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A finite set covering theorem

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Let n, s, t be integers with $s > t > 1$ and $n > (t+2)2^{s-t-1}$. We prove that if n subsets of a set S with s elements have union S then some t of them have union S. The result is best possible.

1. Introduction

Small letters denote non-negative integers and large letters denote sets. In particular 0 is the empty set, and $[i, j]$ denotes the set $\{i, i+1, i+2, \ldots, j\}$. Suppose that $X_1, X_2, \ldots, X_n$ are subsets of the set $S = [1, s]$ which cover (have union) S. We are here concerned with determining the smallest number of X_i which will cover S. Of course s of the X_i will cover S, just take a suitable X_i for each element of S. However can we be sure that t of the X_i will cover S if $t < s$? At the other extreme we could have all the $2^s - 1$ proper subsets of S with no $t = 1$ of them equal to S. So we assume $s > t > 1$, and then an important example is

$$E = \{X;\ X = P \cup Q,\ P \subset [1, t+1],\ |P| \le 1,\ Q \subset S \backslash P\} .$$

Since no set X in E contains two elements of $[1, t+1]$ it is clear that no t sets of E have union S, and the number e of sets in E is

$$e = e(s, t) = (t+2)2^{s-t-1} .$$

We can now state our

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THEOREM. *Let* n, s, t *be integers with* $s > t > 1$ *and let* $N = \{X_1, X_2, \ldots, X_n\}$ *be* n *different subsets* X_i *of* $S = [1, s]$ *with union* S . *Suppose also that no* t *of the* X_i *have union* S . *Then*

(i) $n \le e$, *and*

(ii) if $3 \le t$ *and* $n = e$ *we can obtain* N *from* E *by permuting the elements of* S .

When $t = 2$ we can attain the value e in many ways beside E , for instance

$$F = \{X;\ X = [1] \text{ or } X \subset [2, s],\ X \ne [2, s]\}$$

or

$$G = \{X;\ X = [1] \cup Y \text{ or } X = [2] \cup Y \text{ or } X \subset [3, s],\ Y \subset [4, s]\}$$

and so on. If in an application of the theorem one knew that the X_i have non-empty intersection T one could improve the result by restriction to $S \backslash T$.

2. Proof of (i)

Without loss of generality we strengthen the hypothesis of the theorem by assuming that n is as large as possible. This implies that if X is in N then all subsets of X are in N . When $t = 2$ we can't have a subset X of S and its complement both in N so $n \le \frac{1}{2}2^s = e(s, 2)$ and *(i)* holds. When $t = s - 1$ no set X of N can have more than one element, so $n \le s + 1 = e(s, s-1)$ and again *(i)* holds. We now use double induction on s, t . We suppose $3 \le t \le s - 2$ and that *(i)* holds in the two cases $s - 1, t$ and $s - 1, t - 1$. Then we deduce that *(i)* holds for the case s, t . Clearly some set of N has more than one element so we assume $[1, 2]$ is contained in some set in N .

To define a partition of N , for brevity we write $1, 2$ and $1 \cup 2$ for the sets $[1], [2]$ and $[1, 2]$, and put

$$B = \{X;\ X \cup 1 \cup 2 \notin N,\ X \cup 1 \backslash 2 \in N,\ X \cup 2 \backslash 1 \in N\} .$$

Then the partition is

$$N = A \cup B_0 \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup D$$

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where

$$A = \{X;\ X \cup 1 \cup 2 \in N\}$$

$$B_0 = \{X;\ X \in B,\ 1 \notin X,\ 2 \notin X\}$$

$$B_1 = \{X;\ X \in B,\ 1 \in X\}$$

$$B_2 = \{X;\ X \in B,\ 2 \in X\}$$

$$C_1 = \{X;\ X \cup 1\backslash 2 \in N,\ X \cup 2\backslash 1 \notin N\}$$

$$C_2 = \{X;\ X \cup 1\backslash 2 \notin N,\ X \cup 2\backslash 1 \in N\}$$

and

$$D = \{X;\ X \cup 1\backslash 2 \notin N,\ X \cup 2\backslash 1 \notin N\}\ .$$

Let $a, b_0, b_1, \ldots$ denote the number of elements in the sets $A, B_0, B_1, \ldots$ respectively, even though some of these sets may be empty. If X is in C_1 then all subsets of X are in N so

(1) $$X \cup 1,\ X\backslash 1 \in C_1 \quad \text{for all} \quad X \in C_1\ .$$

By similar reasoning we see that

$$B_1 = \{Y;\ Y = X \cup 1,\ X \in B_0\} \quad \text{and} \quad B_2 = \{Y;\ Y = X \cup 2,\ X \in B_0\}$$

and hence $b_0 = b_1 = b_2$.

Case 1. $b_0 \le d$. In this case we put

$$C_2' = \{X;\ X \in C_2,\ 2 \notin X\} \cup \{Y;\ Y = X \cup 1\backslash 2,\ X \in C_2,\ 2 \in X\}$$

$$D' = \{Y;\ Y = X \cup 1,\ X \in D\}$$

and

$$N' = A \cup B_0 \cup B_1 \cup D' \cup C_1 \cup C_2' \cup D\ .$$

We have chosen N' in such a way that, like in (1), we have

(2) $$X \cup 1,\ X\backslash 1 \in N' \quad \text{for all} \quad X \in N'\ .$$

Since $c_2' = c_2$ and $d' = d$ the number of sets in N' is $n' = n - b_0 + d$. Roughly speaking the sets of N and N' differ only with respect to the elements 1 and 2 . As $[1, 2]$ is in A it is clear that N' covers S .

Since (2) holds we can let $Y_1, Y_2, \ldots, Y_{\frac{1}{2}n'}$ be the sets of the form

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$\{Y;\ Y \in N',\ 1 \notin Y\}$. These sets cover $[2, s]$ but we claim that no t of them do so. For suppose $Y_1, Y_2, \ldots, Y_t$ cover $[2, s]$. Then the element 2 is in one of the sets, Y_1 say, and so Y_1 is in A , because 2 is only in the sets A of N' . Moreover $Y_1 \cup 1 \cup 2$ is also in A . For $2 \le k \le t$ there is a set X_{i_k} of N which differs from Y_k only with respect to the elements 1 and 2 , so in N we have

$$S = \{Y_1 \cup 1 \cup 2\} \cup X_{i_2} \cup X_{i_3} \cup \ldots \cup X_{i_t} ,$$

a contradiction. By our induction hypothesis $\frac{1}{2}n' \le e(s-1,\ t)$, so

$$n \le n - b_0 + d = n' \le 2e(s-1,\ t) = e(s,\ t) ,$$

and *(i)* holds in this case.

Case 2. $b_0 > d$. We show that this case never arises. We put

$$B_3 = \{Y;\ Y = X \cup 1 \cup 2,\ X \in B_0\}$$

$$M = A \cup B_0 \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup B_3$$

$$L = \{[i];\ i \notin M,\ i \in S\}$$

and

$$N'' = L \cup M .$$

If the sets in M cover S then L is empty, but if L is not empty then $L \subset D$ because N covers S . Also $n'' = n + b_0 - d + l > n$, so because the X_i were chosen with n as large as possible, there are t sets $Z_1, Z_2, \ldots, Z_t$ in N'' which cover S . If $L \ne 0$, every set $[i]$ in L is a Z_k , for otherwise the element i would not be covered.

We claim that no Z_k is in A . For otherwise for $1 \le k \le t$ we act as follows:

(α) if $Z_k \in A$ let $X_{i_k} = Z_k \cup 1 \cup 2 \in A$,

(β) if $Z_k \in B_3$ let $X_{i_k} = Z_k \backslash (1 \cup 2) \in B_0$, and

(γ) in all other cases let $X_{i_k} = Z_k$.

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Then we have the contradiction that $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$ cover S in N.

Next we claim that no two of the Z_k lie in

$$H = B_0 \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 .$$

Elements 1 and 2 must be covered by sets in H because they are not covered by sets in L. So suppose Z_1, Z_2 are in H and cover 1 and 2. If $Z_1, Z_2 \notin (B_0 \cup B_3)$ they are in N and we let X_{i_1}, X_{i_2} be them. If $Z_1, Z_2 \in (B_0 \cup B_3)$ we put $X_{i_1} = Z_1 \cup 1 \backslash 2 \in B_1 \subset N$ and $X_{i_2} = Z_2 \cup 2 \backslash 1 \in B_2 \subset N$. Finally if Z_1 is not in $B_0 \cup B_3$ but Z_2 is in, we put $X_{i_1} = Z_1 \in N$ and $X_{i_2} = (Z_2 \backslash [1, 2]) \cup j \in N$, where j is that one of the elements 1, 2 which is not in Z_1. Then for $3 \le k \le t$ we act as in (β) and (γ) above to obtain t sets in N which cover S, a contradiction.

Thus we conclude that $Z_1, Z_2, \ldots, Z_t$ consist of one set in B_3 and $t - 1$ sets in L, so $l = t - 1$. Without loss of generality assume these $t - 1$ sets to be $[s-t+2], [s-t+3], \ldots, [s]$. We now observe that firstly, no set in N contains more than one element of $[s-t+2, s]$, and secondly, no set in N contains 1 or 2 together with an element of $[s-t+2, s]$. Otherwise we easily get t sets of N which cover S. Hence sets in N containing the element s must be of the form $W \cup s$ with $W \subset [3, s-t+1]$. The set $[3, s-t+1] \cup s$ itself cannot be in N or again we would get t sets of N covering S.

It now follows that the element s is in less than 2^{s-t-1} sets of N. No $t - 1$ of the remaining sets cover $[1, s-1]$ so the number of these, by our induction hypothesis, is not greater than $e(s-1, t-1)$. Therefore $n < 2^{s-t-1} + e(s-1, t-1) = e(s, t)$ and this is fewer sets than we get with example E, contradicting our assumption that n was maximal. Thus this case is impossible, and *(i)* holds by induction.

3. Proof of (ii)

If $t = s - 1$ then N has no set with 2 elements, so N is E, and *(ii)* holds in this case. We now use induction on s. In Section 2

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we showed that no t of the sets $V = \{Y_1, Y_2, \ldots, Y_{\frac{1}{2}n'}\}$ cover $[2, s]$, and we now have $n' = e$. Thus by our induction hypothesis V is of the same form as example E, and it is important to know whether or not the element 2 is in the set corresponding to P. Before discussing the cases we observe that if $2 < i < j \le s$ and no set in V contains both i and j then no set in N contains both i and j.

Case 1. By permuting $[3, s]$ in S we get

$$V = \{Y;\ Y = P \cup Q,\ P \subset [s-t, s],\ |P| \le 1,\ Q \subset [2, s]\backslash P\}\ ,\quad 2 \notin P\ .$$

Then after the permutation, no set in N contains two elements of $[s-t, s]$, and since N has e sets, we must have

$$N = \{X;\ X = P \cup Q,\ P \subset [s-t, s],\ |P| \le 1,\ Q \subset S\backslash P\}\ .$$

Case 2. By permuting $[3, s]$ in S we get

$$V = \{Y;\ Y = P \cup Q,\ P \subset [2, t+2],\ |P| \le 1,\ Q \subset [2, s]\backslash P\}\ ,\quad 2 \in P\ .$$

Consider any two elements of $[3, t+2]$, say 3 and 4. Now for $5 \le k \le t+2$ there is a set X_k, say, in N which contains the set $k \cup [t+3, s]$ of V. We claim that there are not two sets X_3, X_4, say, in N with $1, 3 \in X_3$ and $2, 4 \in X_4$. Otherwise the t sets $X_3, X_4, \ldots, X_{t+2}$ cover S in N. Hence, because the elements 3, 4 were chosen arbitrarily, either

$$N = \{X;\ X = P \cup Q,\ P \subset [1] \cup [3, t+2],\ |P| \le 1,\ Q \subset S\backslash P\}$$

or

$$N = \{X;\ X = P \cup Q,\ P \subset [2, t+2],\ |P| \le 1,\ Q \subset S\backslash P\}\ ,$$

and the theorem follows inductively.

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