A Symbolic proof of Euler's Addition Theorem for Elliptic Functions.

By W. SADDLER.

(*Received 5th October 1925. Read 5th June 1925*).

§1. Let $a_0 x^4 + 4\,a_1 x^3 + 6\,a_2 x^2 + 4\,a_3 x + a_4 \equiv a_x^4 \equiv b_x^4$ be a binary quartic; then I propose to show symbolically that the solution of Euler's differential equation

$$\frac{dy}{\sqrt{a_y^4}} = \frac{dz}{\sqrt{a_z^4}},$$

can be written in the form

$$k\,a_y^2\,a_z^2 + \lambda\,h_y^2\,h_z^2 - \frac{\mu}{2}(y\,z)^2 = 0$$

where

$$k^2 - \lambda\,\mu - \tfrac{1}{6}\,\lambda^2\,i = 0,$$

$$i = (a\,b)^4 = 2\,(a_0\,a_4 - 4\,a_1\,a_3 + 3a_2^2) \text{ and } k,\ \lambda,\ \mu$$

are otherwise arbitrary.

Let $h_x^4 = (a\,b)^2\,a_x^2\,b_x^2$ be the Hessian of the quartic, and first let y and z satisfy $a_y^2\,a_z^2 = 0$, namely the second polar of the quartic in y, with respect to z.

Since $$(y\,z)\,(a\,b) = a_y\,b_z - a_z\,b_y\,,$$

therefore
$$\begin{aligned}(y\,z)^2\,(a\,b)^2\,a_y^2\,b_y^2 &= (a_y\,b_z - a_z\,b_y)^2\,a_y^2\,b_y^2,\\ &= a_y^4\,b_y^2\,b_z^2 - 2\,a_y^3\,a_z\,b_y^3\,b_z + a_y^2\,a_z^2\,b_y^4,\\ &= -2\,(a_y^3\,a_z)^2,\end{aligned}$$

since $$a_y^2\,a_z^2 = 0 \qquad \text{(i)}$$

Thus $$(y\,z)^2\,(a\,b)^2\,a_y^2\,b_y^2 = -2\,(a_y^3\,a_z)^2,$$

and similarly $$(y\,z)^2\,(a\,b)^2\,a_z^2\,b_z^2 = -2\,(a_z^3\,a_y)^2,$$

whence $$\frac{\sqrt{h_y^4}}{\sqrt{h_z^4}} = \pm\,\frac{a_y^3\,a_z}{a_z^3\,a_y} \qquad \text{(ii)}$$

From (i), by differentiation, $a_1\,a_y\,a_z^2 + \dfrac{dz_1}{dy_1}\,a_1\,a_y^2\,a_z = 0$

or $$(a_y a_z^3 - a_2 a_y a_z^2 z_2) + \frac{dz_1}{dy_1}(a_y^2 a_z^2 - a_2 z_2 a_z a_y^2) = 0$$

or $$(a_y a_z^3 - a_2 a_y a_z^2 z_2) - \frac{dz_1}{dy_1} a_2 z_2 a_z a_y^2 = 0.$$

Similarly $$-a_2 a_y a_z^2 y_2 + \frac{dz_1}{dy_1}(a_y^3 a_z - a_2 y_2 a_y^2 a_z) = 0.$$

Since y_2 and z_2 are introduced to make the functions homogeneous we may replace them by unity. By subtraction we obtain

$$a_y a_z^3 - \frac{dz_1}{dy_1} a_y^3 a_z = 0.$$

Thus $$\frac{dy_1}{a_y^3 a_z} = \frac{dz_1}{a_z^3 a_y} \quad \text{or} \quad \frac{dy_1}{\sqrt{h_y^4}} = \pm \frac{dz_1}{\sqrt{h_z^4}}.$$

Let $f_{k\lambda} \equiv kf + \lambda h \equiv k a_x^4 + \lambda h_x^4$, and next let y and z satisfy $k a_z^2 a_y^2 + \lambda h_z^2 h_y^2 = 0$. Since the Hessian of $kf + \lambda h$ * is

$$k^2 h + \tfrac{1}{3} i k \lambda f + \lambda^2 (\tfrac{1}{3} j f - \tfrac{1}{6} i h),$$

where $i = (ab)^4$, $j = (ab)^2 (bc)^2 (ca)^2$, choose $k^2 - \frac{1}{6}\lambda^2 i = 0$.

Then $\dfrac{dz_1}{\sqrt{a_z^4}} = \pm \dfrac{dy_1}{\sqrt{a_y^4}}$ where y and z are connected by $k a_z^2 a_y^2 + \lambda h_z^2 h_y^2 = 0$, $k^2 = \frac{1}{6}\lambda i$.

§ 2. **Again, if y and z satisfy,**

$$a_y^2 a_z^2 - \frac{\mu}{2}(yz)^2 = 0,$$

where μ is a constant and

$$(yz)^2 = (y_1 z_2 - y_2 z_1)^2,$$

then by proceeding as before,

$$(yz)^2 (ab)^2 a_y^2 b_y^2 = -2(a_y^3 a_z)^2 + \mu (yz)^2 a_y^4,$$

thus

$$(yz)^2 \{(ab)^2 a_y^2 b_y^2 - \mu a_y^4\} = -2(a_y^3 a_z)^2;$$

similarly

$$(yz)^2 \{(ab)^2 a_z^2 b_z^2 - \mu a_z^4\} = -2(a_z^3 a_y)^2.$$

---

* GRACE and YOUNG, *Algebra of Invariants*, p. 198.

Hence $$\frac{a_y^3 a_z}{\sqrt{h_y^4 - \mu a_y^4}} = \pm \frac{a_z^3 a_y}{\sqrt{h_z^4 - \mu a_z^4}}$$

Differentiating $$a_y^2 a_z^2 = \frac{\mu}{2}(yz)^2,$$

we obtain

$$2 a_1 a_y a_z^2 + 2 a_y^2 a_1 a_z \frac{dz_1}{dy_1} = \mu (yz) \left\{ z_2 - y_2 \frac{dz_1}{dy_1} \right\}.$$

Multiplying by y_1, we have

$$2 a_y^2 a_z^2 - 2 a_2 y_2 a_y a_z^2 + 2 a_y^3 a_z \frac{dz_1}{dy_1} - 2 a_2 y_2 a_y^2 a_z \frac{dz_1}{dy_1}$$
$$= \mu (yz) \left\{ (yz) + y_2 z_1 - y_1 y_2 \frac{dz_1}{dy_1} \right\}$$

Using $a_y^2 a_z^2 = \frac{\mu}{2}(yz)^2$,

$$- 2 a_2 y_2 a_y a_z^2 + 2 a_y^3 a_z \frac{dz_1}{dy_1} - 2 a_2 y_2 a_y^2 a_z \frac{dz_1}{dy_1}$$
$$= \mu \left(y_2 z_1 - y_1 y_2 \frac{dz_1}{dy_1} \right) (yz).$$

Similarly on multiplying by z_1,

$$2 a_y a_z^3 - 2 a_2 z_2 a_y a_z^2 - 2 a_y^2 a_2 z_2 a_z \frac{dz_1}{dy_1} = \mu \left(z_1 z_2 - y_1 z_2 \frac{dz_1}{dy_1} \right) (yz).$$

Then, subtracting and replacing y_2 and z_2 by unity,

$$\frac{dz_1}{a_y a_z^3} = \frac{dy_1}{a_y^3 a_z},$$

or $$\frac{dz_1}{\sqrt{h_z^4 - \mu a_z^4}} = \pm \frac{dy_1}{\sqrt{h_y^4 - \mu a_y^4}} \quad \text{.........................(iii)}$$

In particular, if $\mu a_0 = a_0 a_2 - a_1^2$, the quartic is depressed to a cubic, while if μ is one of the roots of

$$k^3 - \tfrac{1}{2} i k \lambda^2 - \tfrac{1}{3} j \lambda^3 = 0 \, *$$

the equation (iii) reduces to the rational form

$$\frac{dz_1}{\text{quadratic}} = \frac{dy_1}{\text{quadratic}}.$$

---

* GRACE and YOUNG, *loco. cit.*, p. 198.

§ 3. Now let $f_{k\lambda} = kf + \lambda h$,
and let k, λ, μ satisfy

$$k\, a_z^2 a_y^2 + \lambda\, h_z^2 h_y^2 - \frac{\mu}{2}\, (y\, z)^2 = 0.$$

By substituting in equation (iii), with the notation

$$F(x) \equiv h\,(k^2 - \lambda\,\mu - \tfrac{1}{6}\,\lambda^2\, i) + f(\tfrac{1}{3}\, i\, k\, \lambda + \tfrac{1}{6} j\, \lambda^2 - k\,\mu),$$

we have proved

$$\frac{dz_1}{\sqrt{F(z)}} = \pm\, \frac{dy_1}{\sqrt{F(y)}},$$

whence *if* $$k^2 - \lambda\,\mu - \tfrac{1}{6}\,\lambda^2\, i = 0,$$

then $$\frac{dz_1}{\sqrt{a_z^4}} = \pm\, \frac{dy_1}{\sqrt{a_y^4}}$$

has an integral of the form

$$k\, a_z^2 a_y^2 + \lambda\, h_z^2 h_y^2 - \frac{\mu}{2}\, (y\, z)^2 = 0$$

where $$k^2 - \lambda\,\mu - \tfrac{1}{6}\,\lambda^2\, i = 0.$$

§ 4. *Canonical Form of the binary cubic.*

Consider the binary cubic

$$a_0 x^3 + 3\, a_1 x^2 + 3\, a_2 x + a_3 \equiv a_x^3 \equiv b_x^3 = \ldots$$

The Hessian is $h_x^2 = (a\, b)^2 a_x b_x$, and the first polar with respect to x' is $a_x^2 a_{x'}$.

Then if y and z are connected by the relation

$$a_y^2 a_z = 0,$$

by methods exactly similar to § 1 we obtain

$$\frac{dy_1}{a_y^3} = \frac{dz_1}{2\, a_y a_z^2} \quad \text{or} \quad \frac{dy_1}{h_y^2} = -\, \frac{dz_1}{2\, h_z^2}\,.$$

In particular, if $h_x^2 \equiv (x - \alpha)(x - \beta)$,

then $$\frac{dy}{(y - \alpha)(y - \beta)} + \frac{dz}{2\,(z - \alpha)(z - \beta)} = 0$$

has an integral of the form

$$\left(\frac{y - \alpha}{y - \beta}\right)^2 \left(\frac{z - \alpha}{z - \beta}\right) = \text{constant.}$$

But $a_y^2 a_z = 0$ was the relation between y and z.

https://doi.org/10.1017/S0013091500034301 Published online by Cambridge University Press

Hence $a_y^2 a_z$ can be written in the form

$$A\,(y-\alpha)^2(z-\alpha)+B\,(y-\beta)^2(z-\beta),$$

which reduces to a sum of two cubes, the well known canonical form of the cubic when $y=z$.

Not only so, for if we take the form as,

$$k\,a_x^3+\lambda\,t_x^3$$

where t_x^3 is the cubicovariant of a_x^3,

then if $$k\,a_y\,a_x^2+\lambda t_y\,t_x^2=0,$$

it follows that since the Hessian of $k\,a_x^3+\lambda\,t_x^3$

is $$(k^2+\tfrac{1}{2}\Delta\,\lambda^2)\,h_x^2$$

where Δ is the discriminant of the cubic, the cubic covariant has an exactly similar canonical form.

§ 5. *Application to Double Integrals.*

Consider a double binary (2, 2) form,

$$f=a_x^2\,\alpha_\xi^2=a_{00}\,x^2\,\xi^2+2\,a_{01}\,x^2\,\xi+2\,a_{10}\,x\,\xi^2+4\,a_{11}\,x\,\xi$$
$$+\,a_{02}\,x^2+a_{20}\,\xi^2+2\,a_{12}\,x+2\,a_{21}\,\xi+a_{22}$$

which takes the form $$k\,x^2\,\xi^2+l\,(x^2+\xi^2)+4\,m\,x\,\xi+k\,*$$

for the canonical case.

The (1, 1) transvectant $\tfrac{1}{2}J=\tfrac{1}{2}(a\,b)\,(\alpha\,\beta)\,a_x\,b_x\,\alpha_\xi\,\beta_\xi$

$$=\tfrac{1}{2}\,\theta_x^2\,\phi_\xi^2$$

$$=\tfrac{1}{4}\begin{vmatrix}\dfrac{\partial^2 f}{\partial x_1\,\partial\xi_1} & \dfrac{\partial^2 f}{\partial x_1\,\partial\xi_2}\\[2ex] \dfrac{\partial^2 f}{\partial x_2\,\partial\xi_1} & \dfrac{\partial^2 f}{\partial x_2\,\partial\xi_2}\end{vmatrix}$$

$$=\begin{vmatrix}a_{00}\,x\,\xi+a_{01}\,x+a_{10}\,\xi+a_{11} & a_{01}\,x\,\xi+a_{02}\,x+a_{11}\,\xi+a_{12}\\ a_{10}\,x\,\xi+a_{11}\,x+a_{20}\,\xi+a_{21} & a_{11}\,x\,\xi+a_{12}\,x+a_{21}\,\xi+a_{22}\end{vmatrix}$$

or in canonical form,

$$k\,m\,(x^2\,\xi^2+1)+(k^2-l^2)\,x\,\xi-l\,m\,(x^2+\xi^2).$$

---

* Prof. H. W. TURNBULL, *Proc. Roy. Soc., Edinb.*, 44 (1924), 23–50.

https://doi.org/10.1017/S0013091500034301 Published online by Cambridge University Press

Let y and z be cogredient variables, likewise η and ξ.

Then the first polar of $a_y^2 \alpha_\eta^2$ with respect to z, ζ is,

$$a_y a_z \alpha_\eta \alpha_\zeta \text{ or non-symbolically,}$$

$$a_{00}\, y z \zeta \eta + a_{01}\, y z (\zeta + \eta) + a_{10}\, \zeta \eta (y + z) + a_{11} (y + z)(\zeta + \eta)$$
$$+ a_{20}\, \zeta \eta + a_{02}\, y z + a_{21} (\eta + \zeta) + a_{12} (y + z) + a_{22}\ ,$$

or in the canonical form,

$$k\, y z \zeta \eta + m (y + z)(\zeta + \eta) + l (\zeta \eta + y z) + k\, .$$

Let $a_y a_z \alpha_\eta \alpha_\zeta = \frac{\rho}{2} (y z)(\eta \zeta)$, where ρ is a constant,

since

$$(y z)(\zeta \eta) \,.\, (a b)(\alpha \beta)\, a_y b_y \alpha_\eta \beta_\eta = 2\, a_y^2 \alpha_\eta \,.\, b_y b_z \beta_\eta \beta_\zeta$$
$$- 2\, a_y^2 \alpha_\eta \alpha_\zeta \,.\, b_y b_z \beta_\eta^2$$

then

$$(y z)(\eta \zeta) \{(a b)(\alpha \beta)\, a_y b_y \alpha_\eta \beta_\eta - \rho a_y^2 \alpha_\eta^2\} = -2\, a_y^2 \alpha_\eta \alpha_\zeta \,.\, b_y b_z \beta_\eta^2.$$

Similarly,

$$(y z)(\eta \zeta) \{J_{z,\zeta} - \rho f_{z,\zeta}\} = -2\, a_z^2 \alpha_\eta \alpha_\zeta \,.\, b_y b_z \beta_\zeta^2.$$

Hence $$\frac{J_{y_1 \eta} - \rho f_{y_1 \eta}}{J_{z_1 \zeta} - \rho f_{z_1 \zeta}} = \frac{a_y^2 \alpha_\eta \alpha_\zeta \cdot b_y b_z \beta_\eta^2}{a_z^2 \alpha_\eta \alpha_\zeta \,.\, b_y b_z \beta_\zeta^2}\, .$$

By the differentiation of $a_y a_z \alpha_\eta \alpha_\zeta = \frac{\rho}{2} (y z)(\eta \zeta)$, (i)

with respect to y_1 considering z_1 as a function of y, and ζ, η as independent of y, we obtain,

$$a_1 a_z \alpha_\eta \alpha_\zeta + a_1 a_y \alpha_\eta \alpha_\zeta \frac{dz_1}{dy_1} = \frac{\rho}{2} z_2 (\eta \zeta) - \frac{\rho}{2} y_2 (\eta \zeta) \frac{dz_1}{dy_1}.$$

Multiplying both sides by y_1 and using (i), we obtain,

$$- a_2 y_2 a_z \alpha_\eta \alpha_\zeta + a_y^2 \alpha_\eta \alpha_\zeta \frac{dz_1}{dy_1} - a_2 y_2 \alpha_\eta \alpha_\zeta \frac{dz_1}{dy_1}$$
$$= \frac{\rho}{2} y_2 z_1 (\eta \zeta) - \frac{\rho}{2} y_1 y_2 (\eta \zeta) \frac{dz_1}{dy_1}.$$

Similarly,

$$a_z^2 \alpha_\eta \alpha_\zeta - a_2 z_2 a_z \alpha_\eta \alpha_\zeta - a_2 z_2 a_y \alpha_\eta \alpha_\zeta \frac{dz_1}{dy_1}$$
$$= \frac{\rho}{2} z_1 z_2 (\eta \zeta) - \frac{\rho}{2} y_1 z_2 (\eta \zeta) \frac{dz_1}{dy_1}\, .$$

https://doi.org/10.1017/S0013091500034301 Published online by Cambridge University Press

On subtraction and replacing y_2, z_2 by unity, we obtain

$$a_y^2 \alpha_\eta \alpha_\zeta \frac{dz_1}{dy_1} - a_z^2 \alpha_\eta \alpha_\zeta = 0.$$

Similarly

$$a_y a_z \alpha_\zeta^2 \frac{d\eta_1}{d\zeta_1} - a_y a_z \alpha_\eta^2 = 0.$$

Hence

$$\frac{dy_1 \, d\eta_1}{a_y^2 \alpha_\eta \alpha_\zeta \, . \, a_y a_z \alpha_\zeta^2} = \frac{dz_1 \, d\zeta_1}{a_z^2 \alpha_\eta \alpha_\zeta \, . \, a_y a_z \alpha_\eta^2} .$$

Thus

$$\frac{dz_1 \, d\zeta_1}{P_{z,\zeta}} = \frac{dy_1 \, d\eta_1}{P_{y,\eta}}$$

where $P_{y,\eta} \equiv (a\, b)\, (\alpha\, \beta)\, a_y\, b_y\, \alpha_\eta\, \beta_\eta - \rho\, a_y^2\, \alpha_\eta$

$$\equiv \theta_y^2 \phi_\eta^2 - \rho\, a_y^2\, \alpha_\eta^2.$$

I. Let $\rho = 0$.

Then if $a_y a_z \alpha_\eta \alpha_\zeta = 0$,

$$\frac{dy_1 \, d\eta_1}{\theta_y^2 \phi^2} = \frac{dz_1 \, d\zeta_1}{\theta_z^2 \phi_\eta^2} ,$$

which reduces to the result in § 1 when $y = \eta$, $z = \zeta$; $\alpha = a$: or, in the canonical form, *if*,

$$y \eta (k z \zeta + m) + y (l z + m \zeta) + \eta (l \zeta + m z) + m z \zeta + k \equiv 0$$

then,
$$\frac{dy \, d\eta}{k m (y^2 \eta^2 + 1) + (k^2 - l^2) y \eta - l m (y^2 + \eta^2}$$

$$= \frac{dz \, d\zeta}{\text{corresponding expression in } z, \zeta} .$$

II. Since $(J J)_{1,1} = \frac{1}{3} B \, . \, f$ *

where $B = \frac{1}{6} (a\, b)\, (b\, c)\, (c\, a)\, (\alpha\, \beta)\, (\beta\, \gamma)\, (\gamma\, a)$

shewing that the relation between f and J is reciprocal,

then, if $\theta_y \theta_z \phi_\eta \phi_\zeta = 0$,

$$\frac{dy_1 \, d\eta_1}{a_y^2 \alpha_\eta^2} = \frac{dz_1 \, d\zeta_1}{a_z^2 \alpha_\zeta^2}$$

---

* *Proc. R.S.E.*, *loco. cit.*

https://doi.org/10.1017/S0013091500034301 Published online by Cambridge University Press

III. Consider now instead of $a_x^2\,\alpha_\xi^2 = f$ the form

$$F = kf + \lambda\,\theta + \mu\,p\,^*$$

where k, λ, μ are constants, and $p \equiv p_x^2\,\pi_\xi^2 \equiv (f,\,\theta)_{1,1}$.

Then
$$\begin{aligned}(F,\,F)_{1,1} = f\left(\tfrac{1}{12}\,A\,B\,\mu^2 + \tfrac{1}{6}\,k\,\mu\,B + \tfrac{1}{3}\,\lambda^2\,B + \tfrac{1}{2}\,\lambda\,\mu\,C\right)\\ + \theta\left(k^2 + \tfrac{1}{4}\,\mu^2\,C + \tfrac{1}{2}\,k\,\mu\,A + \tfrac{1}{6}\,\lambda\,\mu\,B\right)\\ + p\left(2\,k\,\lambda - \tfrac{1}{6}\,B\,\mu^2\right) \qquad = L_{x,\xi}\end{aligned}$$

This results from the use of the following transvectants,

$$\begin{aligned}&(f,\,p)_{1,1} = \tfrac{1}{4}\,A\,\theta + \tfrac{1}{12}\,Bf\\ &(p,\,\theta)_{1,1} = \tfrac{1}{4}\,Cf + \tfrac{1}{12}\,B\,\theta\\ &(p,\,p)_{1,1} = \tfrac{1}{12}\,A\,Bf + \tfrac{1}{4}\,C\,\theta - \tfrac{1}{6}\,Bp\\ &(\theta,\,\theta)_{1,1} = \tfrac{1}{3}\,Bf\\ &A = (f,\,f)_{2,2}\,;\quad B = (f,\,\theta)_{2,2}\,;\quad C = (\theta,\,\theta)_{2,2}\,.\end{aligned}$$

If now

$$k\,a_y\,a_z\,\alpha_\zeta\,\alpha'_\eta + \lambda\,\theta_y\,\theta_z\,\phi_\eta\,\phi_\zeta + \mu\,p_y\,p_z\,\pi_\eta\,\pi_\zeta = \frac{\rho}{2}\,(y\,z)\,(\zeta\,\eta)$$

then
$$\frac{dy_1\,d\eta_1}{L_{y,\,\eta} - \rho F_{y,\,\eta}} = \frac{dz_1\,d\zeta_1}{L_{z,\,\zeta} - \rho F_{z,\,\zeta}}\,.$$

Making the coefficients of θ, and p, vanish, then *the integral of*

$$\frac{d\,y_1\,d\eta_1}{a_y^{\,2}\,\alpha_\eta^{\,2}} = \frac{d\,z_1\,d\,\zeta_1}{a_z^{\,2}\,\alpha_\zeta^{\,2}}\,,$$

is $\quad k\,a_y\,a_z\,\alpha_\zeta\,\alpha_\eta + \lambda\,\theta_y\,\theta_z\,\phi_\eta\,\phi_\zeta + \mu\,p_y\,p_z\,\pi_\eta\,\pi_\zeta = \dfrac{\rho}{2}\,(y\,z)\,(\zeta\,\eta)$,

where $\quad k^2 + \tfrac{1}{4}\,C\,\mu^2 + \tfrac{1}{2}\,k\,\mu\,A + \tfrac{1}{6}\,\lambda\,\mu\,B - \rho\,\lambda = 0$,

and $\quad 2\,k\,\lambda - \tfrac{1}{6}\,B\,\mu^2 - \rho\,\mu = 0$.

Non-Symbolically:

If $f(y,\,\eta)$ *is a general double quadratic,*

$$a_{00}\,y^2\,\eta^2 + 2\,a_{01}\,y^2\,\eta + \;\dots\; + a_{22}\,,$$

in two independent variables $(y,\,\eta)$, *and if* $\theta\,(y,\,\eta)$, $p\,(y,\,\eta)$ *are its*

---

* **The notation used is that of Peano, quoted in *Proc. R. S. E., loco. cit.***

https://doi.org/10.1017/S0013091500034301 Published online by Cambridge University Press

two (2, 2) *covariants, then an algebraic solution, involving one arbitrary constant, of the differential equation*

$$\frac{dy\,d\eta}{f(y, \eta)} = \frac{dz\,d\zeta}{f(z, \zeta)}$$

exists and is expressible as a quadrilinear relation between the variables $y, \eta\,;\ z, \zeta$:

namely $$2\,kf' + 2\,\lambda\,\theta' + 2\,\mu\,p' = \rho\,(y - z)\,(\eta - \zeta),$$

k, λ, μ, ρ being constants, connected by two relations involving the invariants of f.

Here f,' θ', p' are the quadrilinear polar forms of f, θ, p, respectively.