AN ALTERNATIVE PROOF OF DIESTEL'S THEOREM

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Introduction. We show that Diestel's theorem on weak compactness of subsets of $L_1(\mu, X)$ can be derived as a simple corollary of James's theorem. It is a pleasure to acknowledge several stimulating conserversations with Dave Emmons and the remarks of an anonymous referee. Errors are, of course, solely mine.

Let $(T, \mathcal{T}, \mu)$ be a finite measure space and X a Banach space. Denote by $L_1(\mu, X)$ the Banach space of (equivalence classes of) μ-strongly measurable X-valued Bochner integrable functions $f: T \to X$ normed by $\|f\|_1 = \int_T \|f(t)\| \, d\mu(t)$. In [**3**] Diestel has proved through the use of the factorization method in [**2**] the following result.

THEOREM. *Let K be a weakly compact convex subset of X and*

$$\tilde{K} = \{f \in L_1(\mu, X) : f(t) \in K \quad \text{for almost all } t \text{ in } T\};$$

then $\tilde{K}$ is weakly compact in $L_1(\mu, X)$.

In this note, we offer an alternative proof of Diestel's theorem which relies instead on James's theorem [**5**] and on Brooks's extension of the classical Vitali's theorem [**1**].

Before presenting our proof, we recall that $V_\infty(\mu, X^*)$, where X^* is the continuous dual of X, is isometrically isomorphic to $L_1(\mu, X)^*$ with the correspondence between $F \in V_\infty(\mu, X^*)$ and $\phi \in L_1(\mu, X)^*$ given by $\phi(f) = \int f \, dF$. (For an explanation and properties of $V_\infty(\mu, X^*)$ see [**3**] and his references.)

Proof. Pick an arbitrary $\phi \in (L_1(\mu, X))^*$. If we can show that ϕ attains its supremum on $\tilde{K}$, James's theorem [**5**, Theorem 5] assures us that $\tilde{K}$ is relatively weakly compact. Since $\tilde{K}$ is convex and closed (hence by Mazur's theorem weakly closed) in $L_1(\mu, X)$, the proof is then finished.

Let $F \in V_\infty(\mu, X^*)$ correspond to ϕ. Towards showing that ϕ attains its supremum on $\tilde{K}$, select a pairwise disjoint sequence of elements $T_i \in \mathcal{T}$ such that each T_i has positive measure and $\bigcup_{i=1}^{\infty} T_i = T$. Consider the finite partition $\pi_n = \left\{T_1, T_2, \ldots, T_{n-1}, \bigcup_{i=n}^{\infty} T_i\right\}$ in which T_n will denote $\bigcup_{i=n}^{\infty} T_i$. Let $\pi = \{\pi_n\}_{n \geq 1}$. It is clear that for all integers n, π_{n+1} is a refinement of π_n.

For any partition π_n construct the function $f_n \in L_1(\mu, X)$ such that

$$f_n(t) = x_i \quad \text{for all} \quad t \in T_i, \qquad (i = 1, \ldots, n),$$

where x_i is characterized by the equality $\langle x_i, F(T_i)\rangle = \sup_{y \in K} \langle y, F(T_i)\rangle$. Since K is nonempty and weakly compact, certainly $x_i \in K$ for all i. We are now going to take a suitable limit of a subsequence of these functions.

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By our construction, for any t in T_1, we can let $f_n(t) = f_n(T_1)$, where $f_n(T_1) \in K$ for all n. Since K is weakly compact, the Eberlein–Šmulian Theorem guarantees a subsequence $f_n^{(1)}(T_1)$ which converges weakly to an element, say $f(T_1)$, in K. The procedure is now clear. We can now manufacture a function $f: T \to K$ such that f is the almost everywhere limit of $f_n^{(n)}$, where for any $i = 1, \dots, n-1$, $f_n^{(i+1)}$ is a subsequence of $f_n^{(i)}$ such that for all n, $f_n^{(i)}(t) = f_n^{(i)}(T_{i+1})$ for all t in T_{i+1} and $f_n^{(i+1)}(T_{i+1})$ converges weakly to an element, say $f(T_{i+1})$, in K.

Since K is weakly compact, for all $x \in K$ there exists $M > 0$ such that $\|x\| \leq M$. Using this fact it is now easy to show that the sequence $f_n^{(n)}$ is bounded and uniformly integrable. We can therefore apply Brooks's extension [**1**, Theorem 3] of Vitali's convergence theorem to claim that $f \in L_1(\mu, X)$ and hence $f \in \tilde{K}$ and that $\|f - f_n^{(n)}\|_1 \to 0$. Then, certainly $\int f_n^{(n)}\, dF \to \int f\, dF$.

We now claim that ϕ attains its supremum on $\tilde{K}$ at f. Suppose not; that is there exists $z \in \tilde{K}$ such that

$$\phi(z) = \int_T z\, dF > \int_T f\, dF = \phi(f). \tag{1}$$

For each partition π in Π, define the linear operator $E_\pi : L_1(\mu, X) \to L_1(\mu, X)$ by

$$E_\pi(z) = \sum_{T_i \in \pi} \left[\frac{1}{\mu(T_i)} \int_{T_i} z(t)\, d\mu(t)\right] \chi_{T_i},$$

where χ_A is the characteristic function of A and the $0/0 = 0$ convention is in force. By the mean value theorem for the Bochner integral [**4**, Corollary 8, p. 48] and the convexity of K, certainly $\frac{1}{\mu(T_i)} \int_{T_i} z(t)\, d\mu(t) \in K$. Thus $E_\pi(z) \in \tilde{K}$. We can now apply Lemma 1 in [**3**, p. 67] to assert that $\|E_{\pi_n}(z) - z\|_1 \to 0$. Then certainly $\int_T E_{\pi_n}(z)\, dF \to \int_T z\, dF$.

However by construction,

$$\int_T E_{\pi_n}(z)\, dF \leq \int_T f_n^{(n)}\, dF.$$

By taking limits on both sides, we obtain our sought-after contradiction to (1).

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