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ATOMIC MEASURE SPACES AND ESSENTIALLY NORMAL COMPOSITION OPERATORS

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The adjoint of a composition operator C_T on the L^2-space of an atomic measure is computed and a characterization for an operator to be a composition operator is given in this short note. The dimensions of kernel and co-kernel of C_T are calculated in order to characterise Fredholm composition operators. Finally, essentially normal composition operators are studied on l^2 .

1. Introduction

Let (X, S, λ) be a sigma-finite measure space and T be a non-singular measurable transformation from X into itself. Then the mapping C_T on $L^p(\lambda)$ which takes f into $f \circ T$ is a linear transformation. If the range of C_T is in $L^p(\lambda)$ and C_T is bounded, then we call it a composition operator on $L^p(\lambda)$ induced by T . It is known that the composition transformation C_T is bounded if and only if there exists an $M > 0$ such that $\lambda T^{-1}(E) \le M\lambda(E)$ for all E in S . From this it follows that if C_T is bounded then the induced measure λT^{-1} is absolutely continuous with respect to the measure λ . Hence, by the Radon-Nikodym theorem, there exists a positive measurable function f_0

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such that $\lambda T^{-1}(E) = \int_E f_0 d\lambda$ for every E in S. The function f_0 is called the Radon-Nikodym derivative of the measure λT^{-1} with respect to λ.

A measurable set E is called an atom if $\lambda(E) \neq 0$ and if $F \in S$ and $F \subset E$, then either $\lambda(F) = 0$ or $\lambda(F) = \lambda(E)$. A measure λ is called atomic if every element $E \in S$ such that $\lambda(E) \neq 0$ contains an atom and in this case we say that (X, S, λ) is an atomic measure space. In this paper the adjoint of a composition operator on $L^2(\lambda)$ is obtained and the necessary and sufficient condition for an operator on $L^2(\lambda)$ to be a composition operator is discussed when the underlying measure λ is atomic. Also dimensions of kernel of C_T and kernel of C_T^* are given. Finally essentially normal composition operators on l^2 are characterised.

2. Composition operators and atomic measure spaces

If (X, S, λ) is a sigma-finite atomic measure space, then we can write X as $\bigcup_{i=1}^{\infty} E_i$, where the E_i's are disjoint atoms of finite measure [7]. These atoms are unique in the sense that if $X = \bigcup_{i=1}^{\infty} F_i$, where the F_i's are disjoint atoms of finite measures, then for every F_i there exists an E_j such that $\lambda(E_j \Delta F_i) = \lambda\{(E_j \backslash F_i) \cup (F_i \backslash E_j)\} = 0$. If a non-singular measurable transformation T on X takes one part of an atom E_j to a subset of an atom E_k and the other part of E_j to a subset of another atom E_1, then anyone of the above parts of E_j has to be a null set. As it is obvious that the image of an atom under a non-singular transformation cannot be a null set, we can consider a non-singular measurable transformation $T : X \to X$ as a transformation taking atoms into atoms. Hereafter we denote the atom E_j by j and by $T(j)$ the atom to which E_j is carried over by T. We say that an atom j is in the range of T if $j \in \{T(i) : i \in \mathbb{N}\}$.

A non-singular measurable transformation $T : X \to X$ is called one-to-one almost everywhere if the inverse image of every atom under T contains at most an atom. It is called onto almost everywhere if the inverse image of every atom under T contains at least one atom. If T is one-to-one almost everywhere and onto almost everywhere then it is called invertible almost everywhere. Also every function $f \in L^2(\lambda)$ is constant almost everywhere on an atom. Hence the span of the characteristic functions $\{X_i : i \in \mathbb{N}\}$ form a dense subset of $L^2(\lambda)$. Let $K_i = X_i/\lambda(i)$. Then the set of functions $\{K_i : i \in \mathbb{N}\}$ forms an orthonormal basis for $L^2(\lambda)$. The symbol $B(H)$ stands for the C^*-algebra of all bounded operators on the Hilbert space H . Throughout this paper we assume that (X, S, λ) is an atomic sigma-finite measure space. The following theorem computes the adjoint of C_T .

THEOREM 1. *Let* $C_T \in B\left(L^2(\lambda)\right)$ *and let* A *be defined as* $(Af)(i) = 1/\lambda(i) \int_{T^{-1}(i)} f d\lambda$ *almost everywhere for* $f \in L^2(\lambda)$ *and for every atom* i . *Then* $A = C_T^*$.

Proof. Let $f, g \in L^2(\lambda)$. Then

$$\begin{aligned} \langle C_T f, g\rangle &= \int_X \left(C_T f\right)\bar{g} d\lambda \\ &= \sum_{i=1}^{\infty} \int_{T^{-1}(i)} f \circ T\bar{g} d\lambda \\ &= \sum_{i=1}^{\infty} f(i)\lambda(i)(\overline{Ag})(i) \\ &= \langle f, Ag\rangle . \end{aligned}$$

Hence $A = C_T^*$. Hence the proof is completed.

The following theorem gives a necessary and sufficient condition for an operator to be a composition operator.

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THEOREM 2. *Let* $A \in B(L^2(\lambda))$. *Then* A *is a composition operator if and only if the set* $\{K_i : i \in \mathbb{N}\}$ *is invariant under* A^* . *In this case* T *is determined by* $A^*(K_i) = K_{T(i)}$.

Proof. The proof follows from the above theorem and [3].

The above theorem shows that the functions $\{K_i\}$ play the role of kernel functions for $L^2(\lambda)$. In the following theorem we compute the dimension of $\ker C_T$.

THEOREM 3. *Let* $C_T \in B(L^2(\lambda))$. *Then* $\dim \ker C_T$ *equals the number of atoms in* $X \setminus \{T(i) : i \in \mathbb{N}\}$.

Proof. If an atom i is not in the range of T , then $X_i \in \ker C_T$ since $\lambda T^{-1}(i) = 0$. If an atom j is in the range of T , then $\lambda T^{-1}(j) > 0$ and hence $C_T X_j \neq 0$. Hence $\ker C_T$ is equal to the closure of the span of the set $\{X_k : k \text{ is not in the range of } T\}$. Hence the proof is completed.

Let β_n denote one less than the number of atoms in $T^{-1}(n)$ if $T^{-1}(n)$ has more than one atom, otherwise zero.

THEOREM 4. *Let* $C_T \in B(L^2(\lambda))$. *Then* $\dim \ker C_T^* = \sum_{n=1}^{\infty} \beta_n$.

Proof. From Theorem 2 it follows that

$$C_T^*\left(\sum_{i \in T^{-1}(k)} a_i X_i\right) = \left(\sum_{i \in T^{-1}(k)} b_i\right) X_k \text{ , where } b_i = \frac{\lambda(i)}{\lambda(T(i))} a_i \text{ .}$$

From this it is clear that when the cardinality of $T^{-1}(k) = p > 1$, C_T^* kills $(p-1)$ basis vectors of the closed subspace spanned by the characteristic functions $\left\{X_i : i \in T^{-1}(k)\right\}$. Since $L^2(\lambda)$ is the direct sum of such closed subspaces, we get $\dim \ker C_T^* = \sum_{n=1}^{\infty} \beta_n$. Hence the

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theorem is proved.

DEFINITION. An operator $A \in B(H)$ is called Fredholm if A has closed range and dimensions of kernel of A and co-kernel of A are finite.

Let $X_0 = \{x : x \in X \text{ and } f_0(x) = 0\}$. The following theorem gives a characterization for Fredholm composition operators.

THEOREM 5. *Let $C_T \in B(L^2(\lambda))$. Then C_T is Fredholm if and only if f_0 is bounded away from zero on the complement of X_0 , range of T contains all but finitely many atoms of X and T is one-to-one almost everywhere on the complement of a set with finitely many atoms.*

Proof. The proof follows from Theorems 3 and 4.

SOME CONSEQUENCES

1. C_T is an injection if and only if T is onto almost everywhere.

2. C_T has dense range if and only if T is one-to-one almost everywhere.

3. C_T is invertible if and only if T is invertible almost everywhere and f_0 is bounded away from zero.

4. If $C_T \in B(l^2)$, then C_T is Fredholm if and only if $f_0 = 1$ except for a finite number of points of $\mathbb{N}$.

In [6] it has been proved that in case of a general finite measure space unitary and normal composition operators coincide and isometries and quasinormal composition operators coincide on $B(L^2(\lambda))$. If the measure space is atomic, then all the above coincide.

THEOREM 6. *Let (X, S, λ) be a finite atomic measure space and $C_T \in B(L^2(\lambda))$. Then the following are equivalent:*

(i) C_T *is unitary;*

(ii) C_T *is normal;*

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(iii) C_T *is an isometry;*

(iv) C_T *is quasinormal;*

(v) C_T *is a co-isometry.*

Proof. By Theorems 1 and 2 of [6], *(i)* and *(ii)* are equivalent, and *(iii)* and *(iv)* are equivalent. Now suppose C_T is an isometry. Then T is measure preserving and hence $\lambda T^{-1}(i) = \lambda(i)$ for every atom i in X. Let S_i denote the set of all atoms in X which have the same measure as i. Then each S_i will be a finite set. Also C_T is an isometry implies that T is onto almost everywhere. Since $T(S_i) \subset S_i$ for every i, T/S_i is one-to-one almost everywhere and hence T is one-to-one almost everywhere. Since an isometry has closed range, this implies that C_T is invertible and hence C_T is unitary. This gives the equivalence of *(i)* and *(iii)*. To prove the equivalence of *(v)* and *(i)*, suppose C_T is a co-isometry. Then C_T has dense range and hence T is one-to-one almost everywhere. Also $f_0 \circ T = 1$ almost everywhere. This implies $\lambda T^{-1}(T(i)) = \lambda(i) = \lambda(T(i))$ for every i in X. Considering the set S_i as above we have $T(S_i) \subset S_i$ for every i in X which implies that T is onto almost everywhere. Hence C_T is invertible and hence C_T is unitary. This completes the proof of the theorem.

COROLLARY 6.1. *If the atoms in the finite atomic measure space* (X, S, λ) *are such that* $\lambda(i) \neq \lambda(j)$ *when* i *is different from* j, *then all the above composition operators coincide with the identity operator.*

3. Essentially normal composition operators on l^2

DEFINITION. Let H be a Hilbert space, $C(H)$ denote the ideal of compact operators in $B(H)$ and π from $B(H)$ to the Calkin algebra $B(H)/C(H)$ be the canonical epimorphism. Then an operator A in $B(H)$ is said to be essentially normal, essentially unitary or an essential isometry if $\pi(A)$ is normal, unitary or an isometry respectively in the C^*-algebra

$B(H)/C(H)$ (refer [1]). We say that A is quasi-unitary if $A^*A - I$ and $AA^* - I$ are finite rank operators. A is Fredholm is equivalent to saying that $\pi(A)$ is invertible in the Calkin algebra.

We know from [4] that invertible, normal, unitary and isometric composition operators are not different in l^2. The same is true about Fredholm, essentially normal, essentially unitary and essential isometric composition operators. This we shall exhibit in the following theorem.

THEOREM 7. *Let* $C_T \in B(l^2)$. *Then the following are equivalent:*

(i) C_T *is Fredholm;*

(ii) C_T *is essentially normal;*

(iii) C_T *is quasi-unitary;*

(iv) C_T *is essentially unitary;*

(v) C_T *is an essential isometry.*

Proof. First we prove the equivalence of *(i)* and *(ii)*. Let C_T be Fredholm. Then $f_0 = 1$ except for a finite number of points of $\mathbb{N}$. Now

$$C_T^*C_T - C_TC_T^* = \begin{cases} M_{f_0 - f_0 \circ T} & \text{on } \overline{\operatorname{Ran} C_T}, \\ M_{f_0} & \text{on } \ker C_T^* = (\operatorname{Ran} C_T)^{\perp}. \end{cases}$$

Since C_T is Fredholm $f_0 = f_0 \circ T$ except for a finite number of points of $\mathbb{N}$ and $\ker C_T^*$ is finite dimensional. Hence $C_T^*C_T - C_TC_T^*$ is of finite rank and hence C_T is essentially normal. Conversely, suppose C_T is essentially normal. If possible let dimension of $\ker C_T = \infty$. Then by Theorem 3 there exists an infinite subset $S_1 = \{n_1, n_2, \ldots\}$ of $\mathbb{N}$ such that $f_0(n_i) = 0$ for $i = 1, 2, \ldots$. For $n \in \mathbb{N}$ let $e^{(n)}$ be the function on $\mathbb{N}$ taking value zero at points different from n and value 1 at n. Then the sequence $\{e^{(n)}\}$ tends to zero weakly. But the sequence

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$\left\{\left\|\left(C_T^*C_T-C_TC_T^*\right)e^{(n)}\right\|\right\}$ does not tend to zero. This shows that $\left(C_T^*C_T-C_TC_T^*\right)$ is not compact. Thus C_T is not essentially normal which is a contradiction. Hence $\dim \ker C_T < \infty$. Similarly, if $\ker C_T^*$ is infinite dimensional, then we can prove that $\left(C_T^*C_T-C_TC_T^*\right)$ is not compact. Hence $\dim \ker C_T^* < \infty$. Thus C_T is Fredholm, since every composition operator on l^2 has closed range.

If C_T is Fredholm, then $f_0 = 1$ except for a finite number of points of $\mathbb{N}$. Hence $C_T^*C_T - I = M_{f_0-1}$ is of finite rank. Also

$$C_TC_T^* - I = \begin{cases} M_{f_0 \circ T-1} & \text{on } \overline{\operatorname{Ran} C_T} , \\ -I & \text{on } \left(\operatorname{Ran} C_T\right)^{\perp} = \ker C_T^* . \end{cases}$$

Since $f_0 \circ T = 1$ except for a finite number of points of $\mathbb{N}$ and $\ker C_T^*$ is finite dimensional, $C_TC_T^* - I$ is also of finite rank and hence C_T is quasi-unitary. Hence *(i)* implies *(iii)*. Obviously *(iii)* implies *(iv)* and *(iv)* implies *(v)*. Now, let C_T be an essential isometry. Then $C_T^*C_T - I$ is compact which implies M_{f_0-1} is compact. This implies $f_0 = 1$ except for a finite number of points of $\mathbb{N}$. Hence C_T is Fredholm. Thus *(v)* implies *(i)* and the theorem is proved.

THEOREM 8. *Let* C_T *be a quasinormal composition operator on* l^2 . *Then* C_T *is essentially normal if and only if it is normal.*

Proof. Since a quasinormal composition operator on l^2 is an injection, it is essentially normal if and only if $\ker C_T^*$ is finite dimensional. Also $f_0 = f_0 \circ T$. This, in the light of Theorem 3, implies that C_T is essentially normal if and only if $\ker C_T^*$ is zero dimensional. Thus C_T is essentially normal if and only if C_T is normal. Hence the theorem is proved.

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