ON PROJECTIVE CHARACTERS OF PRIME DEGREE

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All groups G considered in this paper are finite and all representations of G are defined over the field of complex numbers. The reader unfamiliar with projective representations is referred to [**9**] for basic definitions and elementary results.

0. Introduction. Let $\text{Proj}(G, \alpha)$ denote the set of irreducible projective characters of a group G with cocycle α. In a previous paper [**3**] the author showed that if G is a (p, α)-group, that is the degrees of the elements of $\text{Proj}(G, \alpha)$ are all powers of a prime number p, then G is solvable. However Isaacs and Passman in [**8**] were able to give structural information about a group G for which $\xi(1)$ divides p^e for all $\xi \in \text{Proj}(G, 1)$, where 1 denotes the trivial cocycle of G, and indeed classified all such groups in the case $e = 1$. Their results rely on the fact that G has a normal abelian p-complement, which is false in general if G is a (p, α)-group; the alternating group A_4 providing an easy counter-example for $p = 2$.

The aim of this paper is to at least give a full classification of p-groups G whose irreducible projective characters with cocycle α all have degree p. In Section 1 we shall show that a (p, α)-group G which does not possess a normal abelian p-complement may be considered unusual, and we shall assume thereafter that G does have such a complement. Under this assumption Isaacs and Passman's results for ordinary characters still hold for projective characters, and our interest is focussed on the necessary changes in the corresponding proofs.

In Section 2 we obtain the following theorem which is the exact analogue of Theorem II of [**8**].

THEOREM 1. *Let p be an odd prime number, and G be a group with a normal abelian p-complement. Then every irreducible projective character of G with cocycle α has degree dividing p if and only if*

(i) *G is abelian and the cohomology class of α is trivial*; *or*

(ii) *G has an abelian normal subgroup A with the cohomology class of α_A trivial of minimal index p*; *or*

(iii) *G/U is a group of order p^3 and exponent p, where U denotes the set of α-regular elements of G contained in the centre $Z(G)$ of G.*

The case $p = 2$ is exceptional and is dealt with in Section 3. Let C_n and D_n denote the cyclic group of order n and the dihedral group of order $2n$ respectively. Then with notation as above our results culminate in the following theorem.

THEOREM 2. *Let $p = 2$, and G be a group with a normal abelian 2-complement. Suppose every irreducible projective character of G with cocycle α has degree dividing 2. Then G satisfies* (i), (ii), *or* (iii) *of Theorem* 1 *or*

(iv) *$U = Z(G)$ and $G/U \cong C_2 \times C_2 \times C_2 \times C_2$, or $D_4 \times C_2$, or*

$$R = \langle x, y, z : x^4 = y^2 = z^2 = 1, xy = yx, yz = zy, xz = zx^{-1}y \rangle.$$

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1. Prime power degrees. Let $[\alpha]$ denote the cohomology class of the cocycle α in the Schur multiplier $M(G)$ of the group G. We start by noting that the degrees of projective representations are unaffected under projective equivalence, so that if G is a (p, α)-group then it is also a (p, β)-group for $[\beta]=[\alpha]$. Thus in what follows it is no loss to assume that the cocycle α under consideration is a class-preserving cocycle, such that $o([\alpha])=n$ if and only if α^n is the trivial cocycle of G. We also state and use without further reference the fact that $o([\alpha])$ divides $\xi(1)$ for all $\xi \in \mathrm{Proj}(G, \alpha)$. Finally for the remainder of this paper let p be a fixed prime number.

DEFINITION 1.1. A group G is said to have p.r.x.(e, α) (projective representation exponent e) if there exists a cocycle α of G such that $\xi(1)$ divides p^e for all $\xi \in \mathrm{Proj}(G, \alpha)$.

For convenience we quote, summarize, and generalize 2.4, 2.7 and 2.8 of [**8**], and 2.2, 2.3, 2.4, and Theorem B of [**3**], the proofs where needed can easily be derived from those given in the relevant paper.

LEMMA 1.2. *Let G have* p.r.x.(e, α), $N \trianglelefteq G$, $\zeta \in \mathrm{Proj}(N, \alpha_N)$, *and $I_G(\zeta)$ denote the inertia group of ζ in G. Then*

(i) *N has* p.r.x.(e, α_N);

(ii) *if G/N is non-abelian, then N has* p.r.x.$(e-1, \alpha_N)$;

(iii) *$I_G(\zeta)/N$ has* p.r.x.(e, β) *for some cocycle β of $I_G(\zeta)/N$, and $[G:I_G(\zeta)]$ divides p^e.*

LEMMA 1.3. *Let $N \trianglelefteq G$ with G/N a p-group. Suppose G has* p.r.x.(e, α) *and N has* p.r.x.$(e-1, \alpha_N)$. *Then F has* p.r.x.$(e-1, \alpha_N)$, *where F is the inverse image in G of the Frattini subgroup of G/N.*

LEMMA 1.4. *Let G have* p.r.x.(e, α), *$L \leq G$ such that $[G:L]$ is coprime to p, and $\zeta \in \mathrm{Proj}(L, \alpha_L)$. Then ζ extends to G.*

THEOREM 1.5. *Let G have* p.r.x.(e, α). *Then G is solvable and has abelian Hall p'-subgroups.*

LEMMA 1.6. *Let G have* p.r.x.(e, α) *and suppose G has a normal abelian p-complement. Let $H \trianglelefteq K \leq G$ with K/H an abelian group of order coprime to p. Then*

(i) *K has* p.r.x.(e, α_K);

(ii) *if H has* p.r.x.(f, α_H), *then K has* p.r.x.(f, α_K).

Our next aim is to relate the results on projective and ordinary representation exponents, the following proposition providing the crucial link.

PROPOSITION 1.7. *Let G have* p.r.x.(e, α), $p^a=\min\{\xi(1): \xi \in \mathrm{Proj}(G, \alpha)\}$, *and suppose G has a normal abelian p-complement. Then G has* p.r.x.$(e+a, \alpha^n)$ *for any integer n.*

Proof. Let S be a Sylow p-subgroup of G, and $\zeta \in \mathrm{Proj}(S, \alpha_S)$ of minimum degree. Then $\zeta(1)=p^a$ by Proposition 1 of [**2**]. Now since S is a PM-group there exists a subgroup T of S with $[S:T]=p^a$ and $\lambda \in \mathrm{Proj}(T, \alpha_T)$ with $\lambda^S=\zeta$.

Let N be the normal abelian p-complement of G, then by 1.6(i) TN has p.r.x.(e, α_{TN}), and so by 1.4 λ extends to $\mu \in \mathrm{Proj}(TN, \alpha_{TN})$. Thus $[\alpha_{TN}]=[1]$, so that G

is a (p, α^n)-group for any integer n by Theorem 2 of **[11]**, and since $[G:TN]=p^a$ it follows that G has p.r.x.$(e+a, \alpha^n)$.

The above result allows us to generalize Theorem I of **[8]** as follows.

THEOREM 1.8. *Let G have* p.r.x.(e, α) *and suppose G has a normal abelian p-complement. Then G has a series of subgroups*

$$A_0 \trianglelefteq A_1 \trianglelefteq \ldots \trianglelefteq A_e = G,$$

such that A_0 is abelian, $[\alpha_{A_0}]=[1]$, and A_i/A_{i-1} is an elementary abelian p-group with not more than $2(i+a)+1$ generators; where $p^a = \min\{\xi(1): \xi \in \mathrm{Proj}(G, \alpha)\}$. Hence G has an abelian subgroup A_0 with $[\alpha_{A_0}]=[1]$ whose index divides $p^{e(e+2a+2)}$.

Proof. We proceed by induction on e, noting that if A is a subnormal subgroup of G then $\min\{\zeta(1): \zeta \in \mathrm{Proj}(A, \alpha_A)\}$ divides p^a. Thus it suffices to prove that there exists a normal subgroup A_{e-1} of G, such that A_{e-1} has p.r.x.$(e-1, \alpha_{A_{e-1}})$ and G/A_{e-1} is an elementary abelian p-group of order $\leq p^{2(e+a)+1}$.

Suppose G is abelian. Then $\mathrm{Proj}(G, \alpha)$ all have the same degree. Let U denote the set of α-regular elements of G and $\xi \in \mathrm{Proj}(G, \alpha)$. Then U has p.r.x.$(0, \alpha_U)$, $\xi_U = \xi(1)\lambda$ for some $\lambda \in \mathrm{Proj}(U, \alpha_U)$, and $\xi(g)=0$ for $g \in G-U$. Thus $[\alpha_U]=[1]$ and $[G:U]=p^{a^2}$. If $U=G$ there is nothing to prove, whereas if $U<G$ we may let A_{e-1} be a maximal subgroup of G with $U \leq A_{e-1} < G$. Then $[G:A_{e-1}]=p$, and A_{e-1} has p.r.x.$(e-1, \alpha_{A_{e-1}})$ from above.

Suppose G is non-abelian. Let $N \triangleleft G$ be maximal such that G/N is non-abelian and set $\bar{G}=G/N$. Let H be the inverse image in G of the normal abelian p-complement of $\bar{G}$. Then N has p.r.x.$(e-1, \alpha_N)$ by 1.2, and H has p.r.x.$(e-1, \alpha_H)$ by 1.6.

Case 1: $\bar{H}=H/N$ is a non-trivial subgroup of $\bar{G}$.
In this case $\bar{G}$ is a Frobenius group with an abelian Frobenius complement, $\bar{G}'$ is the Frobenius kernel and is an elementary abelian group. It follows that $\bar{G}' \leq \bar{H}$, and $\bar{H}$ is both a maximal abelian normal subgroup of $\bar{G}$ and a q-group for some prime $q \neq p$. By 2.9 of **[8]** and 12.3 of **[7]** we conclude that $\bar{H}=\bar{G}'$. Now it follows from 12.4 of **[7]** that if $\zeta \in \mathrm{Proj}(H, \alpha_H)$, then $[G:H]\zeta(1)$ is the degree of an element of $\mathrm{Proj}(G, \alpha)$. Thus $[G:H] \leq p^e$.

Finally let A_{e-1} be the inverse image in G of the Frattini subgroup of G/H. Then by 1.3 A_{e-1} has p.r.x.$(e-1, \alpha_{A_{e-1}})$, and G/A_{e-1} is an elementary abelian p-group of order $\leq p^e$.

Case 2: $\bar{G}$ is a p-group.
In this case let A_{e-1} be the inverse image in G of the Frattini subgroup of $\bar{G}$. Then by 1.3 A_{e-1} has p.r.x.$(e-1, \alpha_{A_{e-1}})$. Also by Case 2 p. 451 of **[8]** and 1.7, G/A_{e-1} is an elementary abelian p-group of order $\leq p^{2(e+a)+1}$.

We now state a derivative of the theorem which we shall use in Section 2.

COROLLARY 1.9. *Let G have* p.r.x.(e, α), *and suppose G has a normal abelian p-complement and an abelian Sylow p-subgroup. Then G has a series of subgroups*

$$A_0 \trianglelefteq A_1 \trianglelefteq \ldots \trianglelefteq A_e = G,$$

such that A_0 is abelian, $[\alpha_{A_0}]=[1]$, and A_i/A_{i-1} is an elementary abelian p-group with not more than i generators. Hence G has an abelian subgroup A_0 with $[\alpha_{A_0}]=[1]$ whose index divides $p^{e(e+1)/2}$.

Proof. The result follows from the proof of 1.8 by strengthening the inductive hypothesis and noting that Case 2 of that proof cannot occur.

Let G have p.r.x.(e, α). Then the previous two results have relied on the assumption that G has a normal abelian p-complement. The last two results in this section deal with the circumstances under which this assumption is tenable.

It is convenient in what follows to call a group H an F_i-group if H is a Frobenius group of order $p^i q$, where q is a prime number such that q divides $p^i - 1$, and the Frobenius kernel of H is an elementary abelian group of order p^i.

LEMMA 1.10. *Let G be an F_i-group with Frobenius kernel S, and suppose G is a (p, α)-group. Then* $\text{Proj}(G, \alpha)$ *consists of q-extensions of the unique element of* $\text{Proj}(S, \alpha_S)$.

Proof. Let $\zeta \in \text{Proj}(S, \alpha_S)$. Then it follows from 12.4 of [7] that $\zeta(1)^2 = |S|$. Also ζ extends to G by 1.4.

To illustrate the above lemma we may take $p = 2$ and then A_4 is the only example of an F_2-group which also has p.r.x.$(1, \alpha)$, for $[\alpha]$ the non-trivial element of $M(A_4)$. We also note for future reference that if α is a cocycle of a group G such that $|\text{Proj}(G, \alpha)| = 1$, then G is said to be of α-central type.

If N is a normal subgroup of a group G, we shall let inf denote the inflation homomorphism from $M(G/N)$ into $M(G)$.

PROPOSITION 1.11. *Let G be a group of minimal order which has* p.r.x.(e, α) *but does not possess a normal abelian p-complement. Let $K \lhd G$ be maximal such that G/K is non-abelian. Then G/K is an F_i-group for $1 \leq i \leq 2a$, where* $p^a = \min\{\xi(1) : \xi \in \text{Proj}(G, \alpha)\}$.

In particular if $a = 1$, then either G/K is an F_2-group which has p.r.x.$(1, \beta)$ *for some cocycle β of G/K with* $\text{inf}([\beta]) = [\alpha]$, *or G/K is an F_1-group.*

Proof. Let $L/K = (G/K)'$. Then by 12.3 and 12.4 of [7] and 1.5, all the non-linear irreducible ordinary characters of G/K have equal degree f, G/K is a Frobenius group with an abelian Frobenius complement of order f and an elementary abelian Frobenius kernel L/K; also if $\zeta \in \text{Proj}(L, \alpha_L)$ then $V(\zeta) \leq K$ and $[L:K]$ divides $\zeta(1)^2$, where $V(\zeta)$ denotes the vanishing-off subgroup of ζ. Thus $[L:K]$ divides p^{2a} and f divides $[L:K] - 1$. Now let T be a maximal normal subgroup of G containing L. Then by 1.2, T and hence T/K have normal abelian p-complements. Since f is coprime to p, it follows that $T = L$ and $f = q$ for some prime $q = [G:L]$.

It remains to show that if $a = 1$ and G/K is an F_2-group, then G/K has p.r.x.$(1, \beta)$ for some cocycle β of G/K with $\text{inf}([\beta]) = [\alpha]$. Let $\xi \in \text{Proj}(G, \alpha)$ with $\xi(1) = p$. Then $\xi_L = \zeta$ for some $\zeta \in \text{Proj}(L, \alpha_L)$, since $[G:L] = q$. However, the inner product $\langle \zeta_K, \zeta_K \rangle = p^2$ since $V(\zeta) \leq K$, and so $\xi_K = p\lambda$ for some $\lambda \in \text{Proj}(K, \alpha_K)$. Thus $I_G(\lambda) = G$, and the desired result follows from 1.2 and 1.10.

2. Prime degree. Throughout Section 2 we shall assume that G is a group having p.r.x.$(1, \alpha)$ and a normal abelian p-complement.

DEFINITION 2.1. A subgroup A of G is said to be special if

(i) A is an abelian normal subgroup of G such that $[\alpha_A] = [1]$;

(ii) G/A is an elementary abelian p-group;

(iii) if $A < B$, then either B is non-abelian or B is abelian but $[\alpha_B] \neq [1]$.

It is convenient to call the special subgroup A more-special if $A<B$ implies B is non-abelian, and less-special otherwise. We note that with the assumptions made above 1.8 yields a special subgroup of G of index dividing p^5.

LEMMA 2.2. *Suppose p is odd, and let A be a special subgroup of G. Then each element $a \in A$ has $C_\alpha(a)=A$ or G, where $C_\alpha(a)=\{x \in C_G(a): \alpha(a,x)=\alpha(x,a)\}$.*

Proof. If $a \in A$, then $A \le C_\alpha(a)$ since $[\alpha_A]=[1]$. Thus the result is trivial if $[G:A] \le p$.

Now suppose $a \in A$ with $A < C_\alpha(a) < G$. Choose $x \in C_\alpha(a) - A$ and $y \in G - C_\alpha(a)$, and set $K=\langle A, x, y\rangle$. By (ii) of 2.1 it is clear that $[K:A]=p^2$. Since $x \notin A$, $B=\langle A, x\rangle$ is either non-abelian or is abelian with $[\alpha_B] \ne [1]$. In either case there exists $b \in A$ with $x \notin C_\alpha(b)$. Let $\bar{z}$ denote the element $z \in G$ viewed as an element of the twisted group algebra $\mathbb{C}G_\alpha$. Then $u=(\bar{x})^{-1}(\bar{b})^{-1}\bar{x}\bar{b}$ and $v=(\bar{y})^{-1}(\bar{a})^{-1}\bar{y}\bar{a}$ are non-identity elements of $\mathbb{C}A_{\alpha_A}$. Now by working in $\mathbb{C}A_{\alpha_A}$ the proof of 3.3 of **[8]** carries over to our situation to give a contradiction, provided that $\bar{1}+uv \ne u+v$. However writing $u=c\bar{w}$ and $v=k\bar{z}$, where $w=[x,b]$, $z=[y,a]$ and c, k are pth roots of unity; we have that $\bar{1}+uv=u+v$ if and only if $z^{-1}=w=z\ne 1$, $k=-c$, $p=2$, and $\alpha(w,w)=1$.

We note in the context of 2.2 that if A is more-special then replacing $C_\alpha(a)$ by $C_G(a)$ in the proof yields that $C_G(a)=A$ or G for each $a \in A$, provided that $\bar{1}+uv\ne u+v$. This observation coupled with the lemma allows us to describe the α-regular elements of a special subgroup of G. Let $U=\{z \in Z(G): z \text{ is } \alpha\text{-regular}\}$, the reader may refer to **[12]** for various characterizations of U.

LEMMA 2.3. *Let A be a special subgroup of G and suppose $[G:A]\ne p$. Then $A \cap Z(G)=U$. In particular if A is more-special then $U=Z(G)$.*

Proof. Clearly $U<A$ by definition of A, and also $Z(G)<A$ if A is more-special. Let $K=A \cap Z(G)$. Then by 2.2 (and its proof in the case $p=2$) each $a \in K$ has $C_\alpha(a)=A$ or G. Let $\lambda \in \text{Proj}(K, \alpha_K)$, then $\lambda^x=\lambda$ if and only if $\alpha(a,x)=\alpha(x,a)$ for all $a \in K$. Thus $I_G(\lambda)=\bigcap_{a\in K} C_\alpha(a)=A$ or G. If $I_G(\lambda)=A$, then λ has $[G:A]$ conjugates. Since G has p.r.x.$(1,\alpha)$ we conclude that $K=U$.

For the rest of this section we shall assume that if A is a special subgroup of G then $C_\alpha(a)=A$ or G for each $a \in A$, we shall also assume that $C_G(a)=A$ or G for each $a\in A$ when A is more-special. Of course these assumptions are certainly valid for $p\ne 2$ by 2.2, and will be discussed in detail for $p=2$ in Section 3.

PROPOSITION 2.4. *Let A be a special subgroup of G and suppose $[G:A]\ne p$. Then every element of A is α-regular if A is more-special, whereas the elements of U are the α-regular elements of A if A is less-special.*

Proof. Suppose $a \in A-U$. Then $C_\alpha(a)=A$. However using 2.3 we have that $C_G(a)=A$ if A is more-special, whereas $A<C_G(a)$ if A is less-special.

We next show that U has index p in a special subgroup A with $[G:A]>p$. This will enable us to classify G/U in subsequent results.

PROPOSITION 2.5. *Let A be a special subgroup of G of index p^t.*

(a) *If $t>1$ or $t=1$ and A is less-special, then $[A:U]=p$.*

(b) *If $t=1$ and A is more-special, then $p\,|Z(G)|\,|G'|=|G|$.*

Proof. Suppose first that A is less-special. Let B be an abelian group with $[B:A]=p$. Then $[\alpha_B]\neq[1]$, so that all elements of $\mathrm{Proj}(B,\alpha_B)$ have degree p and vanish on $B-A$. Let T denote the set of α_B-regular elements of B. Then since B is abelian it follows as in the proof of 1.8 that $[A:T]=p$, and all elements of $\mathrm{Proj}(B,\alpha_B)$ are non-zero exactly on T. From 2.4 (or trivially if $B=G$) we conclude that $T=U$, since every element of $\mathrm{Proj}(G,\alpha)$ restricts irreducibly to B.

Now suppose A is more-special. Suppose also $t>1$. Then by 1.8 of [**3**], 2.3, 2.4, and the proof of 3.5 of [**8**] we obtain the equation

$$|U|\,(p^t-1)=|A|\,(p^{t-1}-1)+k(p^t-p^{t-1}), \tag{1}$$

where k is the number of G-invariant elements of $\mathrm{Proj}(A,\alpha_A)$. Suppose $k=0$, then $p^{t-1}-1$ divides p^t-1, which is impossible. Thus there exists a G-invariant element of $\mathrm{Proj}(A,\alpha_A)$. Let Q be the Sylow p-subgroup of A, then it follows from 1.2 that G/Q has p.r.x.$(1,\beta)$ for some cocycle β of G/Q such that $\inf([\beta])=[\alpha]$. The proof of 3.4 of [**8**], and 1.9 now yield that the normal abelian p-complement of G is central. Thus since $[A:U]$ is a power of p, we obtain from (1) that $[A:U]=p$. Suppose now $t=1$. Then G is non-abelian and has p.r.x.(1, 1), so that (b) is just 3.5(b) of [**8**]. (Note: the proof of 3.5(b) of [**8**] is independent of the assumption that G is a p-group.)

LEMMA 2.6. *Suppose A is a special subgroup of G of minimal index p^t, where $t>1$ or $t=1$ and A is less-special. Then*

(i) *G/U has exponent p;*

(ii) *G/T is abelian for any T with $U<T\trianglelefteq G$;*

(iii) *if G/U is non-abelian, then $A/U=(G/U)'$ is the unique minimal normal subgroup of G/U.*

Proof. (i) Suppose there exists an element x of order p^2 in G/U. Then $x^p\in A-U$, so that A is a proper subgroup of $\langle x,U\rangle$ since $[A:U]=p$ by 2.5. But $\langle x,U\rangle$ is a special subgroup of G, contrary to the minimality of $[G:A]$.

(ii) Suppose there exists $T\triangleleft G$ with $U<T$ such that G/T is non-abelian. Then T is abelian with $[\alpha_T]=[1]$ by 1.2. Let F be the inverse image in G of the Frattini subgroup of G/T. Then F is abelian with $[\alpha_F]=[1]$ by 1.3, but $[G:F]<[G:A]$ by 2.5, a contradiction.

(iii) If G/U is non-abelian, then by (ii) U is a maximal normal subgroup of G such that G/U is non-abelian. Thus $(G/U)'=A/U$ is the unique minimal normal subgroup of G.

We can now prove Theorem 1, noting that the proof we shall give still holds in the case $p=2$, provided that all special subgroups of G of minimal index satisfy the assumptions of this section.

Proof of Theorem 1. Let A be a special subgroup of G of minimal index p^t, so that $t\leq 5$ by 1.8. If $t=0$ or 1, then G satisfies (i) or (ii) respectively. Also if $[\alpha]=[1]$, then G satisfies (i), (ii) or (iii) by Theorem II of [**8**]. So suppose $t>1$ and $[\alpha]\neq[1]$. Then by 2.5 and 2.6, G/U has exponent p and order p^{t+1}, and if G/U is non-abelian (so $p\neq 2$) it must be an extra-special p-group of order p^3 or p^5, since in this case $Z(G/U)$ is cyclic.

Stage 1: If $t\geq 3$, G/U is not elementary abelian.

Let $x\in G-U$. Then $\langle x,U\rangle$ is a special subgroup of G, so that $C_\alpha(x)=A$ from 2.2. Thus every α-regular conjugacy class of G contains either 1 or p^t elements. Let $u=|U|$

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and $[r]$ temporarily denote the integral part of the real number r. Then G has at most $u+[(|G|-u)/p^t]=u+[u(p^{t+1}-1)/p^t]$ α-regular conjugacy classes. But we know that G has exactly $|G|/p^2=up^{t-1}$ α-regular conjugacy classes, since G has p.r.x.$(1,\alpha)$ and $[\alpha]\neq[1]$. So we certainly require that $1+p^{-t}(p^{t+1}-1)\leq p^{t-1}$ i.e. $p^t-1\geq p^{t+1}(p^{t-2}-1)$, which is clearly impossible for $t\geq 3$.

Stage 2: G/U is not an extra-special group of order p^5.

Let $\bar{G}=G/U$ and $\bar{Z}=A/U$. Then from the proof of 3.3.6 of [**10**], $\inf: M(\bar{G}/\bar{Z})\to M(\bar{G})$ is a surjection with kernel of order p. Let β be a cocycle of $\bar{G}/\bar{Z}$ such that $\ker(\inf)=\langle[\beta]\rangle$ and $\beta^p=1$. Then by V.16.14 of [**6**] only the identity element of $\bar{G}/\bar{Z}$ is β^i-regular for $1\leq i\leq p-1$. Now each $\xi\in \mathrm{Proj}(G,\alpha)$ has $\xi_U=p\lambda$ for some $\lambda\in \mathrm{Proj}(U,\alpha_U)$. It follows from 1.2 both that there exist cocycles $\bar{\alpha}$ of G/U with $[\bar{\alpha}]$ inflated to G equal to $[\alpha]$, and that G/U has p.r.x.$(1,\bar{\alpha})$ for each such $\bar{\alpha}$. Similarly, since every element of $\bar{Z}$ is $\bar{\alpha}$-regular, there exist cocycles $\beta^i\gamma$ for $0\leq i\leq p-1$ with $\inf([\gamma])=[\bar{\alpha}]$ for which $\bar{G}/\bar{Z}$ has p.r.x.$(1,\beta^i\gamma)$. We thus require that $\bar{G}/\bar{Z}$ contains exactly p^2 $\beta^i\gamma$-regular elements for each i with $0\leq i\leq p-1$.

Let $\bar{G}/\bar{Z}=\langle x_1\rangle\times\ldots\times\langle x_4\rangle$, and $\beta(x_i,x_j)=\omega^{c_{ij}}$, where ω is a non-trivial pth root of unity. Let C be the skew-symmetric matrix whose (i,j)th entry is c_{ij} for $j>i$. Then $x_1^{b_1}\ldots x_4^{b_4}$ is β-regular if and only if $C\vec{b}=\vec{0}$ in $\mathbb{Z}_p^4$, where $\vec{b}=[b_1,\ldots,b_4]^T$. Now since no non-trivial element of $\bar{G}/\bar{Z}$ is β-regular we have that C has rank 4, and so there exists $M\in \mathrm{GL}(4,p)$ with

$$M^TCM=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}.$$

Let D be the matrix constructed from γ in the same way, then we require that M^TDM+iM^TCM has rank 2 for $0\leq i\leq p-1$. So if

$$M^TDM=\begin{bmatrix}0&a&b&c\\-a&0&d&e\\-b&-d&0&f\\-c&-e&-f&0\end{bmatrix},$$

we must have that $(a+i)(f+i)+be-cd=0$ for $0\leq i\leq p-1$. Setting $i=-a$ gives $be-cd=0$, and we are left with $(a+i)(f+i)=0$ for $0\leq i\leq p-1$, which is impossible for $p\geq 3$.

Stages 1 and 2 thus yield that G/U is of type (iii).

The converse statement in the theorem is proved in essentially the same way as that of Theorem II of [**8**].

As an immediate and perhaps surprising corollary of Theorem 1 we obtain the following stronger version of 1.7.

COROLLARY 2.7. *Let p be odd and let G have* p.r.x.$(1,\alpha)$. *Suppose G has a normal abelian p-complement. Then G has* p.r.x.$(1,\alpha^n)$ *for any integer n.*

To conclude this section we note that if we allow $p=2$ in the matrix calculations of stage 2 above, then we find that the extra-special 2-groups G of order 32 have p.r.x.$(1,\alpha)$

for 10 cohomology classes out of the 32 in $M(G)$. Since these groups are not classified by Theorem 1 (with $p=2$), they provide counter-examples to the results of 2.2 and will be dealt with in the following section.

3. Degree two. In this section we shall deal exclusively with a group G having p.r.x.$(1, \alpha)$ and a normal abelian p-complement which is not classified by Theorem 1. In particular then $p=2$, and we can assume that there exists a special subgroup A of G of minimal index $2^t>2$, with an element $a \in A$ such that either $A<C_\alpha(a)<G$ or A is more-special and $A<C_G(a)<G$. With this notation fixed for the duration of this section we can now prove.

PROPOSITION 3.1. *A is more-special and G/A is an elementary abelian group of order* 4.

Proof. Let $C(a)$ denote $C_\alpha(a)$ or if A is more-special $C_G(a)$. Then we may treat the two separate possibilities above simultaneously. Now the proof of 2.2 yields for all $x \in C(a)-A$, all $y \in G-C(a)$, and all $b \in A$ with $x \notin C(b)$, that $z^{-1}=w=z \neq 1$, where $w=[x, b]$ and $z=[y, a]$. If $C(a)=C_\alpha(a)$ this gives that a is α-regular, and then that A must be more-special, otherwise we can obtain that z or $w=1$ respectively. Thus A is more-special and $C(a)=C_G(a)$.

Now for any choice of $y_1, y_2 \in G-C_G(a)$ we must have that $[y_1, a]=[y_2, a]$ i.e. $y_1 y_2^{-1} \in C_G(a)$, so that $[G:C_G(a)]=2$. By 2.5 of **[8]** there exists a subgroup T of G of index 2 with $A<T$, such that T has p.r.x.$(1, \alpha_T)$ and $[\alpha_T]=[1]$. Clearly A is a special subgroup of T, so that by 3.3 of **[8]** or 2.2 $C_{\alpha_T}(a)=C_T(a)=A$ or $C_{\alpha_T}(a)=C_T(a)=T$.

In the former case there exists $g \in G-T$ with $g \in C_G(a)$, and since $g^2 \in T$ we have that $g^2 \in A$. Thus in this case it follows that $C_\alpha(a)=C_G(a)=\langle A, g\rangle$ or $C_\alpha(a)=A$ and $C_G(a)=\langle A, g\rangle$.

In the latter case we conclude by 2.3 that $C_\alpha(a)=C_G(a)=T$. In this case if T is the unique subgroup of G satisfying 2.5 of **[8]** then T must be abelian, contrary to the fact that A is more-special. So we may let S be another subgroup of G of index 2 with the same properties as T. Then as before $C_S(a)=A$ or S. Since $ST=G$, we must have that $C_S(a)=A$, and the desired results follows as in the first case.

By 3.1 each element b of A is classified into one of four types according to whether 1. $b \in U=Z(G)$; 2. $C_\alpha(b)=A=C_G(b)$; 3. $C_\alpha(b)=A$ and $[G:C_G(b)]=2$; 4. $C_\alpha(b)=C_G(b)$ and $[G:C_G(b)]=2$. Let z, r, s, and t denote respectively the number of elements of each type contained in A. Also let t_1, t_2, and t_3 be the number of elements of type 4 centralized by $T_1=\langle A, x\rangle$, $T_2=\langle A, y\rangle$, and $T_3=\langle A, xy\rangle$ respectively; where $G/A=\langle Ax, Ay\rangle$ and $C_G(a)=\langle A, x\rangle$. Using this notation we now show.

PROPOSITION 3.2. (i) *G/U has order* 16 *and exponent* 2 *or* 4.

(ii) *G/U has* p.r.x.$(1, \beta)$ *for any cocycle β of G/U with* $\inf([\beta])=[\alpha]$.

(iii) *G/U is of γ-central type for some cocycle γ of G/U with* $\inf([\gamma])=[1]$.

Proof. We first note by 3.1 that A is a special subgroup of T_i of index 2 with respect to the trivial cocycle of T_i, so that from 3.5 of **[8]** $|A|=|Z(T_i)|\,|T_i'|$ for $1 \leq i \leq 3$.

Case 1: A contains an element of type 3.

We may assume for notational convenience that a is of type 3. Now the number of x-invariant elements of $\operatorname{Proj}(A, \alpha_A)$ is 0, since a is not α-regular. Let $\lambda \in \operatorname{Proj}(A, \alpha_A)$, then we may assume without loss of generality that $I_G(\lambda) = T_2$. If this is true for all elements of $\text{Proj}(A, \alpha_A)$ then T_2 is abelian, contrary to the fact that A is more-special. So there exists $\lambda' \in \text{Proj}(A, \alpha_A)$ with $I_G(\lambda') = T_3$. Now let b be any type 3 element, then by the arguments above $C_G(b) = T_1$. So by 1.8 of [**4**], $0 = z - s + t_1$; and similarly by considering y and xy, $2z + t_2 + t_3 = |A| = z + r + s + t$, so that $z = r + s + t_1$. Thus $r + 2t_1 = 0$, and hence $r = t_1 = 0$, $z = s$, and $|Z(T_1)| = 2z$.

We now consider T_1'. We note from the proof of 3.1 that $[x', b]$ is the same element w of order 2 for all $b \in A - Z(T_1)$ and all $x' \in T_1 - A$. Let $c, d \in A$, then

$$[cx, dx] = \begin{cases} 1, & \text{if } c, d \in Z(T_1) \text{ or } c, d \notin Z(T_1); \\ w, & \text{otherwise.} \end{cases}$$

Similar calculations show that $[cx, d]$ or $[c, dx]$ is 1 or w, and hence T_1' is the group of order 2 generated by w. Thus $[A:U] = 4$.

Case 2: A contains no element of type 3.

In this case every element of A is α-regular, and a is of type 4. As in the proof of 3.1 we have for c a type 4 element that for all $x' \in C_G(c) - A$, all $y' \in G - C_G(c)$, and all $b \in A$ with $x' \notin C_G(b)$, that $z^{-1} = w = z \neq 1$; where $w = [x', b]$ and $z = [y', c]$. Suppose c has $C_G(c) = T_2$, then it follows that for any type 2 element 'b' that $[y, b] = [x, b]$, so that $xy^{-1} \in C_G(b) = A$, a contradiction. We obtain a similar contradiction if we assume that $C_G(c) = T_3$. Thus either $r = 0$, or all type 4 elements c have $C_G(c) = T_1$.

Now $|A| = z + r + t$, and A contains $z + \frac{r}{4} + \frac{t}{2}$ conjugacy classes of G. Suppose G fixes k elements and has m orbits of length 2 in its action on $\text{Proj}(A, \alpha_A)$. Then $|A| = k + 2m$, and it follows from 1.8 of [**4**] that $k + m = z + \frac{r}{4} + \frac{t}{2}$. Thus $k = z - \frac{r}{2}$, and so $z \geq \frac{r}{2}$. Now $|Z(T_i)| = z + t_i$ and $U \leq Z(T_i)$, so that z divides t_i for $1 \leq i \leq 3$. Thus z divides t and hence r. We conclude that $r = 0$, z, or $2z$.

Suppose $r = 0$. Then as in Case 1 we may now show that T_i' has order 2 for $1 \leqslant i \leqslant 3$. So $z + t_1 = t_2 + t_3$, $z + t_2 = t_1 + t_3$, $z + t_3 = t_1 + t_2$, and hence adding we obtain that $t = 3z$. Thus $[A:U] = 4$.

Suppose now $r > 0$. Then $|Z(T_1)| = z + t$. If $r = 2z$, then $|A| = 3z + t$. So $z + t$ divides $2z$, and since z divides t we conclude that $t = z$. Thus $[A:U] = 4$. Finally if $r = z$, then we obtain similarly that $z + t$ divides z, which is impossible.

We have thus proved that G/U has order 16, also G/U has p.r.x.$(1, \beta)$ for any cocycle β of G/U with $\inf([\beta]) = [\alpha]$ from 1.2. Now since A is a more-special subgroup of G of minimal index 4 and $[A:U] \neq 2$, we have that G has p.r.x.(2, 1) but not p.r.x.(1, 1) by 3.5 of [**8**] and 2.3. Thus there exists a cocycle γ of G/U with $\inf([\gamma]) = [1]$ for which G/U is of γ-central type. Finally suppose Ug has order 8 in G/U. Then $\langle U, g \rangle$ is special and has index 2 in G, contrary to the definition of A. Thus G/U has exponent 2 or 4.

We can now immediately proceed to classify G/U, and hence prove Theorem 2. The proof we shall give actually contains additional information about the various cohomology classes of G/U.

Proof of Theorem 2. Suppose firstly that G/U is abelian. Then by 3.2, G/U has exponent 2 or 4 and order 16, so $G/U \cong C_2 \times C_2 \times C_2 \times C_2$, or $C_4 \times C_4$, or $C_4 \times C_2 \times C_2$.

Suppose $G/U \cong C_2 \times C_2 \times C_2 \times C_2$. The set of skew-symmetric 4×4 matrices over $\mathbb{Z}_2$ consists of 1 matrix of rank 0, 28 of rank 2, and 35 of rank 4. These correspond as in the proof of Theorem 1 to 1, 28, and 35 cohomology classes of G/U for which G/U has minimal p.r.x. 0, 1, and 2 respectively.

Suppose $G/U \cong C_4 \times C_4$. Then $M(G/U) \cong C_4$. Now consider $\inf: M(G/U) \to M(G)$. We have that $[\alpha] = \inf([\beta])$ for some $[\beta] \in M(G/U)$, and so ker(inf) must be of order 2, since G does not have p.r.x.$(1, 1)$. Thus $[\beta]$ has order 4, and G/U is of β-central type, contrary to 3.2(ii).

Finally suppose $G/U \cong C_4 \times C_2 \times C_2$. Then by 5.4 of **[12]** the set of elements which are γ-regular for all cocycles of G/U is isomorphic to C_2. Thus in this case G/U is not of γ-central type for any cocycle γ of G/U, contrary to 3.2(iii). Alternatively Lemma 2 of **[1]** also gives this result.

Now suppose G/U is non-abelian. Then again since G/U has exponent 4 and order 16 we have that G/U is isomorphic to one of the following five (non-isomorphic) groups: 1. $D_4 \times C_2$; 2. R, as in the statement of Theorem 2; 3. $Q \times C_2$, Q the quaternion group; 4. $\langle x, y, z : x^4 = y^2 = z^2 = 1, xy = yx, zx = xz, yz = zx^2y \rangle$; 5. $C_4 \ltimes C_4$.

In Case 1, $M(D_4 \times C_2) \cong C_2 \times C_2 \times C_2$ from p. 378 of **[13]**. Here we may consider normal cocycles γ of $D_4 \times C_2$ as in Proposition 1 of **[13]**, and show that the elements of order 4 in D_4 are γ-regular for exactly 6 classes $[\gamma]$ of $M(D_4 \times C_2)$. For these classes $D_4 \times C_2$ cannot be of γ-central type and so must have p.r.x.$(1, \gamma)$. It follows that $D_4 \times C_2)$ has p.r.x. 1 for 6 cohomology classes of $D_4 \times C_2$, and has minimal p.r.x. 2 for the remaining 2 classes.

In Case 2, $M(R) \cong C_2 \times C_2$ from p. 378 of **[13]**, and from **[5]** R has p.r.x. 1 for 3 cohomology classes of R and has minimal p.r.x. 2 for the remaining class.

In Cases 3, 4, or 5 we may consider groups of order 32 with centre of order 2 (see **[14]**), and we conclude by using 3.5 of **[12]** that G/U is not of γ-central type for any cocycle γ of G/U, contrary to 3.2(iii).

Finally we give examples to show that groups satisfying Theorem 2 do exist, noting that examples satisfying Theorem 1 have already been given on p. 456 of **[8]**.

EXAMPLES 1. Let G be either of the extra-special 2-groups of order 32. Then G has an ordinary character of degree 4 by V.16.14 of **[6]**. We indicated at the end of Section 2 how to show that G has p.r.x.$(1, \alpha)$ for 10 cohomology classes $[\alpha]$ out of the 32 in $M(G)$. Also for any such class $[\alpha]$, $U = Z(G)$, so that G/U is elementary abelian of order 16.

2. Let G be either of the groups G_1, G_2 of order 32 described in **[5]**. Then from **[5]** $G/Z(G) \cong R$, and G has both an ordinary character of degree 4 and p.r.x.$(1, \alpha)$ for the unique non-trivial cohomology class of G. Also $U = Z(G)$ for $[\alpha]$.

3. Let G be either of the two groups of order 32 with $G/Z(G) \cong D_4 \times C_2$. Then G' is cyclic of order 4 from **[14]**. Also G has an ordinary character of degree 4 by using 3.5 of **[8]**, or since G has 11 conjugacy classes from **[14]**. It follows that the kernel of the inflation homomorphism from $G/Z(G)$ into G has order 2. Now from the proof of Theorem 2 exactly four classes of $M(D_4 \times C_2)$ will inflate to give two classes $[\alpha]$ of $M(G)$ for which G has p.r.x.$(1, \alpha)$. Lastly $U = Z(G)$ for both such $[\alpha]$.

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