ their finest weak topologies. In the dual $(W', V')$ of an algebraic $\mathbf{C}^N$-system $(V, W)$, the spaces $W'$, $V'$ contain all linear functionals on $W$ and $V$ respectively. Note that if $(V, W)$ is not finite-dimensional, then $(W', V')$ fails to be an algebraic $\mathbf{C}^N$-system, owing to the $\sigma(W', W)$ topology of $W'$, and the similar one on $V'$. Every topological $\mathbf{C}^N$-system gives an underlying algebraic $\mathbf{C}^N$-system upon forgetting the topologies of the spaces involved.

**4. Copure subsystems of finite codimension.** Aronszajn and Fixman's work in [**2**] is about algebraic $\mathbf{C}^2$-systems. Upon noting that "spectral" and "quasi-spectral" were used in [**2**] instead of "direct summand" and "pure", we restate Theorem 5.5 of [**2**] not just for $N = 2$ but for any $N$. The proof in [**2**] still holds with trivial changes.

THEOREM 3 (Aronszajn-Fixman). *Every finite-dimensional pure subsystem of an algebraic $\mathbf{C}^N$-system is a direct summand.*

As noted in the introduction, the dual of this result (in our present terminology) is the following.

THEOREM 4. *Every copure subsystem of finite codimension in an algebraic $\mathbf{C}^N$-system is a direct summand.*

To prove Theorem 4 we need the criterion of [**4**, Theorem 4.5] for a finite-dimensional subsystem of a *topological* $\mathbf{C}^N$-system to be a direct summand.

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To restate it, let us assume $(V, W)$ is again a topological $\mathbf{C}^N$-system and $(X, Y)$ a finite-dimensional subsystem. Choose bases $\{x_1, \ldots, x_m\}$ of $X$ and $\{y_1, \ldots, y_n\}$ of $Y$, and let $\{x_1', \ldots, x_m'\}$, $\{y_1', \ldots, y_n'\}$ be their dual bases in $X'$ and $Y'$. Consider the tensor product spaces $V \otimes X'$ and $W \otimes Y'$, along with the finest topologies on them making each of the tensor mappings $V \times X' \to V \otimes X'$ and $W \times Y' \to W \otimes Y'$ separately continuous. Let $R$ be the subspace of the topological direct sum $V \otimes X' \oplus W \otimes Y'$ generated algebraically by all terms of the form $(v \otimes ey', -ev \otimes y')$, with $v \in V$, $y' \in Y'$ and $e \in \mathbf{C}^N$. We denote by $\bar{R}$ the closure of $R$ in $V \otimes X' \oplus W \otimes Y'$. The terms

$$\{(x_i \otimes x_j', 0), (0, y_k \otimes y_l') \text{ for } i, j = 1, \ldots, m; k, l = 1, \ldots, n\}$$

in $V \otimes X' \oplus W \otimes Y'$ are *independent of $\bar{R}$ up to a zero trace* in case every pair of linear combinations

$$(\sum c_{ij} x_i \otimes x_j', \sum d_{kl} y_k \otimes y_l'),$$

which belongs to $\bar{R}$ must satisfy $\sum c_{ii} + \sum d_{kk} = 0$. According to [**4**, Theorem 4.5] this trace condition is necessary and sufficient in order that $(X, Y)$ be a direct summand of $(V, W)$.

It is clearly "easier" for $(X, Y)$ to be a direct summand of the underlying algebraic $\mathbf{C}^N$-system of $(V, W)$. This happens if and only if the terms

$$\{(x_i \otimes x_j', 0), (0, y_k \otimes y_l')\}$$

are independent of $R$ (no closure) up to a zero trace. Thus, whenever $R = \bar{R}$, the system $(X, Y)$ is a direct summand of the *topological* $\mathbf{C}^N$-system $(V, W)$ if and only if $(X, Y)$ is a direct summand of the underlying *algebraic* $\mathbf{C}^N$-system.

LEMMA 5. *If $(Q, P)$ is an algebraic $\mathbf{C}^N$-system, then its dual $(P', Q')$ (which is topological) is such that, for any finite-dimensional subsystem $(X, Y)$, the subspace $R$ of $P' \otimes X' \oplus Q' \otimes Y'$ generated algebraically by the terms $(p' \otimes ey', -ep' \otimes y')$ is already closed in $P' \otimes X' \oplus Q' \otimes Y'$.*

*Proof.* The plan is to show that any convergent net from $R$ has its limit inside $R$. Let $e^1, \ldots, e^N$ be a base of $\mathbf{C}^N$, $\{x_1, \ldots, x_m\}$, $\{y_1, \ldots, y_n\}$ bases of $X$ and $Y$ respectively and $\{x_1', \ldots, x_m'\}$, $\{y_1', \ldots, y_n'\}$ the dual bases in $X'$ and $Y'$. A typical element of $P' \otimes X' \oplus Q' \otimes Y'$ can be written in the form

$$\left(\sum_{i=1}^{m} p_i' \otimes x_i', \sum_{j=1}^{n} q_j' \otimes y_j'\right).$$

According to [**4**, Proposition 4.6], any element of $R$ is of the form

$$\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{k=1}^{N} e_{ji}{}^{k} p_{jk}'\right) \otimes x_i', -\sum_{j=1}^{n}\left(\sum_{k=1}^{N} e^k p_{jk}'\right) \otimes y_j'\right),$$

where the $nN$ $p_{jk}'$'s belong to $P'$, and the matrix $(e_{ij}{}^{k})$ is defined by $e^k x_i = \sum_{j=1}^{n} e_{ji}{}^{k} y_j$. Denote any such element of $R$ by the symbol $[p_{jk}']$.

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Suppose a net $([p_{jk}'(\alpha)])_{\alpha \in D}$ from $R$ (with $D$ a directed set) converges in $P' \otimes X' \oplus Q' \otimes Y'$ to $(\sum_{i=1}^{m} p_i' \otimes x_i', \sum_{j=1}^{n} q_j' \otimes y_j')$. We need to find an $n$ by $N$ array of functionals $p_{jk}': P \to \mathbf{C}$ with $j = 1, \ldots, n$, $k = 1, \ldots, N$ such that the limit point equals $[p_{jk}']$. Owing to [**4**, Proposition 4.7], the convergence of the above net means that

$$\sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}^{k} p_{jk}'(\alpha) \to p_i' \text{ in } P', \quad \text{for } i = 1, \ldots, m, \text{ and}$$

$$\sum_{k=1}^{N} e^k p_{jk}'(\alpha) \to -q_j' \text{ in } Q', \quad \text{for } j = 1, \ldots, n.$$

We recall here that $P'$ and $Q'$ consist of *all* linear functionals on $P$ and $Q$ (because $(Q, P)$ is an algebraic $\mathbf{C}^N$-system); and that the above limits are taken with the $\sigma(P', P)$ and $\sigma(Q', Q)$ topologies on $P'$ and $Q'$ respectively.

There is a natural one-to-one correspondence between $n$ by $N$ arrays of functionals $p_{jk}': P \to \mathbf{C}$, $j = 1, \ldots, n$, $k = 1, \ldots, N$, and single functionals $F$: $P \otimes Y \otimes \mathbf{C}^N \to \mathbf{C}$, as follows. Express each element of $P \otimes Y \otimes \mathbf{C}^N$ as $\sum_{j=1}^{n} \sum_{k=1}^{N} p_{jk} \otimes y_j \otimes e^k$, with the $p_{jk}$'s uniquely determined by the element. The functionals $p_{jk}'$ determine $F: P \otimes Y \otimes \mathbf{C}^N \to \mathbf{C}$ via

$$F\left( \sum_{j=1}^{n} \sum_{k=1}^{N} p_{jk} \otimes y_j \otimes e^k \right) = \sum_{j=1}^{n} \sum_{k=1}^{N} \langle p_{jk}, p_{jk}' \rangle.$$

The inverse map attaches to $F: P \otimes Y \otimes \mathbf{C}^N \to \mathbf{C}$ the $n$ by $N$ array of functionals $(p_{jk}')$ defined by

$$\langle p, p_{jk}' \rangle = F(p \otimes y_j \otimes e^k), \quad \text{for all } p \in P.$$

For each $\alpha \in D$ let $F_\alpha$ be the functional on $P \otimes Y \otimes \mathbf{C}^N$ corresponding to the array $(p_{jk}'(\alpha))$. Now for $p \in P$ and $i = 1, \ldots, m$,

$$F_\alpha\left( \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}^{k} p \otimes y_j \otimes e^k \right) = \sum_{j=1}^{n} \sum_{k=1}^{N} \langle e_{ji}^{k} p, p_{jk}'(\alpha) \rangle$$

$$= \langle p, \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}^{k} p_{jk}'(\alpha) \rangle \to \langle p, p_i' \rangle,$$

as $\alpha$ runs over $D$. Similarly for $q \in Q$ and $j = 1, \ldots, n$,

$$F_\alpha\left( \sum_{k=1}^{N} e^k q \otimes y_j \otimes e^k \right) = \sum_{k=1}^{N} \langle e^k q, p_{jk}'(\alpha) \rangle$$

$$= \sum_{k=1}^{N} \langle q, e^k p_{jk}'(\alpha) \rangle = \langle q, \sum_{k=1}^{N} e^k p_{jk}'(\alpha) \rangle \to \langle q, -q_j' \rangle,$$

as $\alpha$ runs over $D$. Thus $(F_\alpha)_{\alpha \in D}$ is a net of functionals on $P \otimes Y \otimes \mathbf{C}^N$, which converges pointwise on the subspace $E$ generated by terms of the form $\sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}^{k} p \otimes y_j \otimes e^k$ and $\sum_{k=1}^{N} e^k q \otimes y_j \otimes e^k$, with $p \in P, q \in Q, i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Let $F$: $P \otimes Y \otimes \mathbf{C}^N \to \mathbf{C}$ be any extension of the pointwise limit of the functionals $F_\alpha|_E$: $E \to \mathbf{C}$. The desired array of functionals $(p_{jk}')$ inside $P'$ is then given by

$$\langle p, p_{jk}' \rangle = F(p \otimes y_j \otimes e^k)$$

for each $p \in P$, $j = 1, \ldots, n$, $k = 1, \ldots, N$. For then we have

$$\langle p, p_i' \rangle = \lim_\alpha F_\alpha \left( \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}{}^k p \otimes y_j \otimes e^k \right)$$

$$= F\left( \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}{}^k p \otimes y_j \otimes e^k \right)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}{}^k \langle p, p_{jk}' \rangle = \langle p, \sum_{j=1}^{n} \sum_{k=1}^{N} e_{ji}{}^k p_{jk}' \rangle$$

for all $p \in P$, $i = 1, \ldots, m$. Thus $p_i' = \sum_{j=1}^n \sum_{k=1}^N e_{ji}{}^k p_{jk}'$. Similarly,

$$\langle q, -q_j' \rangle = \lim_\alpha F_\alpha \left( \sum_{k=1}^{N} e^k q \otimes y_j \otimes e^k \right) = F\left( \sum_{k=1}^{N} e^k q \otimes y_j \otimes e^k \right)$$

$$= \sum_{k=1}^{N} \langle e^k q, p_{jk}' \rangle = \langle q, \sum_{k=1}^{N} e^k p_{jk}' \rangle$$

for all $q \in Q$, $j = 1, \ldots, n$. Thus $q_j' = -\sum_{k=1}^N e^k p_{jk}'$.
This means exactly that

$$\left( \sum_{i=1}^{m} p_i' \otimes x_i', \sum_{j=1}^{n} q_j' \otimes y_j' \right) = [p_{jk}'],$$

which is the desired result.

*Proof of Theorem* 4. Let $(Q, P)$ be an algebraic $\mathbf{C}^N$-system with a copure subsystem $(T, S)$ of finite codimension in $(Q, P)$. It is easy to see that inside the dual system $(P', Q')$ the polar subsystem $(S^0, T^0)$ is finite-dimensional; and, due to Proposition 2, it is pure in $(P', Q')$. Clearly $(S^0, T^0)$ is also pure in the algebraic $\mathbf{C}^N$-system underlying $(P', Q')$. By Theorem 3 $(S^0, T^0)$ is a direct summand of the *algebraic* $\mathbf{C}^N$-system underlying $(P', Q')$. Then Lemma 5 and the comment preceding it yield that $(S^0, T^0)$ is a direct summand in the *topological* $\mathbf{C}^N$-system $(P', Q')$.

So there are $\sigma(P', P)$- and $\sigma(Q', Q)$-continuous projections $P' \to S^0$ and $Q' \to T^0$ such that their kernels $K$ and $L$ respectively form a subsystem $(K, L)$ of $(P', Q')$. These projections afford a decomposition of $P'$ as the direct sum of $S^0$ and $K$, and of $Q'$ as the direct sum of $T^0$ and $L$. That is,

$$P' = S^0 + K \quad \text{and} \quad 0 = S^0 \cap K$$
$$Q' = T^0 + L \quad \text{and} \quad 0 = T^0 \cap L.$$

The dual of $P'$ is $P$ (since every $\sigma(P', P)$-continuous functional on $P'$ identifies with an element of $P$). Take polars in $P$ of $S^0$ and $K$. The polar of $S^0$ is $S$;

and since $K$ and $S^0$ are closed, we infer from the above lattice relations that

$$0 = S \cap K^0 \quad \text{and} \quad P = S + K^0.$$

Similarly,

$$0 = T \cap L^0 \quad \text{and} \quad Q = T + L^0.$$

From these lattice conditions we have projections $Q \to T$ and $P \to S$ with kernels $L^0$ and $K^0$ respectively. In addition $(L^0, K^0)$ is a subsystem of $(Q, P)$ because $(K, L)$ was a subsystem of $(P', Q')$. Thus $(T, S)$ is a direct summand of $(Q, P)$.

**5. Equivalence of purity and copurity.** Theorems 3 and 4 together yield the following.

THEOREM 6. *A subsystem of an algebraic* $\mathbf{C}^N$*-system is pure if and only if it is copure.*

*Proof.* Let $(V, W)$ be an algebraic $\mathbf{C}^N$-system and $(X, Y)$ a subsystem. Suppose $(X, Y)$ is pure in $(V, W)$. Testing for its copurity let $(U, Z)$ be of finite codimension in $(X, Y)$. Then the finite-dimensional quotient system $(X/U, Y/Z)$ is pure in $(V/U, W/Z)$, (see [**2**, Proposition 5.3 (d)]). By Theorem 3 $(X/U, Y/Z)$ is a direct summand of $(V/U, W/Z)$. Hence $(X, Y)$ is copure in $(V, W)$.

Conversely, suppose $(X, Y)$ is copure in $(V, W)$. To test for purity let $(U, Z)$ be a finite-dimensional extension of $(X, Y)$ in $(V, W)$. It is easy to see that $(X, Y)$, of finite codimension in $(U, Z)$, is also copure in $(U, Z)$. By Theorem 4 $(X, Y)$ is a direct summand of $(U, Z)$, and thereby pure in $(V, W)$.

**6. A counterexample.** It is to be noted that Theorems 3, 4 and 6 fail for topological $\mathbf{C}^N$-systems in general. For example, to negate Theorem 3 let $(V, W)$ be a topological $\mathbf{C}^N$-system such that $\overline{\mathbf{C}^N V} \backslash \mathbf{C}^N V \neq \emptyset$. (Here $\mathbf{C}^N V$ stands for the space in $W$ generated by all terms $ev$, where $e \in \mathbf{C}^N$, $v \in V$). For a chosen $w \in \overline{\mathbf{C}^N V} \backslash \mathbf{C}^N V$, the subsystem $(0, \mathbf{C}w)$ of $(V, W)$ is pure in $(V, W)$ but not a direct summand of $(V, W)$.

Indeed, no pair of continuous projections $V \to 0$ and $W \to \mathbf{C}w$ exists such that their kernels $K = V$ and $L$ form a subsystem $(K, L)$ of $(V, W)$. For then $L$, being closed in $W$, would have to contain $\overline{\mathbf{C}^N V}$ and hence $w$.

On the other hand, if $(X, Y)$ is a finite-dimensional extension of $(0, \mathbf{C}w)$, then $\overline{\mathbf{C}^N X} = \mathbf{C}^N X$ and $w \notin \mathbf{C}^N X$. Take any projection of $Y$ onto $\mathbf{C}w$ with $\mathbf{C}^N X$ inside its kernel $L$. This kernel, along with the kernel $K = X$ of the only projection $X \to 0$, give a subsystem $(K, L)$ which is supplementary to $(0, \mathbf{C}w)$ in $(X, Y)$. Thus $(0, \mathbf{C}w)$ is a direct summand of $(X, Y)$; and pure in $(V, W)$.

An example to negate Theorem 4 is found by simply dualizing the above example. We take for our $\mathbf{C}^N$-system $(W', V')$. The polar of $(0, \mathbf{C}w)$, namely $((\mathbf{C}w)^0, V')$, is a subsystem of finite codimension in $(W', V')$. It can be shown that, since $(0, \mathbf{C}w)$ is pure but not a direct summand of $(V, W)$, then $((\mathbf{C}w)^0, V')$ is copure but not a direct summand of $(W', V')$.

Finally, Theorem 6 is negated by these examples as well, since Theorem 6 trivially implies Theorems 3 and 4.

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