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CHANGING THE SCALAR MULTIPLICATION ON A VECTOR LATTICE

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Introduction

Throughout this paper only abelian l-groups will be considered and G will denote an abelian l-group. G is *large* in the l-group H or H is an *essential extension* of G if G is an l-subgroup of H and for each l-ideal $L \neq 0$ of H we have $L \cap G \neq 0$. A *v-hull* of G is a minimal vector lattice that contains G and is an essential extension of G. Each G admits a v-hull (Conrad (1970)). We shall be interested in the following properties of G.

I. G admits a scalar multiplication so that it is a vector lattice.

II. Any two scalar multiplications of G are connected by an l-automorphism of G.

III. G admits a unique v-hull.

Suppose that G satisfies I and let $\cdot$ be a scalar multiplication for G. Then each l-automorphism ϕ determines a new scalar multiplication Φ.

$$r\Phi g = (r \cdot (g\phi))\phi^{-1} \text{ for each } r \in R \text{ and } g \in G.$$

Note that ϕ is a linear l-isomorphism of (G, Φ) onto $(G, \cdot)$ and so connects the two scalar multiplications. Thus if G satisfies II then G admits essentially only one scalar multiplication.

Two l-automorphisms α and β of G determine the same scalar multiplication.

if and only if $(r \cdot (g\alpha))\alpha^{-1}\beta = r \cdot (g\beta)$ for all $r \in R$ and $g \in G$
if and only if $(r \cdot h)\alpha^{-1}\beta = r \cdot (h\alpha^{-1}\beta)$ for all $r \in R$ and $h \in G$
if and only if $\alpha^{-1}\beta$ is linear with respect to $\cdot$.

Now let $\mathscr{L}$ be the group of all l-automorphisms of G and let $\mathscr{L}^{\cdot} = \{\alpha \in \mathscr{L} \mid \alpha$ is linear with respect to $\cdot\}$. If G satisfies II then there exists a one to one map of the set of all scalar multiplications of G onto the set of all left cosets of $\mathscr{L}^{\cdot}$ in $\mathscr{L}$.

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We shall show that II is satisfied by a large class of vector lattices and that each l-group can be embedded in a vector lattice that satisfies II. Whether or not each l-group satisfies II remains an open and very difficult question, even for totally ordered vector lattices.

If G is a vector lattice with respect to two scalar multiplications, then the l-ideals of G are subspaces under both multiplications. In the unordered case there is no such preassigned set of subspaces and Example 5.1 shows that if $U \neq 0$ is an unordered real vector space then II is not satisfied.

An endomorphism α of an l-group G is a *p-endomorphism* (or a polar preserving endomorphism) if

$$x, y \in G \text{ and } x \wedge y = 0 \text{ imply } x\alpha \wedge y = 0.$$

The set S of all p-endomorphisms of G is a semiring. Thus the subring $\mathscr{P}(G)$ of the endomorphism ring of G that is generated by S is a directed po-ring with positive cone S. $\mathscr{P}(G)$ is called the ring of *polar preserving endomorphisms* of G. If G is archimedean then $\mathscr{P}(G)$ is an archimedean f-ring (see Bigard and Keimel (1969) or Conrad and Diem (1971)). A subring of $\mathscr{P}(G)$ that contains the identity e and is o-isomorphic to R will be called a *real subfield* of $\mathscr{P}(G)$.

PROPOSITION. *There is a natural one to one correspondence between the real subfields of $\mathscr{P}(G)$ and the scalar multiplications on G. In particular, G satisfies* I *if and only if $\mathscr{P}(G)$ is a po real vector space.*

PROOF. If $(G, \cdot)$ is a vector lattice and $a \in R$ then define $\cdot a \in \mathscr{P}(G)$

$$(\cdot a)g = a \cdot g \text{ for all } g \in G.$$

The map $a \to \cdot a$ is an o-isomorphism of R onto a real subfield of $\mathscr{P}(G)$. Since the only automorphism of the field R is the identity, distinct scalar multiplications of G map onto distinct real subfields of $\mathscr{P}(G)$. Thus the map η of $\cdot$ onto the real subfield $\cdot R$ is one to one.

Now let D be a real subfield of $\mathscr{P}(G)$ and let π be the o-isomorphism of R onto D. For each $r \in R$ and $g \in G$ define

$$r \cdot g = (\pi r)g.$$

Then $(G, \cdot)$ is a vector lattice and $\cdot R = D$. Thus η is a one to one map of the scalar multiplication of G onto the real subfields of $\mathscr{P}(G)$.

Finally each real subfield of $\mathscr{P}(G)$ determines a scalar multiplication of $\mathscr{P}(G)$ so that it is a po real vector space. Thus G satisfies I if and only if $\mathscr{P}(G)$ contains a real subfield if and only if $\mathscr{P}(G)$ is a po real vector space.

Let $(G, \cdot)$ and $(G, *)$ be vector lattices and let α be a group automorphism of G.

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COROLLARY. *α is a linear map of $(G, \cdot)$ onto $(G, *)$ if and only if $\alpha \cdot R\alpha^{-1} = {}^*R$.*

PROOF. $(\rightarrow)$ For each $a \in R$ and $g \in G$

$$(\alpha \cdot a)g = \alpha(\cdot ag) = \alpha(a \cdot g) = a * (\alpha g) = {}^*a(\alpha g) = ({}^*a\alpha)g.$$

Thus $\alpha \cdot a = {}^*a\alpha$ and hence $\alpha \cdot a\alpha^{-1} = {}^*a$. Therefore $\alpha \cdot R\alpha^{-1} = {}^*R$.

$(\leftarrow)$ The map $\cdot a \xrightarrow{\tau} \alpha \cdot a\alpha^{-1}$ is an isomorphism of $\cdot R$ onto ${}^*R$, and since R admits only one automorphism, τ is the o-isomorphism $\cdot a \rightarrow {}^*a$. Thus

$$\alpha \cdot a\alpha^{-1} = {}^*a \text{ or } \alpha \cdot a = {}^*a\alpha \text{ for all } a \in R.$$

Thus for $a \in R$ and $g \in G$

$$\alpha(a \cdot g) = \alpha(\cdot ag) = (\alpha \cdot a)g = ({}^*a\alpha)g = {}^*a(\alpha g) = a * (\alpha g).$$

Therefore any results we obtain about I or II for G have applications to $\mathscr{P}(G)$ and conversely.

1. Archimedean l-Groups

Throughout this section let G be an archimedean l-group. In Conrad (1970), it is shown that G admits a unique v-hull G^v, and Bleier (1971) proves that G^v is the smallest archimedean vector lattice that contains G. Thus G satisfies III. Also G satisfies II since it admits at most one scalar multiplication. For if $(G, \cdot)$ and $(G, *)$ are vector lattices then the identity automorphism of G is linear (see Conrad (1970)).

Iwasawa (1943) showed that if G is divisible and complete then G satisfies I. Thus if G is essentially closed then it satisfies I. If G has a basis and is laterally complete then G is a cardinal product ΠT_α of archimedean o-groups T_α and hence G satisfies I if and only if each convex o-subgroup is o-isomorphic to R. If G is a subdirect sum of integers then the Dedekind-MacNeille completion $G^\wedge$ of G is a vector lattice if and only if each $0 < g \in G$ is unbounded (see Conrad (1970)).

PROPOSITION 1.1. *G satisfies* I *if and only if each principal l-ideal $G(g)$ satisfies* I.

PROOF. If G satisfies I each l-ideal is a subspace. Now $G \subseteq G^v$. Thus since G^v is archimedean each $G(g)$ is a subspace of G^v (see Conrad (1970)) and hence $G = \bigcup_{\in G g} G(g)$ is a subspace of G^v.

Now $\mathscr{P}(G)$ is an archimedean f-ring and hence squares are positive. Thus a subring K of $\mathscr{P}(G)$ that is isomorphic to R is a totally ordered subring of $\mathscr{P}(G)$ and hence a real subfield provided that $e \in K$.

PROPOSITION 1.2. *If S is an archimedean f-ring with identity e then there*

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exists a largest o-subring of S that contains e. In particular, S contains at most one real subfield.

PROOF. By Bernau's embedding theorem (Bernau (1965)) we may assume that S is an l-subring of the ring $D(X)$ of almost finite continuous functions on a Stone space X and e is the identity for $D(X)$. Let F be an o-subring of S that contains e. Then F consists of constant functions—for otherwise there exists $f \in F$ such that $0 < f(x) < f(y) < \infty$ for some pair $x, y \in X$. Thus there are positive integers m and n such that

$$nf(x) < me < nf(y).$$

Therefore nf and me are not comparable, a contradiction.

COROLLARY. *An archimedean l-group G satisfies* I *if and only if the largest o-subring of $\mathscr{P}(G)$ that contains e is a real subfield. Since $\mathscr{P}(G)$ contains at most one real subfield, G admits at most one scalar multiplication.*

THEOREM 1.3. *An archimedean l-group G contains a largest l-subgroup H that is a vector lattice. H is the largest subspace of G^v contained in G and H is an l-characteristic subgroup of G.*

PROOF. If A and B are l-subgroups of G and vector lattices then they are subspaces of G^v (Conrad (1970)). We show that the l-subgroup C of G generated by A and B is also a subspace of G^v and hence a vector lattice.

The group $A+B$ is a subspace of G^v and if $c \in C$ then

$$c = \bigvee_X \bigwedge_Y t_{xy}$$

where the t_{xy} belong to $A+B$ and X and Y are finite. Thus for $r \in R$

$$rc = r(\bigvee \bigwedge t_{xy}) = \bigvee \bigwedge (rt_{xy}) \in C.$$

Thus G contains a largest l-subgroup H that is a vector lattice and H is a subspace of G^v. The above argument shows that if D is a subspace of G^v contained in G then the l-subgroup of G generated by D is also a subspace of G^v. Thus H is the largest subspace of G^v contained in G.

Finally suppose that α is an l-automorphism of G, then $H\alpha$ is an l-subgroup of G and a vector lattice (any l-homomorphism of a vector lattice into G^v is necessarily linear). Therefore $H\alpha \subseteq H$.

REMARK. If G is an arbitrary l-group and an l-subgroup of a vector lattice K then the above proof shows that G contains a largest l-subgroup H that is also a subvector lattice of K, and H is the largest subspace of K contained in G. Example 5.9 shows that even if G is a vector lattice in its own right it need not equal H.

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THEOREM 1.4. *For an archimedean l-group G the following are equivalent.*

1) *G satisfies* I.

2) *Each principal l-ideal G(g) satisfies* I.

3) $\mathscr{P}(G)$ *satisfies* I.

4) *The largest o-subring of* $\mathscr{P}(G)$ *is a real subfield.*

5) *G is divisible and each cut in* Q^+e *contains an element of* $\mathscr{P}(G)$, *where e is the identity for* $\mathscr{P}(G)$.

6) *G is divisible and for an arbitrary* $0 < g \in G$ *each cut in* Q^+g *contains an element of G.*

PROOF. We have shown 1), 2), 3), 4) are equivalent and clearly if G satisfies I then it is divisible. So we shall assume that G and hence $\mathscr{P}(G)$ are divisible.

If $0 < g \in G$ then a cut in Q^+g contains at most one element from G. For suppose that $a, b \in G$ belong to the cut. Then $a, b \in G(g)$. Let M be a maximal l-ideal of $G(g)$. Modulo M a and b determine the same cut in Q^+g and so $a \equiv b \bmod M$ for all such M. Thus $a = b$.

$(4 \to 5)$. $Q^+e \subseteq F \cong R$, where F is the real o-subfield of $\mathscr{P}(G)$. Thus each cut in Q^+e contains an element of F.

$(5 \to 6)$. Let (L,U) be a cut in Q^+g. Then the corresponding cut $(\bar{L}, \bar{U})$ in Q^+e contain a unique element α from $\mathscr{P}(G)$. Thus $g\alpha$ is contained in (L, U).

$(6 \to 1)$. Let a be the element in R determined by the cut (L, U) in Q^+ and let h be the element in G contained in the corresponding cut $(\bar{L}, \bar{U})$ in Q^+g. Define $ag = h$. This determines a scalar multiplication on G so that it is a vector lattice.

PROPOSITION 1.5. *For a vector lattice H the following are equivalent.*

1) *H is archimedean.*

2) *The scalar multiplication on each l-subspace S of H is unique.*

PROOF. $(1 \to 2)$. If S is a vector lattice then it must be a subspace of H (Conrad (1970)).

$(2 \to 1)$. If H is not archimedean then there exists $0 < b \ll a$ in H. The subspace $Ra \oplus Rb$ of H is totally ordered and hence an l-subspace of H. Let f be a homomorphism of Ra into Rb that is not linear and define

$$(r_1a + r_2b)\tau = r_1a + f(r_1a) + r_2b.$$

This is an o-automorphism of $Ra \oplus Rb$ that is not linear and so can be used to define a new scalar multiplication on $Ra \oplus Rb$, but this contradicts (2).

It is an open question whether or not (1) is equivalent to:

3) The scalar multiplication on H is unique. If H is totally ordered then a slight generalization of the above proof shows that (3) implies (1).

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2. The l-Group $V(\Gamma, R)$

Let Γ be a *po*-set such that no incomparable elements have a lower bound — usually called a *root* system. Let $V = V(\Gamma, R)$ be the set of all functions from Γ into the reals whose support satisfies the ACC. A component v_γ of $v \in V$ is maximal if $v_\gamma \neq 0$ and $v_\alpha = 0$ for all $\gamma < \alpha \in \Gamma$. Define $v \in V$ to be positive if each maximal component is positive. Then V is a vector lattice with respect to the natural addition and scalar multiplication (Conrad, Harvey and Holland (1963)).

Let A be an l-subgroup of V. A *v-isomorphism* τ of A into V is an l-isomorphism such that for each $a \in A$, a_α is a maximal component of a if and only if $(a\tau)_\alpha$ is a maximal component of $a\tau$.

LEMMA 2.1. *Each v-isomorphism τ of V into itself is epimorphic.*

PROOF. Consider $\theta < v \in V$ with a maximal component v_α. There is an element $u \in V$ with support α for which $(u\tau)_\alpha = v_\alpha$ since any o-isomorphism of R into R must be a multiplication by a positive real and hence an epimorphism.

Thus $V\tau$ is order dense in V and so τ preserves all infinite joins and intersections that exist in V (Bernau (1966)). Now $V\tau$ is laterally complete (i.e. each disjoint subset of V has a least upper bound) and so the join w of all the $u\tau$ (one for each maximal component of v) belongs to $V\tau$ and is a-equivalent to v ($mw \geqq v$ and $nv \geqq w$ for some positive integers m and n). Thus V is an a-extension of the a-closed l-group $V\tau$ and so $V = V\tau$. For a proof that V and hence $V\tau$ is a-closed see Conrad (1966).

LEMMA 2.2. *If A is an l-subgroup of V and $(A, *)$ is a vector lattice then the scalar multiplication $*$ can be extended to V so that $(V, *)$ is also a* vector *lattice.*

PROOF. There exists a linear v-isomorphism τ of $(A, *)$ into V that can be extended to a v-isomorphism α of V into V. For a proof of this see Conrad (1970) (τ is determined by a Banaschewski map for real subspaces but they are also rational subspaces and so we get α). Now by Lemma 2.1 α is epimorphic. For $r \in R$ and $v \in V$ define

$$r \# v = (r(v\alpha))\alpha^{-1}.$$

This is a scalar multiplication for V and for $a \in A$ we have

$$r \# a = (r(a\tau))\alpha^{-1} = ((r * a)\tau)\alpha^{-1} = ((r * a)\alpha)\alpha^{-1} = r * a$$

so $\#$ extends $*$.

REMARK. Example 5.4 shows that A need not be a subspace of V.

An *n-automorphism* of V is a v-automorphism that induces the identity on each V^γ / V_γ where

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$$V^\gamma = \{v \in V \mid v_\alpha = 0 \text{ for all } \alpha > \gamma\}, \text{ and}$$

$$V_\gamma = \{v \in V \mid v_\alpha = 0 \text{ for all } \alpha \geqq \gamma\}.$$

THEOREM 2.3. *Each n-characteristic l-subgroup A of V satisfies* II; *in fact any two scalar multiplications on A are connected by an n-automorphism of V. A satisfies* I *if and only if A is a subspace of V.*

PROOF. Let $*$ be a scalar multiplication so that $(A, *)$ is a vector lattice. By Lemma 2.2 $*$ can be extended to V. Thus (see Conrad (1970)) there exists a linear v-isomorphism α of $(V, *)$ into V and by Lemma 2.1 α is epimorphic. Now $V^\gamma\alpha = V^\gamma$ and $V_\gamma\alpha = V_\gamma$ so α induces an o-automorphism on each V^γ/V_γ. But $V^\gamma/V_\gamma \cong R$ and so these o-automorphisms are multiplications by positive reals. Let $\bar{\alpha}$ be the v-automorphisms of V determined by these multiplications. Then $\alpha\bar{\alpha}^{-1}$ is a linear n-automorphism of $(V, *)$ onto V and since A is n-characteristic

$$(A, *)\alpha\bar{\alpha}^{-1} = A.$$

In particular, A is a subspace of V.

COROLLARY I. *Each l-group can be embedded in a vector lattice that satisfies* II.

PROOF. The main theorem in Conrad, Harvey and Holland (1963) asserts that each l-group can be embedded in a suitable $V(\Gamma, R)$.

COROLLARY II. *Each l-ideal of V satisfies* I *and* II.

PROOF. It suffices to show that if $\theta < v \in V$ then the principal l-ideal $V(v)$ generated by v is n-characteristic. For each l-ideal of V is the join of a directed set of principal l-ideals and hence is n-characteristic.

Let τ be an n-automorphism of V. Then clearly v and $v\tau$ are a-equivalent and hence

$$V(v)\tau = V(v\tau) = V(v).$$

COROLLARY III. *Let* $\{A_\lambda \mid \lambda \in \Lambda\}$ *be a set of a-closed o-groups (that is, Hahn groups). Then the cardinal sum* $\sum A_\lambda$ *and the cardinal product* $\prod A_\lambda$ *of the* A_λ *satisfy* I *and* II.

PROOF. $\prod A_\lambda = V(\Delta, R)$ when Δ is the join of the $\Gamma(A_\lambda)$ and $\sum A_\lambda$ is an l-ideal of $\prod A_\lambda$.

COROLLARY IV. *If G is an n-characteristic l-subgroup of V then any two real subfields of* $\mathscr{P}(G)$ *are conjugate by a p-automorphism of G.*

PROOF. This follows from the theory in the introduction and the fact that an n-automorphism of V is a p-automorphism.

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Let N be the group of the n-automorphisms of V. If $*$ is a scalar multiplication of V then Theorem 2.3 asserts that there exist $\alpha \in N$ such that

$$(rv)\alpha = r*(v\alpha) \text{ for all } r \in R \text{ and } v \in V.$$

Thus each scalar multiplication of V is determined by an $\alpha \in N$ and the scalar multiplications of V determined by $\alpha, \beta \in N$ agree if and only if $\alpha\beta^{-1}$ is linear.

Let A be a vector lattice. Then we may assume that A is an l-subspace of $V = V(\Gamma, R)$ for a suitable Γ. Suppose that $*$ is another scalar multiplication for A. Then we can extend $*$ to V and there exists a linear n-automorphism τ of $(V, *)$ onto V. In particular, A and $A\tau$ are subspace of V and $r^*a = (r(a\tau))\tau^{-1}$ for each $r \in R$ and $a \in A$. Conversely if τ is an n-automorphism of V and $A\tau$ is a subspace of V then for each $a \in A$ and $r \in R$ we define $r * a = (r(a\tau))\tau^{-1}$. Then $(A, *)$ is a vector lattice and τ is a linear l-isomorphism of $(A, *)$ onto A.

Therefore the scalar multiplications of A are determined by the n-automorphisms of V that map A onto a subspace of V.

3. The l-Group $\Sigma(\Gamma, R)$

Let $V = V(\Gamma, R)$ be the vector lattice investigated in the last section. Let

$$\Sigma = \Sigma(\Gamma, R) = \{v \in V \mid \text{support of } v \text{ is finite}\}$$

$$F = F(\Gamma, R) = \{v \in V \mid \text{support of } v \text{ lies on a finite number of chains in } \Gamma\}$$

A *value* of an element g of an l-group G is an l-ideal of G that is maximal without containing g. G is *finite valued* if each $g \in G$ has only a finite number of values. The set $\Gamma = \Gamma(G)$ of all the values of elements in G is a root system.

In Conrad (1974) it is shown that if A is a finite valued vector lattice with countable dimension then there exists a linear l-isomorphism of A onto $\Sigma(\Gamma, R)$, where Γ is the index set for the set of all the regular subgroups of the l-group A. In particular, A is completely determined by the root system Γ.

THEOREM 3.1. *If A is a finite valued l-group then any two scalar multiplications of A for which the dimension of A is countable are connected by a v-automorphism of A.*

PROOF. Let $*$ and $\#$ be two such scalar multiplications. Then $(A, *) \cong \Sigma(\Gamma, R) \cong (A \#)$.

COROLLARY. *Let Γ be a countable root system and let $\Sigma = \Sigma(\Gamma, R)$ with the natural scalar multiplication. If $*$ is a new scalar multiplication for Σ then Σ and $(\Sigma, *)$ are connected by a v-automorphism if an only if $(\Sigma, *)$ has countable dimension.*

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REMARK. We have been unable to determine whether or not $(\Sigma, *)$ always has countable dimension. If so, then of course Σ satisfies II.

THEOREM 3.2. *Suppose that G is a finite valued l-group and $\Gamma(G)$ satisfies the DCC.*

1) $\Sigma(\Gamma, R) = F(\Gamma, R)$ *is the unique a-closure of G.*
2) $\Sigma(\Gamma, R)$ *is the unique a-extension of G that is a vector lattice.*
3) $\Sigma(\Gamma, R)$ *is the unique v-hull of G that is also an a-extension.*

PROOF. Recall that H is an a-extension of G if H is an l-group, G is an l-subgroup if H and each $0 < h \in H$ is an a-equivalent to an element $0 < g \in G$ or equivalently $L \to L \cap G$ is a one to one mapping of the set of l-ideals of H onto the l-ideals of G. An *a-closure* of G is an a-extension of G that does not admit a proper a-extension. Each group admits an a-closure but usually not a unique one (Conrad (1966) or Wolfenstein).

We first show that each a-extension H of G is finite valued. Here we do not need the fact that Γ satisfies the DCC. For $0 < h \in H$ there is an element $0 < g \in G$ such that $\eta g > h$ and $nh > g$ for some $n > 0$. Let $\{H_\lambda \mid \lambda \in \Lambda\}$ be the set of all values of h in H. Then they are also values of g. Thus $\{H_\lambda \cap G \mid \lambda \in \Lambda\}$ is a set of values of g in G and hence Λ is finite.

(1) Let K be an a-closure of G. Then since K is finite valued, divisible and $\Gamma(K)$ satisfies the DCC there is a value preserving l-isomorphism σ of K such that

$$K\sigma = \Sigma\,(\Gamma, K^\gamma/K_\gamma)$$

where K^γ is the intersection of all the l-ideals of K that properly contain K_γ (see Theorem 4.9 in Conrad (1970)). In particular, each K^γ/K_γ is o-isomorphic to a subgroup S_γ of R and so there exists an l-isomorphism τ of K so that

$$K\tau = \Sigma\,(\Gamma, S_\gamma) \subseteq \Sigma\,(\Gamma, R).$$

But clearly $\Sigma\,(\Gamma, R)$ is an a-extension of $\Sigma\,(\Gamma, S_\gamma)$ and so since $K\tau$ is a-closed, $K\tau = \Sigma\,(\Gamma, R)$. Now $F(\Gamma, R)$ is always an a-closure of $\Sigma\,(\Gamma, R)$ (Conrad (1966) p. 147) and so in our case $F = \Sigma$.

(2) Suppose that K is an a-extension of G that is a vector lattice. Then each $K^\gamma/K_\gamma \cong R$ and so

$$K\tau = \Sigma\,(\Gamma, S_\gamma) = \Sigma\,(\Gamma, R).$$

(3) Since a v-hull of G is a vector lattice this is a special case of (2).

COROLLARY. *For a root system Γ the following are equivalent.*

a) $\Sigma\,(\Gamma, R)$ *is a-closed.*
b) $\Sigma\,(\Gamma, R) = F(\Gamma, R)$.
c) Γ *satisfies the DCC.*

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Proof. We have shown c) → a) → b) and since F is always a-closed (b → a).

(a → c) If $\gamma_1 > \gamma_2 > \cdots$ is an inversely well ordered descending chain in Γ then let a be the element in $V(\Gamma, R)$ such that

$$a_\gamma = \begin{cases} 1 & \text{if } \gamma = \gamma_i \quad \text{for some } i \\ 0 & \text{otherwise.} \end{cases}$$

Then an easy computation shows that $[a] \oplus \Sigma$ is an a-extension of Σ and hence the chain must be finite.

Remark. In Conrad (1970) it is shown that for a totally ordered group G of finite rank a v-hull need not be an a-extension so G need not have a unique v-hull. Example 5.5 shows that $\Sigma(\Gamma, R)$ need not be an n-characteristic subgroup of $V(\Gamma, R)$ so we cannot conclude from the theory in the last section that $\Sigma(\Gamma, R)$ satisfies II.

Theorem 3.3. *If G is a finite valued l-group and $\Gamma(G)$ satisfies the DCC then the following are equivalent.*

1) *G satisfies* I.
2) *Each G^γ/G_γ is o-isomorphic to R.*
3) $G \cong \Sigma(\Gamma, R)$.
4) *G is a-closed.*

If this is the case then G satisfies II.

Proof. 1 → 2, and 3 → 1 and 2 are obvious. By Theorem 3.2 Σ is the unique a-closure of G and hence $3 \leftrightarrow 4$.

(2 → 3). Since each G^γ/G_γ is divisible there exists a v-isomorphism σ such that

$$\Sigma(\Gamma, G^\gamma/G_\gamma) \subseteq G\sigma \subseteq V(\Gamma, G^\gamma/G_\gamma)$$

(Conrad (1970)) and since $G^\gamma/G_\gamma \cong R$ for each $\gamma \in \Gamma$ we may assume

$$\Sigma(\Gamma, R) \subseteq G\sigma \subseteq V(\Gamma, R).$$

Now Γ satisfies the DCC and so by the proof of Theorem 4.9 in Conrad (1970) we have $\Sigma(\Gamma, R) = G\sigma$.

Now suppose that $*$ is another scalar multiplication for Σ and for each $\gamma \in \Gamma$ define $e(\gamma)$

$$e(\gamma)_\alpha = \begin{cases} 1 & \text{if } \alpha = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Then $E = \{e(\gamma) \mid \gamma \in \Gamma\}$ is a basis for Σ and an independent subset of $(\Sigma, *)$ so the identity map on E can be lifted to linear v-isomorphism τ of Σ into $(\Sigma, *)$. Since Σ is a-closed and $(\Sigma, *)$ is an a-extension of $\Sigma\tau$, τ is epimorphic and hence G satisfies II.

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THEOREM 3.4. *If G is an a-closed l-group that satisfies*

(F) *each bounded disjoint subset is finite,*

Then G satisfies I *and* II.

PROOF. $F(\Gamma, R)$ is the unique a-closure of an l-group that satisfies (F) (see Conrad (1966)). Thus $G \cong F$ and so G satisfies I.

In Conrad (1966) it is shown that if G is a vector lattice that satisfies (F) then there exists a linear v-isomorphism τ such that $\Sigma(\Gamma, R) \subseteq G\tau \subseteq F(\Gamma, R)$ and hence F is an a-extension of $G\tau$. Thus if G is a-closed $G\tau = F$ and so II is satisfied.

REMARK. If G satisfies (F) then $F(\Gamma, R)$ is an l-ideal of $V(\Gamma, R)$ and so Theorem 3.4 follows immediately from Corollary II of Theorem 2.3. Byrd (1966) gives an example that shows that in general $F(\Gamma, R)$ need not be n-characteristic in $V(\Gamma, R)$.

Note that if $\Gamma(G)$ is finite then G satisfies the hypothesis of Theorem 3.2 and 3.3. Also if G satisfies these hypotheses then any two real subfields of $\mathscr{P}(G)$ are conjugate by a p-automorphism of G.

4. Totally Ordered Groups

Throughout this section G will denote a totally ordered group with Γ the index set for the set of components G^γ/G_γ of G. $V(\Gamma, R)$ is the unique a-closure of G (Hahn 1907)). Thus if G is a-closed then $G \cong V$ and so by Theorem 2.3 G satisfies I and II and so does each l-ideal of V. The next Proposition shows that this is all we can conclude from Theorem 2.3.

PROPOSITION 4.1. *An n-characteristic subgroup L of V that is also a subspace is convex and conversely.*

PROOF. We show that $V(g) \subseteq L$ for each $0 < g \in L$. Let g_γ be the maximal component of g and consider $0 < h \in V(g)$. If $V(h) \neq V(g)$ then $h \ll g$ and o there exists an n-automorphism of V that maps g onto $g+h$. Then $h = g + h - g \in L$. If $V(h) = V(g)$ then there exists $r \in R$ such that $h_\gamma = rg_\gamma$ and so there exists an n-automorphism of V that maps rg onto h.

Note that we need the total order of G. For if $H = R \oplus R$ then $\{(x,x) \mid x \in R\}$ is an n-characteristic l-subgroup of H and a subspace but it is not an l-ideal.

Now suppose that Γ satisfies the DCC then $V = \Sigma(\Gamma, R)$ and by Theorems 3.1 and 3.2 we have:

V is the unique a-extension of G that is a vector lattice.

V is the unique v-hull of G that is also an a-extension.

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Moreover the following are equivalent: G satisfies I; each $G^\gamma/G_\gamma \cong R$; $G \cong V$; G is a-closed.

PROPOSITION 4.2. *For an o-group G the following are equivalent.*
1) *G satisfies* I.
2) *Each* $G(g)$ *satisfies* I.

PROOF. $(1 \to 2)$ Clear.

$(2 \to 1)$ Let S be the collection of all pairs $(L, *)$ where L is a convex subgroup of G and an ordered vector space with respect to the scalar multiplication $*$ Define $(L,*) \leqq (H, \#)$ if $(L,*)$ is a subspace of $(H, \#)$. Then by Zorn's lemma there exists a maximal element $(M,*)$ in S. We show $M = G$. Suppose by way of contradiction that $0 < g \in G \backslash M$. Then $G(g) \supset M$ and since M is divisible $G(g) = M \oplus D$ a lexicographic extension of M by the o-group D. Now by hypothesis $G(g)$ admits a scalar multiplication $\#$ and since M is contained in $G(g)$, $(M, \#)$ is a subspace. Thus $G(g)/M \cong D$ is also an ordered vector space — say $(D,\circ)$. For $r \in R$ and $m + t \in M \oplus D$ define

$$r \cdot (m+d) = r * m + r \circ d.$$

Then $(G(g), \cdot)$ is an ordered vector space and $(M,*)$ is a subspace, but this contradicts our choice of M and so $M = G$ satisfies I.

5. Examples and open questions

EXAMPLE 5.1. A real non-ordered vector space U does not satisfy II. For let α be a group isomorphism of R onto the direct sum $\oplus_\Lambda R_\lambda$ and for r in the field R and x in the group R define

$$r \circ x = (r(x\alpha))\alpha^{-1}$$

where $r(x\alpha)$ is the natural scalar multiplication in $\oplus_\Lambda R_\lambda$. Then $(R,\circ)$ is a real vector space of dimension $|\Lambda|$. Thus if $|\Lambda| > 1$ then $(R,\circ)$ and $(R,\cdot)$ are not connected by a group automorphism.

EXAMPLE 5.2. $R = D \oplus Q$ lexicographically ordered is a totally ordered group and a real vector space but it does not satisfy I. Also the cardinal sum $D \oplus Q$ is an archimedean l-group and a real vector space that does not satisfy I.

EXAMPLE 5.3. Let G be the subgroup of the cardinal product $\prod_{i=1}^\infty R_i$ generated by $\sum_{i=1}^\infty R_i$ and $(1,1,1,\cdots)$. Then G is an l-group and each $G^\gamma/G_\gamma \cong R$ except $G/\Sigma R_i$, but G does not satisfy I since it is not divisible.

If we totally order $\prod R_i$ by defining $(x_1, x_2, \cdots)$ to be positive if the first non-zero x_i is positive, then G is an o-group with each $G^\gamma/G_\gamma \cong R$ and G/C satisfies I for each non-zero convex subgroup C of G, but G does not satisfy I.

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Let $H = \sum_{i=1}^{\infty} R_i \oplus Q(1,1,1,\cdots)$ the divisible hull of G. Then G admits a scalar multiplication so that it is a real vector space, since its dimension as a rational vector space is large enough. If we impose the cardinal order on H then it does not satisfy I; for then it would have to be a subspace of the vector lattice $\coprod_{i=1}^{\infty} R_i$.

One should be able to show that if we impose the above total order on H then H does not satisfy I. If H does satisfy I then it follows from Lemma 2.2 and Theorem 2.3 that there exists an n-automorphism τ of $\coprod R_i$ such that $H\tau$ is a subspace.

EXAMPLE 5.4. Let $V = \coprod_{i=0}^{\infty} R_i$ be totally ordered as in the last example. Let μ be a group isomorphism of R onto $\coprod_{i=1}^{\infty} Q_i$

$$a \rightarrow (\mu_1(a), \mu_2(a), \cdots).$$

Define τ

$$(a_0, a_1, a_2, \cdots)\tau = (a_0, \mu_1(a_0) + a_1, \mu_2(a_0) + a_2, \cdots).$$

Then τ is an n-automorphism of V. Now $A = (\sum_{i=0}^{\infty} R_i)\tau$ is o-isomorphic to $\sum_{i=0}^{\infty} R_i$ and so it admits a scalar multiplication but it is not a subspace of V. For pick the $a \in R$ for which $a\mu = (1,1,1,\cdots)$. Then $(a,0,0,\cdots)\tau = (a,1,1,\cdots) \in A$ but $r(a,1,1,\cdots) \notin A$ for $r \in R \setminus Q$.

EXAMPLE 5.5. Let $V = \coprod_{i=1}^{\infty} R_i$ totally ordered as above and let $G = \sum_{i=1}^{\infty} R_i$. Then the map

$$(1,0,0,\cdots) \rightarrow (1,1,1,\cdots)$$
$$(0,1,0,\cdots) \rightarrow (0,1,1,\cdots)$$
$$\cdots\cdots\cdots$$

determines a linear v-isomorphism σ of Σ into V such that

$$\Sigma \subset \Sigma\sigma \subset V.$$

The map

$$(1,0,0,\cdots) \rightarrow (1,1,0,0,\cdots)$$
$$(0,1,0,\cdots) \rightarrow (0,1,1,0,\cdots)$$
$$\cdots\cdots\cdots$$

determines a linear v-isomorphism of Σ onto a proper subgroup of itself.

The map

$$(1,0,0,\cdots) \rightarrow (1,1,1,\cdots)$$
$$(0,1,0,\cdots) \rightarrow (0,1,0\cdots)$$
$$(0,0,1,0,\cdots) \rightarrow (0,0,1,0,\cdots)$$
$$\cdots\cdots\cdots$$

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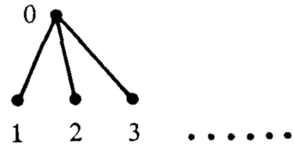
determines a map σ of Σ into V such that $\Sigma \mid \mid \Sigma\sigma$.

EXAMPLE 5.6. Let

$\Gamma =$ 

and let $V = V(\Gamma, R)$. The map

$$(1,0,0,\cdots) \to (1,1,1,\cdots)$$
$$(0,1,0,\cdots) \to (0,1,0,\cdots)$$
$$(0,0,1,0,\cdots) \to (0,0,1,0,\cdots)$$

determines an n-isomorphism σ of Σ into V such that $\Sigma \mid \mid \Sigma\sigma$.

EXAMPLE 5.7. An a-closed archimedean l-group need not satisfy I. Let

$$G = \prod_{i=1}^{\infty} Z_i \subset C \subset \prod_{i=1}^{\infty} R_i$$

cardinally ordered, where C consists of all the elements of the form $g + (x_1, x_2, \cdots)$ where $g \in G$ and $0 \leqq x_i \leqq 1$ and the number of distinct x_i is finite. Thus $C = G+F$, where F is the group of all elements in $\prod R_i$ with finite range. It is shown in Conrad (1966) that C is an a-closure of G. If C is a vector lattice then it must be a subspace of $\prod R_i$, but $\sqrt{2}(1,2,3,\cdots) \notin C$.

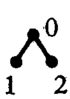
Note also that the v-hull G^v of G is not an a-extension of G. For clearly $G^v \supset C$. Actually

$$G^v = \{a \in \prod R_i \mid \text{there exists reals } r_1, \cdots, r_k \text{ such that each component of } a \text{ is of the form } x_1r_1 + x_2r_2 + \cdots x_kr_k \text{ with } x_i \in Z\}.$$

REMARK. It can be shown that a hyper-archimedean a-closed l-group need not satisfy I.

EXAMPLE 5.8. A minimal vector lattice that contains the o-subgroup $[1] \oplus [\sqrt{2}] \oplus [\pi]$ of R need not be totally ordered. Let f be a homomorphism of R into $R \oplus R$; $f(a) = (f_1(a), f_2(a))$ where

$$f(1) = (1,1)$$
$$f(2) = (\sqrt{2}, \sqrt{2}+1)$$
$$f(\pi) = (\pi+1, \pi).$$

Let $\Gamma =$ 

and let $V = V(\Gamma, R)$.

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Define $(a_0, a_1, a_2)\tau = (a_0, a_1 + f_1(a_0), a_2 + f_2(a_0))$. Then τ is an n-automorphism of V.

Define $r*(x\tau) = (rx)\tau$ for all $x \in V$ and $r \in R$. Then $(V, *)$ is a vector lattice.

$$r*(a_0, a_1 + f_1(a_0), a_2 + f_2(a_0)) = (ra_0, ra_1, ra_2)\tau = (ra_0, ra_1 + f_1(ra_0), ra_2 + f_2(ra_0)).$$

If $a_0 = 1$ and $a_1 = a_2 = -1$ we have

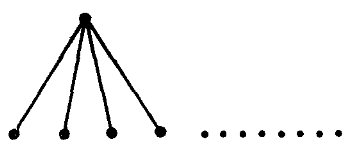
$$r*(1,0,0) = (r, -r + f_1(r), -r + f_2(r)).$$

In particular

$$\sqrt{2}*(1,0,0) = (\sqrt{2}, 0, 1)$$
$$\pi*(1,0,0) = (\pi, 1, 0).$$

Now let G be the o-subgroup of V generated by $(1,0,0)$, $(2,0,0)$ and $(\pi,0,0)$. Then V is a minimal vector lattice that contains G. Of course V is not the v-hull of G.

Example 5.9. A finite valued l-group G with $\Gamma(G)$ satisfying the DCC that admits two non-isomorphic v-hulls. Let Γ be the root system



and let $V = V(\Gamma, R)$. Let f be an isomorphism of R onto $\coprod_{i=1}^{\infty} R_i$ such that $f(1) = (1,0,0,\cdots)$ and in general $f(x) = (f_1(x), f_2(x), \cdots)$. Define

$$(x; x_1, x_2, \cdots)\tau = (x; x_1 + f_1(x), x_2 + f_2(x), \cdots).$$

Then τ is an n-automorphism of V. For $v \in V$ and $r \in R$ define $r \cdot (v\tau) = (rv)\tau$. Then $(V, \cdot)$ is a vector lattice.

$$\begin{aligned} r \cdot (x; f_1(x), f_2(x), \cdots) &= r \cdot (x; 0, 0, \cdots)\tau = (rx; 0, 0, \cdots)\tau \\ &= (rx; f_1(rx), f_2(rx), \cdots). \end{aligned}$$

In particular for $x = 1$ we have

$$r \cdot (1; 1, 0, 0, \cdots) = (r; f_1(r), f_2(r), \cdots).$$

Thus $(V, \cdot)$ is a v-hull of $G = \Sigma(\Gamma, R)$ and G is also a vector lattice with respect to the natural scalar multiplication. Now $G \not\cong V$ since the maximal l-ideal of V is laterally complete but the maximal l-ideal of G is not.

Note, of course, that the v-hull V of G is not finite valued and it is not an a-extension of G.

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Some open questions

1) Does II always hold?

2) If G is an archimedean l-group and each $G^\gamma/G_\gamma \cong R$ then does G satisfy I?

3) If G is an l-group and each $G(g)$ satisfies I then does G satisfy I?

4) If G is a vector lattice with a unique scalar multiplication then is G archimedean?

References

S. Bernau (1965), 'Unique representations of Archimedean lattice groups and normal Archimedean lattice rings', *Proc. London Math. Soc.* **15**, 599–631.

S. Bernau (1966), Orthocompletions of lattice groups', *Proc. London Math. Soc.* **16**, 107–130.

A. Bigard and K. Keimel (1969), 'Sur les endomorphismes conservant les polares d'un groupe réticule archimedien', *Bull. Soc. France* **97**, 381–398.

R. Bleier (1971), 'Minimal vector lattice cover', *Bull. Australian Math. Soc.* **5**, 331–335.

R. Byrd (1966), *Lattices ordered groups* (Dissertation Tulane Univ. 1966).

Paul Conrad (1966), 'Archimedean extensions of lattice-ordered groups', *J. Indian Math. Soc.* **30**, 131–160.

Paul Conrad, J. Harvey and C. Holland (1963), 'The Hahn embedding theorem for abelian lattice ordered groups', *Trans. Amer. Math. Soc.* **108**, 143–169.

Paul Conrad (1970), *Lattice ordered groups* (Lecture Notes, Tulane Univ. 1970).

Paul Conrad and J. Diem (1971), 'The ring of polar preserving endomorphisms of an l-group', *Illinois J. Math.* **15**, 222–240.

Paul Conrad (1974), 'Countable vector lattices', *Bull. Austral. Math. Soc.* **10**, 371–376.

H. Hahn (1907), 'Uber die nichtarchimedischen Grossensysteme', *Sitzungsberichte der Kaiserlchen Akademie der Wissensschaften, Vienna* **116**, 601–653.

K. Iwasawa (1943), 'On the structure of conditionally complete lattice-groups', *Japan J. Math.* **18**, 777–789.

S. Wolfenstein, *Contribution à l'étude des groupes réticules: extensions archimediennes, groupes à valeur normales* (Thèse, U. of Paris).

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