ABSOLUTE CONVERGENCE FACTORS FOR H^p SERIES

JAMES CAVENY

A famous theorem of Hardy asserts that if $f \in H^1$, then the sequence $(\hat{f}(0), \hat{f}(1), \ldots)$ of Fourier coefficients satisfies $\sum_{n=1}^{\infty} n^{-1}|\hat{f}(n)| < \infty$. For this reason we say that the sequence $(1, 1/2, 1/3, \ldots)$ belongs to the multiplier class (H^1, l^1). In this paper, we investigate the multiplier classes (H^p, l^1) for $1 \leqq p \leqq \infty$. Our observations are based on the fact that a sequence $(\lambda(0), \lambda(1), \ldots)$ belongs to (H^p, l^1) independent of the arguments of its terms. We also show that (H^p, l^1) may be thought of as the conjugate space of a certain Banach space.

1. Preliminaries. L^p denotes the space of complex-valued Lebesgue measurable functions f defined on the circle $|z| = 1$ such that

$$||f||_p = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta\right]^{1/p}$$

is finite. L^∞ is the space of essentially bounded complex-valued functions f on $|z| = 1$ with norm $||f||_\infty = \text{ess sup}\, |f(z)|$. For $1 \leqq p \leqq \infty$, the Fourier coefficients of an $f \in L^p$ are given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The Hardy class H^p is the closed subspace of L^p consisting of those functions whose Fourier coefficients vanish for negative indices. H^p may also be described as the space of functions f which are analytic in the unit disc $|z| < 1$ and for which

$$||f||_p = \left[\sup_{0 \leqq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta\right]^{1/p}$$

is finite. For these and other basic properties, (**4**) is a convenient reference. c_0 denotes the space of complex null sequences $\lambda = (\lambda(0), \lambda(1), \ldots)$ with norm given by $||\lambda|| = \max_n |\lambda(n)|$, and l^1 is the space of absolutely summable complex number sequences with its usual norm. If f is an H^p function and $\lambda = (\lambda(0), \lambda(1), \ldots)$ is a c_0 sequence, we write $f * \lambda$ for the analytic function with power series

$$f * \lambda(z) = \sum_{n=0}^{\infty} \hat{f}(n)\lambda(n) z^n, \tag{1.1}$$

Received August 4, 1967.

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

which obviously converges in $|z| < 1$. In general, if $f(z) = \sum \hat{f}(n)z^n$ and $g(z) = \sum \hat{g}(n)z^n$ are any two power series, then their Hadamard product $f * g$ is the function with formal power series $f * g(z) = \sum \hat{f}(n)\hat{g}(n)z^n$. If both the power series $f(z)$ and $g(z)$ converge in $|z| < 1$, then so does $f * g(z)$, and furthermore, we have the representation

$$f * g(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\rho e^{it}) f\left(\frac{r}{\rho} e^{i(\theta - t)}\right) dt \qquad (0 \leqq r < \rho < 1).$$

If $g \in H^p$, then we may put $\rho = 1$ in the above formula, and if $f \in H^q$, $q = p/(p-1)$, it follows that $f * g$ is continuous on the closed disc $|z| \leqq 1$ (see **7**, p. 38).

In addition to these standard classes, we adopt the notation L_+^1 and L_+^∞, respectively, for the classes of functions which are analytic projections of the Fourier series of integrable functions or bounded functions. More precisely, $f \in L_+^\infty$ if, and only if, $f(z) = \sum \hat{f}(n)z^n$ and the coefficients $\hat{f}(n)$ are given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) e^{-in\theta} \, d\theta \qquad (n = 0, 1, 2, \ldots) \tag{1.2}$$

for some $F \in L^\infty$. For $f \in L_+^\infty$ and $g \in H^1$, the representation

$$f * g(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i(\theta - t)}) F(e^{it}) \, dt \tag{1.3}$$

is valid for $0 \leqq r \leqq 1$, and, in particular, $f * g$ is continuous on the closed disc $|z| \leqq 1$. The class L_+^1 is defined in a similar manner, and for $f \in L_+^1$ and $g \in H^\infty$, a representation analogous to (1.3) shows that $f * g$ is continuous on $|z| \leqq 1$.

2. Arguments of coefficients. A multiplier in the class (H^p, l^1) is generally the sequence of coefficients of an H^q function, $q = p/(p-1)$, but much more can be said since a sequence belongs to (H^p, l^1), independent of the arguments of its terms. We use this notion to obtain a simple but convenient characterization of (H^p, l^1).

DEFINITION 2.1. *(H^p, l^1) is the collection of all complex number sequences $\lambda = (\lambda(0), \lambda(1), \ldots)$ such that $(\hat{f}(0)\lambda(0), \hat{f}(1)\lambda(1), \ldots) \in l^1$ for each $f \in H^p$.*

DEFINITION 2.2. *Let f and h be analytic in $|z| < 1$. The function h is called an agitation of f if $|\hat{h}(n)| = |\hat{f}(n)|$ for $n = 0, 1, 2, \ldots$. For $1 \leqq p \leqq \infty$, the class AH^p is the subset of H^p consisting of those functions f such that every agitation of f belongs to H^p.*

Thus, $f \in AH^p$ if, and only if, altering the arguments of the coefficients $\hat{f}(n)$ at random always results in another sequence of coefficients of an H^p function.

THEOREM 2.3. *Let $1 < p < \infty$ and $q = p/(p-1)$. Then $f \in AH^q$ if, and only if, $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^p, l^1)$.*

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

Proof. If $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^p, l^1)$, then certainly each agitation h of f satisfies $(\hat{h}(0), \hat{h}(1), \ldots) \in (H^p, l^1)$. If $g \in H^p$, then $h * g$ is a bounded analytic function in $|z| < 1$. By (**1**, Theorem 1), it must be that $h \in H^q$ and $f \in AH^q$.

If $f \in AH^q$ and g is any H^p function, then define

$$\hat{h}(n) = |\hat{f}(n)| \exp(-i \arg \hat{g}(n)).$$

If $h(z) = \sum \hat{h}(n) z^n$, then $h \in H^q$, and $h * g$ is continuous on $|z| \leqq 1$. But then the limit

$$\lim_{r \to 1} h * g(r) = \lim_{r \to 1} \sum_{n=0}^{\infty} |\hat{f}(n)\hat{g}(n)| r^n$$

is finite; therefore the series $\sum |\hat{f}(n)\hat{g}(n)|$ converges and $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^p, l^1)$.

We can extend the above theorem to the cases $p = 1$ and $p = \infty$ by considering the classes L_+^∞ and L_+^1.

DEFINITION 2.4. *If $p = 1$ or $p = \infty$, then AL_+^p is the collection of those analytic functions f such that every agitation of f belongs to L_+^p.*

THEOREM 2.5. *Let $p = 1$ or ∞ and $q = \infty$ or 1, respectively. Then $f \in AL_+^q$ if, and only if, $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^p, l^1)$.*

Proof. If $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^\infty, l^1)$, h is any agitation of f, and g is any H^∞ function, then the coefficients of $h * g$ are in l^1 and

$$\lim_{r \to 1} \sum_{n=0}^{\infty} \hat{h}(n)\hat{g}(n) r^n$$

exists. By the main theorem in (**5**), it follows that $h \in L_+^1$. If p and q are interchanged, a similar argument can be based on (**1**, Theorem 2) to show that $h \in L_+^\infty$. The other parts are straightforward.

The roles of L_+^q and H^p may be interchanged in Theorem 2.5. To see this we need to observe that any sequence in the multiplier class (L_+^∞, H^∞) is the sequence of Taylor coefficients of an H^1 function, and any sequence in the multiplier class (L_+^1, H^∞) is the sequence of Taylor coefficients of an H^∞ function. To verify the statement about (L_+^∞, H^∞), recall that H^1 is isometrically isomorphic to the conjugate space of the quotient space $C/\bar{A}_0$ (see **4**, p. 137). Here, C denotes the continuous functions on the circle $|z| = 1$, A_0 the continuous analytic functions that vanish at the origin, and $\bar{A}_0$ the complex conjugates of the functions in A_0. Suppose that $\sum_{n=0}^{\infty} \hat{f}(n)\hat{G}(n) z^n$ is in H^∞ for each G in L^∞; then for each fixed $G \in C$, the expression

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{-i\theta}) G(e^{i\theta})\, d\theta \right| \tag{2.1}$$

is bounded for r in the range $0 \leqq r < 1$. The integral in (2.1) is independent of the representative of G in $C/\bar{A}_0$. Thus, if $f_r(\theta) = f(re^{i\theta})$, then the principle

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

of uniform boundedness implies that the norms of the f_r, as linear functionals on $C/\bar{A}_0$, are uniformly bounded, i.e., $||f_r||_1 \leqq M$ and $f \in H^1$. The analogous relation between (L_+^1, H^∞) and H^∞ may be established in a similar manner by treating H^∞ as the conjugate space of $L^1/\bar{H}_0^1$. The technique of Theorems 2.3 and 2.5 may now be used to prove the following result.

THEOREM 2.6. *Let $p = 1$ or ∞ and $q = \infty$ or 1, respectively. Then $f \in AH^q$ if, and only if, $(\hat{f}(0), \hat{f}(1), \ldots) \in (L_+^p, l^1)$.*

Now we return to the case $q = \infty$ of Theorem 2.5 and show that the condition that all agitations of f belong to L_+^∞ can be significantly weakened. In fact, this condition will be satisfied if merely the agitation with positive coefficients belongs to L_+^∞.

THEOREM 2.7. *The function f belongs to AL_+^∞ if, and only if, the function with coefficients $|\hat{f}(n)|$ belongs to L_+^∞.*

Proof. It is trivially true that the agitation h given by $h(z) = \sum|\hat{f}(n)|z^n$ is in L_+^∞ whenever $f \in AL_+^\infty$. Assuming that $h \in L_+^\infty$, one can modify the proof of Hardy's theorem (**4**, p. **70**) to show that $(|\hat{f}(0)|, |\hat{f}(1)|, \ldots)$ belongs to (H^1, l^1), but this implies that $f \in AL_+^\infty$. Indeed, if g is in H^1 with positive coefficients, then

$$\sum_{n=0}^{\infty} \hat{g}(n)|\hat{f}(n)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta})H(e^{-i\theta})\, d\theta \leqq ||g||_1 ||H||_\infty,$$

where H is any L^∞ function whose analytic projection is h. For an arbitrary $g \in H^1$, the fact that g can be factored into the product of H^2 functions shows that there is a G in H^1 with positive coefficients which dominate the coefficients of g, $|\hat{g}(n)| \leqq \hat{G}(n)$ $(n = 0, 1, 2, \ldots)$. Then

$$\sum_{n=0}^{\infty} |\hat{g}(n)|\, |\hat{f}(n)| \leqq \sum_{n=0}^{\infty} \hat{G}(n)|\hat{f}(n)| < \infty.$$

THEOREM 2.8. *The function f belongs to AH^∞ if, and only if, the sequence $(\hat{f}(0), \hat{f}(1), \ldots)$ is in l^1.*

We omit the simple proof, and proceed to investigate the analogous statements for the other values of p. When $1 \leqq p \leqq 2$ the situation is particularly simple.

THEOREM 2.9. *If $1 \leqq p \leqq 2$, then $f \in AH^p$ if, and only if, $(\hat{f}(0), \hat{f}(1), \ldots) \in l^2$.*

Proof. Because of Parseval's formula, if $(\hat{f}(0), \hat{f}(1), \ldots) \in l^2$ and h is any agitation of f, then $h \in H^2 \subset H^p$ since $1 \leqq p \leqq 2$. On the other hand, if $f \in AH^p$, then, for every choice of signs, the series $\sum \pm |\hat{f}(n)|e^{in\theta}$ is the formal boundary series of an H^p function. But then $\sum \pm |\hat{f}(n)| \cos n\theta$ is the Fourier series of an L^p function for arbitrary choice of signs, and $\sum|\hat{f}(n)|^2$ cannot diverge (**7**, Chapter V, (8·14)).

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

COROLLARY 2.10. *If* $2 \leqq q < \infty$, *then* $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^q, l^1)$ *if, and only if,* $f \in H^2$. *Furthermore,* $(\hat{f}(0), \hat{f}(1), \ldots) \in (L_+^\infty, l^1)$ *if, and only if,* $f \in H^2$.

3. (H^p, l^1) **as a dual space.** In this section we shall use a construction technique introduced in (**3**) to build a Banach space which has the multiplier class (H^p, l^1) as its normed conjugate. In the construction we are forced to consider expressions of the form (1.1), where $f \in H^p$ and $\lambda \in c_0$. We cannot in general claim that $f * \lambda$ belongs to a certain Hardy class, but its formal existence is all we require.

DEFINITION 3.1. *Let* $1 \leqq p \leqq \infty$. $H^p \otimes c_0$ *denotes the collection of all functions* f, $f(z) = \sum \hat{f}(n) z^n$ $(|z| < 1)$, *for which there exists a sequence* $(f_0, f_1, f_2, \ldots)$ *of* H^p *functions and a sequence* $(\lambda_0, \lambda_1, \lambda_2, \ldots)$ *of* c_0 *sequences such that*

$$\sum_{k=0}^{\infty} ||f_k||_p ||\lambda_k|| < \infty \tag{3.1}$$

and

$$\hat{f}(n) = \sum_{k=0}^{\infty} \hat{f}_k(n) \lambda_k(n) \qquad (n = 0, 1, 2, \ldots). \tag{3.2}$$

If (3.1) holds, then the series defining the coefficients in (3.2) are uniformly absolutely convergent, and the formal power series $f(z)$ with coefficients $\hat{f}(n)$ is necessarily convergent in $|z| < 1$.

THEOREM 3.2. $H^p \otimes c_0$ *is a linear space of functions, analytic in* $|z| < 1$, *with the usual addition and scalar multiplication. A norm* N_p *may be defined for* $f \in H^p \otimes c_0$ *by*

$$N_p(f) = \inf \sum_{k=0}^{\infty} ||f_k||_p ||\lambda_k||. \tag{3.3}$$

The space $H^p \otimes c_0$ *equipped with the norm* N_p *is a Banach space.*

The infimum in (3.3) is, of course, to be taken over all sequences $(f_0, f_1, \ldots) \subset H^p$ and $(\lambda_0, \lambda_1, \ldots) \subset c_0$ which satisfy (3.2). The proof that $H^p \otimes c_0$ is a linear space and that N_p is a norm is elementary. A straightforward proof that the normed space is complete may be given by showing that absolutely summable series are summable.

If $\alpha = (\alpha(0), \alpha(1), \ldots)$ is any sequence in (H^p, l^1), the closed graph theorem shows that the linear operation $f \to (\hat{f}(0)\alpha(0), \hat{f}(1)\alpha(1), \ldots)$ from H^p into l^1 is continuous. Thus, the multiplier class (H^p, l^1) may be viewed as a Banach space of sequences equipped with the operator norm M_p defined by

$$M_p(\alpha) = \sup \frac{\sum |\hat{f}(n)\alpha(n)|}{||f||_p}, \qquad f \in H^p. \tag{3.4}$$

THEOREM 3.3. *Let* $1 \leqq p \leqq \infty$. *Then* (H^p, l^1) *with the operator norm* (3.4) *is isometrically isomorphic to the dual space of* $H^p \otimes c_0$. *If* χ *is the identity function*

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

$\chi(z) = z$ ($|z| < 1$), *then an isometry is given by* $\alpha \leftrightarrow T$, *where* $T \in (H^p \otimes c_0)^*$ *and* $\alpha \in (H^p, l^1)$ *are related by* $T(\chi^n) = \alpha(n)$, $n = 0, 1, 2, \ldots$.

Proof. If $\alpha = (\alpha(0), \alpha(1), \ldots) \in (H^p, l^1)$, then, formally, we define

$$T(f) = \sum_{n=0}^{\infty} \hat{f}(n)\alpha(n) \quad \text{for } f \in H^p \otimes c_0. \tag{3.5}$$

The series in (3.5), which defines $T(f)$, is absolutely convergent. Indeed, if $(f_0, f_1, f_2, \ldots) \subset H^p$ and $(\lambda_0, \lambda_1, \lambda_2, \ldots) \subset c_0$ satisfy (3.1) and (3.2), then

$$\begin{aligned} |T(f)| &\leqq \sum_{n=0}^{\infty} |\hat{f}(n)\alpha(n)| \\ &\leqq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |\alpha(n)\hat{f}_k(n)|\,|\lambda_k(n)| \\ &\leqq M_p(\alpha) \sum_{k=0}^{\infty} ||f_k||_p||\lambda_k||. \end{aligned}$$

Thus, $|T(f)| \leqq M_p(\alpha)N_p(f)$ and

$$||T|| \leqq M_p(\alpha) \quad \text{for } \alpha \in (H^p, l^1). \tag{3.6}$$

Let $T \in (H^p \otimes c_0)^*$, $\chi(z) = z$, and put $\alpha(n) = T(\chi^n)$ for $n = 0, 1, 2, \ldots$. Let f be a fixed H^p function, and define a linear functional T_0 on c_0 by putting $T_0(\lambda) = T(f * \lambda)$ for $\lambda \in c_0$. Then

$$|T_0(\lambda)| \leqq ||T||N_p(f * \lambda) \leqq ||T||\,||f||_p||\lambda||;$$

thus, T_0 is continuous on c_0 and $||T_0|| \leqq ||T||\,||f||_p$. T_0 is given by an inner product with an l^1 sequence $\beta = (\beta(0), \beta(1), \ldots)$; whence, $T(f * \lambda) = T_0(\lambda) = \sum \lambda(n)\beta(n)$ for all $\lambda \in c_0$, and $\sum|\beta(n)| = ||T_0|| \leqq ||T||\,||f||_p$. Let $\delta_k(n) = 0$ for $k \neq n$ and $\delta_k(k) = 1$, and put $\delta_k = (\delta_k(0), \delta_k(1), \ldots)$. Then $\hat{f}(k)\alpha(k) = \hat{f}(k)T(\chi^k) = T(\hat{f}(k)\chi^k) = T(f * \delta_k) = T_0(\delta_k) = \beta(k)$, and $\sum|\hat{f}(n)\alpha(n)| \leqq ||T||\,||f||_p$. This shows that $\alpha \in (H^p, l^1)$ and

$$M_p(\alpha) \leqq ||T|| \quad \text{for } T \in (H^p \otimes c_0)^*. \tag{3.7}$$

The desired properties of the correspondence $\alpha \leftrightarrow T$ are immediate consequences of (3.6) and (3.7).

In Corollary 2.10 we showed that the multiplier class (H^q, l^1) consists precisely of the square summable sequences when $2 \leqq q < \infty$. Because of the relation between (H^q, l^1) and the dual of $H^q \otimes c_0$, and the fact that $(H^2)^* = H^2$, one would expect that $H^q \otimes c_0$ is just the class of H^2 functions. Even more can be proved. In what follows, $H^q * c_0$ denotes the collection of analytic functions of the form $f * \lambda$, where $f \in H^q$ and $\lambda \in c_0$. It is known that $L^1 * L^q = L^q$ for $1 \leqq q < \infty$ (see **2**). Here, $L^1 * L^q$ is the class of functions which are convolutions of L^q functions with L^1 functions. A simple corollary is the factorization theorem

$$L^1 * H^q = H^q \qquad (1 \leqq q < \infty). \tag{3.8}$$

https://doi.org/10.4153/CJM-1969-018-9 Published online by Cambridge University Press

We shall use this result to obtain a factorization theorem for H^2.

THEOREM 3.4. *Let* $2 \leqq q < \infty$. *Then* $f \in H^q * c_0$ *if, and only if,* $f \in H^2$.

Proof. If $f = g * \lambda$ with $g \in H^q \subset H^2$ and $\lambda \in c_0 \subset l^\infty$, then the sequence of Taylor coefficients of g is square summable and the same is true for the coefficients of f, and so $f \in H^2$.

If $f \in H^2$ and $\hat{f}(n) = a_n + ib_n$ $(n = 0, 1, 2, \ldots)$, then both the sequences $(a_0, a_1, a_2, \ldots)$ and $(b_0, b_1, b_2, \ldots)$ are square summable, and for almost all choices of signs, both the series

$$(3.9) \qquad \sum_{n=0}^{\infty} \pm a_n \cos n\theta \quad \text{and} \quad \sum_{n=0}^{\infty} \pm b_n \sin n\theta$$

have sums in L^r for every $r > 0$ (see **7**, Chapter V, (8.16)). In particular, there exists a choice of signs such that the sum h of the series

$$(3.10) \qquad h(\theta) \sim \sum_{n=0}^{\infty} \pm \hat{f}(n) e^{in\theta}$$

is a complex-valued L^q function, and hence belongs to H^q. By (3.8), there is an H^q function g and an L^1 function F such that $h = g * F$. The Riemann-Lebesgue lemma guarantees that the sequence $(\hat{F}(0), \hat{F}(1), \ldots)$ is in c_0. Since h is an agitation of f, it follows that $f = g * \lambda$ with $|\lambda(n)| = |\hat{F}(n)|$ $(n = 0, 1, 2, \ldots)$. However, $\lambda \in c_0$ and the desired factorization for H^2 functions is established.

Since (H^p, l^1) is a class of coefficient sequences for functions in H^q, it is natural to ask how the multiplier norm M_p and the H^q norm $||\cdot||_q$ are related. In describing the relation, it is convenient to consider the classes AL_+^q $(1 \leqq q \leqq \infty)$ whose definitions are obvious. The norm in L_+^q is the quotient norm of $L^q/\overline{H_0^q}$, and will be denoted by $||\cdot||_{q^+}$. Because of the M. Riesz theorem (**4**, p. 151), the classes AL_+^q and AH^q consist of the same analytic functions for $1 < q < \infty$.

THEOREM 3.5. *Let* $1 \leqq p \leqq \infty$. *If* $(\hat{f}(0), \hat{f}(1), \ldots) \in (H^p, l^1)$, *then* $M_p(f) = \sup ||h||_{q^+}$ $(1/p + 1/q = 1)$, *where the supremum is taken over all agitations* h *of* f.

Proof. Let g be any H^p function. Define h by putting

$$h(z) = \sum \hat{f}(n) \exp(-i \arg \hat{g}(n)) z^n.$$

Then

$$\sum_{n=0}^{\infty} |\hat{f}(n)\hat{g}(n)| = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) g(e^{-i\theta})\, d\theta \leqq ||h||_{q^+} ||g||_p.$$

It follows that $M_p(f) \leqq \sup ||h||_{q^+}$.

For any fixed agitation h of f and any $\epsilon > 0$ there is an H^p function g such that $||g||_p = 1$ and

$$||h||_{q^+} - \epsilon < \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta}) g(e^{-i\theta})\, d\theta.$$

But then

$$||h||_{q^+} - \epsilon < \sum \hat{h}(n)\hat{g}(n) \leqq \sum |\hat{f}(n)\hat{g}(n)| \leqq M_p(f),$$

and

$$||h||_{q^+} \leqq M_p(f).$$

THEOREM 3.6. *Let* $1 < q < \infty$ *and let* $f \in AH^q$. *The set* $A(f)$ *of all possible agitations of* f *is a norm bounded set in* H^q.

Proof. By M. Riesz's theorem there exists a constant A_q such that $||h||_q \leqq A_q||h||_{q^+}$. Thus, for any $h \in A(f)$ we have that $||h||_q \leqq A_q||h||_{q^+} \leqq A_qM_p(h) = A_qM_p(f)$.

4. Duality. In this section we state some results that are obtained by simple duality arguments.

THEOREM 4.1. *For* $1 < p < \infty$ *and* $1/p + 1/q = 1$, *we have that* $(c_0, H^p) = (H^q, l^1)$.

Proof. If $\lambda \in (c_0, H^p)$ and α is any c_0 sequence, then $\alpha * \lambda(z) = \sum \alpha(n)\lambda(n)z^n$ is the power series of an H^p function $\alpha * \lambda$. As usual, the closed graph theorem guarantees the existence of a constant M such that $||\alpha * \lambda||_p \leqq M||\alpha||$. If f is any member of H^q, then $f * (\alpha * \lambda)$ is continuous on the closed disc $|z| \leqq 1$, and we can define a continuous linear functional L on c_0 by putting

$$L(\alpha) = \lim_{r \to 1} \sum \hat{f}(n)\lambda(n)\alpha(n)r^n = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{-it})\alpha * \lambda(e^{it})\, dt$$

and $|L(\alpha)| \leqq (M||f||_q)||\alpha||$. However, L is realized through an l^1 sequence, $L(\alpha) = \sum \alpha(n)\beta(n)$, $\beta = (\beta(0), \beta(1), \ldots) \in l^1$. Appropriate choices of the sequences α show that $\hat{f}(n)\lambda(n) = \beta(n)$ $(n = 0, 1, 2, \ldots)$; thus, $\lambda \in (H^q, l^1)$. This shows that $(c_0, H^p) \subset (H^q, l^1)$, and a similar argument may be used to establish the containment $(H^q, l^1) \subset (c_0, H^p)$.

For the extreme cases we have the following result.

THEOREM 4.2. (i) $(c_0, H^\infty) = (L_+{}^1, l^1)$;
(ii) $(c_0, L_+{}^\infty) = (H^1, l^1)$;
(iii) $(c_0, H^1) = (C_+, l^1) = (L_+{}^\infty, l^1)$.

Proof. Cases (i) and (ii) may be established by means of an argument similar to the proof of Theorem 4.1 since H^∞ and $L_+{}^\infty$ can be associated, respectively, with the duals of $L_+{}^1$ and H^1. In part (iii), C_+ is the class of analytic projections of the continuous functions, and $(c_0, H^1) = (C_+, l^1)$ follows since H^1 can be identified with the conjugate space of $C/\bar{A}_0$. Moreover, a simple argument, similar to those given in § 2, shows that (C_+, l^1) consists of the sequences of Taylor coefficients of AH^1 functions, i.e., the sequences in $(L_+{}^\infty, l^1)$.

5. Questions. We conclude with several questions which these investigations have left unanswered. Theorem 3.4 guarantees that any H^2 function can be factored into the Hadamard product of an H^q function and a c_0 sequence, where q is any finite positive number. Is each H^2 function factorable into the Hadamard product of an H^∞ function and a c_0 sequence?

For each p there are two other classes which arise naturally from the problem (H^p, l^1). $S(H^p)$ denotes the collection of all those power series $f(z)$ such that some agitation of $f(z)$ belongs to H^p. X^p is the maximal sequence space that is mapped to l^1 under inner product with the members of the class (H^p, l^1). Thus, $X^p = ((H^p, l^1), l^1)$. When $2 \leqq p < \infty$ it is easy to see that $f \in S(H^p)$ if, and only if, its coefficient sequence belongs to $X^p = l^2$. For other values of p it is clear that if $f \in S(H^p)$, then $(\hat{f}(0), \hat{f}(1), \ldots) \in X^p$. Does X^p ever contain sequences which are not coefficient sequences of $S(H^p)$ functions?

Added in proof. Professor J. Fournier has pointed out that Corollary 2.10 actually holds for $q = \infty$. For the stronger result, see (R. Paley, *A note on power series*, J. London Math. Soc. *7* (1932), 122–130). For a more recent paper, see (H. Helson, *Conjugate series and a theorem of Paley*, Pacific J. Math. *8* (1958), 437–446).

References

1. D. J. Caveny, *Bounded Hadamard products of H^p functions*, Duke Math. J. *33* (1966), 389–394.

2. P. C. Curtis, Jr. and A. Figà-Talamanca, *Factorization theorems in Banach algebras; Function algebras* (Proc. Internat. Sympos. on Function *Algebras*, Tulane Univ., 1965), pp. 169–185 (Scott Foresman, Chicago, Ill., 1966).

3. A. Figà-Talamanca, *Translation invariant operators in L^p*, Duke Math. J. *32* (1965), 495–501.

4. K. Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, N.J., 1962).

5. G. Piranian, A. L. Shields, and J. H. Wells, *Bounded analytic functions and absolutely continuous measures*, Proc. Amer. Math. Soc. *18* (1967), 818–826.

6. J. H. Wells, *Some results concerning multipliers of H^p* (to appear).

7. A. Zygmund, *Trigonometric Series*, Vol. I, 2nd ed. (Cambridge Univ. Press, New York, 1959).

Florida State University,
Tallahassee, Florida