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RIGHT-ORDERED POLYCYCLIC GROUPS

BY
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1. **Introduction.** One of the features that make right-ordered groups harder to investigate than ordered groups is that their system of convex subgroups may fail to have the following property:

(*) if C and C' are convex subgroups of G and C' covers C, then C is normal in C' and C'/C is order-isomorphic to a subgroup of the naturally ordered additive group of real numbers.

It is therefore natural to pay attention to certain subclasses of the class of right-ordered groups, namely the class $\mathfrak{C}$ of groups admitting at least one right-order satisfying (*) and the class $\mathfrak{C}^*$ of right-ordered groups all of whose right-orders satisfy (*).

J. C. Ault [1] and A. H. Rhemtulla [6] have proved independently that every torsion-free locally nilpotent group is a $\mathfrak{C}^*$-group. (In fact Rhemtulla's proof can easily be slightly modified to show that every right-ordered locally nilpotent-by-periodic group is a $\mathfrak{C}^*$-group.)

It was asked in [6] whether every polycyclic right-ordered group would also necessarily be in $\mathfrak{C}^*$. Our first example gives a negative answer to this question.

The second and third examples bring evidence to the fact that the class $\mathfrak{C}$ is much larger than the class of ordered groups, in fact it has not yet been established whether $\mathfrak{C}$ is indeed smaller than the class of all right-ordered groups. Both examples give instances of properties of ordered-groups which are not shared by $\mathfrak{C}$-groups. The second example is a polycyclic $\mathfrak{C}$-group which is not nilpotent-by-abelian. This contrasts with the situation of ordered groups which must be nilpotent-by-abelian whenever they are polycyclic, as it was shown in [4].

It is well-known that the quotient of an ordered group with respect to its center is again ordered, see for instance [3], while in the case of right-ordered groups such a quotient need not even be torsion-free. Our third example is a polycyclic $\mathfrak{C}$-group whose quotient with respect to the center is not right-ordered, although it is torsion-free.

In section 5 we list a number of equivalent formulations of (*).

2. **Example of a polycyclic $\mathfrak{C}$-group which is not in $\mathfrak{C}^*$.** Let σ and τ be the following order-preserving transformations of the real line:

$$x\sigma = x+1$$

and

$$x\tau = x/\alpha,$$

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where $\alpha=+\sqrt{\frac{5+\sqrt{21}}{2}}$ is a root of the equation $x^4-5x^2+1=0$. The group G generated by σ and τ admits the following presentation:

$$G = \langle \sigma, \tau;\ \sigma^{\tau^4} = (\sigma^5)^{\tau^2}\sigma^{-1},\ [\sigma, \sigma^{\tau^i}] = e \quad \text{for} \quad i = 1, 2, 3\rangle.$$

By considering the normal closure of the subgroup generated by σ, it is easy to verify that G is the extension of a free abelian group of rank 4 by an infinite cyclic group, and therefore it is polycyclic and it belongs to the class $\mathfrak{C}$.(1)

In order to show that G is not a $\mathfrak{C}^*$-group, we will right-order G in such a way that property (3) in section 5 will not be satisfied. Well-order the set R of real numbers letting 0 be the first element and -1 the second, then for any $g\in G$ consider the first $r\in R$ in the given well-ordering such that $rg\neq r$, and define g to be positive if $rg>r$ in the natural order of R. In this way σ, τ, and $\sigma\tau$ turn out to be positive, but $(\sigma\tau)^n\tau(\sigma\tau)^{-1}$ is negative for all positive integers n, because under $(\sigma\tau)^n\sigma^{-1}$ the element 0 is mapped to $[(1-\alpha^n)/(1-\alpha)\alpha^n]-1$ which is less than 0. Thus the right-order that we have imposed on G does not satisfy property (3).

3. Example of a polycyclic $\mathfrak{C}$-group which is not nilpotent-by-abelian. Let $N_i=\langle a_i, b_i;\ a_i^{b_i}=a_i^{-1}\rangle$, $i=1, 2$, and let N be the direct product of N_1 and N_2.

Let G be the split extension of N by an infinite cyclic group $\langle\phi\rangle$ with $a_1^\phi=a_2$, $b_1^\phi=b_2$, and ϕ^2 in the center of G. Since $e \lhd \langle a_1\rangle \lhd N_1 \lhd \langle N_1, a_2\rangle \lhd N \lhd G$ is a normal chain all of whose factors are infinite cyclic, G is a polycyclic $\mathfrak{C}$-group.

Let $a=[a_1, \phi]$ and $b=[b_1, \phi]$. The subgroup of G generated by a and b admits the presentation

$$\langle a, b;\ a^b = a^{-1}\rangle,$$

and it is easy to verify that such a group is not nilpotent. Therefore, G' is not nilpotent either, which implies that G is not nilpotent-by-abelian.

4. Example of a $\mathfrak{C}$-group whose quotient with respect to the center is torsion-free, but not right-orderable. Let $G=\langle x, y, z;\ x^2y^{-2}x^2y=y^2x^{-1}y^2x=z,\ xz=zx,\ yz=zy\rangle$. G has a normal chain $G=G_0 \rhd G_1 \rhd G_2 \rhd G_3 \rhd G_4=\langle e\rangle$ with infinite cyclic factors, where $G_1=\langle x^2y^2z^{-1},\ y^4z^{-1},\ xy^{-1}\rangle$, $G_2=\langle x^2y^2z^{-1},\ y^4z^{-1}\rangle$, $G_3=\langle x^2y^2z^{-1}\rangle$, therefore it is a polycyclic $\mathfrak{C}$-group.

It has been shown in [5] that $G/\langle z\rangle$ is torsion-free but not right-orderable, thus the only thing left to verify is that $\langle z\rangle$ coincides with the center of G. This follows trivially from the fact that $G/\langle z\rangle$, which is isomorphic to the group $\langle x, y;\ y^2x^{-1}y^2x=x^2y^{-1}x^2y=e\rangle$, has no center. The verification of the last assertion is straightforward.

(1) The group G has also the property that every two-sided partial order can be extended to a two-sided total order, in particular G is orderable.

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5. **Equivalent forms of property (*).** Given two positive elements a and b, we say that a is infinitely smaller than b and write $a \ll b$ if $a^n < b$ for all positive integers n. If a and b are negative, $a \ll b$ means that $a < b^n$ for all positive integers n. Let P be the positive cone of a right-order on a group G, then the following are equivalent:

(1) $\forall a, b \in P\ \exists n \in N: (ab)^n > ba$,

(2) $\forall a, b \in P,\ a < b\ \exists n \in N: ab^n a^{-1} > b$,

(3) $\forall a, b \in P\ \exists n \in N: a^n b > a$,

(4) $= (*)$,

(5) $\forall a, b \in P^{-1}\ \exists n \in N: (ab)^n < ba$,

(6) $\forall a, b \in P^{-1},\ a > b\ \exists n \in N: ab^n a^{-1} < b$,

(7) $\forall a, b \in P^{-1}\ \exists n \in N: a^n b < a$,

(8) $\forall a, b \in P\ a \ll b \Leftrightarrow b^{-1} \ll a^{-1}$,

(9) $\forall a, b \in P\ a \ll b \Rightarrow b^{-1} < a^{-1}$,

(10) $\forall a, b \in P^{-1}\ b \ll a \Rightarrow a^{-1} < b^{-1}$,

(11) $\forall a, b \in G,\ c \in P\ |a| \ll c$ and $|b| \ll c \Rightarrow |ab| \ll c$,

(12) $\forall a \in P$ the set $\{x \in G : |x| \ll a\}$ is a convex subgroup of G.

Note that (5), (6), (7), and (10) are respectively the dual of (1), (2), (3), and (9), i.e. they can be deduced from each other simply by substituting the right-order P with its opposite P^{-1}.

The equivalence of (1), (2), (3), and (4) was proved by P. Conrad in [2]. Let us show, for example, the following implications:

$$(1)+(5) \Rightarrow (8) \Rightarrow (9) \Rightarrow (3).$$

Let $a, b \in P$, $a \ll b$. We must show $b^{-1} \ll a^{-1}$. If this were not so, then for some positive integer m we would have $a^{-m} < b^{-1}$, and by applying property (1) to the two elements b and $b^{-1}a^m$ we would obtain $a^{mn} > b^{-1}a^m b > b$, which is a contradiction. In the same way, using property (5) instead of property (1), we see that $b^{-1} \ll a^{-1}$ implies $a \ll b$.

Obviously (9) is a consequence of (8). To prove that (9) implies (3), consider two positive elements a and b. If $a \le b$, property (3) is satisfied with $n=1$. Let $a > b$ and suppose that for all $n \in N$ $a^n b < a$. Then $a \ll ab^{-1}$ and by property (9) $ba^{-1} < a^{-1}$, contradicting the fact that b is positive.

The rest can be proved by similar techniques.

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