H. Tsumura
Nagoya Math. J.
Vol. 164 (2001), 185–197

ON MODIFICATION OF THE q-L-SERIES AND ITS APPLICATIONS

HIROFUMI TSUMURA

Abstract. We slightly modify the definitions of q-Hurwitz ζ-functions and q-L-series constructed by J. Satoh. By using these modified functions, we give some relations for the ordinary Dirichlet L-series. Especially we give an elementary proof of Katsurada's formula on the values of Dirichlet L-series at positive integers.

Introduction

Satoh defined q-L-series $L_q(s,\chi)$ in [S-1], which interpolated Carlitz's q-Bernoulli numbers at non-positive integers. His result was a response to Koblitz's problem suggested in [Ko]. In fact, $L_q(s,\chi)$ could be regarded just as what Koblitz required. $L_q(s,\chi)$ was essentially defined as a sum of two q-series. This causes difficulty in studying $L_q(s,\chi)$.

In [T-3], we considered the modified q-Riemann ζ-function, which is an example of Satoh's recent result (see [S-2]). By elementary calculations of q-series, we proved the formulas for $\zeta(2k+1)$ given by Cvijović and Klinowski ([C-K]).

In the present paper, corresponding to our previous work in [T-3], we modify the definition of q-L-series. In Section 1, we consider the modified q-Hurwitz ζ-function. In Section 2, we define the modified q-L-series. By investigating their properties, we prove some relations for the values of modified q-L-series (see Lemma 7). By letting $q \to 1$ in these relations, we prove some relations between the values of ordinary Dirichlet L-series at positive integers (see Proposition 2). Furthermore we give another proof of Katsurada's recent result on the values of Dirichlet L-series at positive integers (see Proposition 3). His result was proved by using the Mellin transformation technique ([Ka]).

Received October 22, 1999.
Revised March 31, 2000.
2000 Mathematics Subject Classification: Primary 11M41, 11M06. Secondly 11B68 11M06, 11M35, 33B15.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

The author would like to thank the referee for his valuable suggestions and comments.

§1. q-Hurwitz ζ-function

For $q \in \mathbf{R}$ with $0 < q < 1$, let $[z] = [z; q] = (1 - q^z)/(1 - q)$ for an indeterminate z. Note that $\lim_{q \to 1}[z] = z$. The modified q-Bernoulli numbers $\{\widetilde{\beta}_n(q)\}$ can be defined by

$$F_q(t) = \sum_{n=0}^{\infty} \widetilde{\beta}_n(q) \frac{t^n}{n!},$$

where $F_q(t)$ is determined as a solution of the following q-difference equation

$$F_q(t) = e^t F_q(qt) - t, \quad F_q(0) = \frac{q-1}{\log q},$$

(see [T-1]). Moreover we let $F_1(t) = t/(e^t - 1)$, and $\widetilde{\beta}_n(1) = B_n$ which is the ordinary Bernoulli number. If $0 < q < 1$ then the following series representation for $F_q(t)$ holds:

$$F_q(t) = \frac{q-1}{\log q} e^{t/(1-q)} - t \sum_{n=0}^{\infty} q^n e^{[n]t}, \tag{1.1}$$

(see [S-2],[T-3]). By above considerations, we can see that $F_q(t)$ is continuous as a function of (q, t) on $(0, 1] \times \{t \in \mathbf{C} \mid |t| < 2\pi\}$. As generalizations, we defined the modified q-Bernoulli polynomials by

$$F_q(q^x t) e^{[x]t} = \sum_{n=0}^{\infty} \widetilde{\beta}_n(x, q) \frac{t^n}{n!}.$$

Note that

$$F_q(q^x t) e^{[x]t} = \frac{q-1}{\log q} e^{t/(1-q)} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]t}.$$

We define the modified q-Hurwitz ζ-function by

$$\widetilde{\zeta}_q(s, x) = \frac{(1-q)^s}{(1-s)\log q} + \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]^s}, \tag{1.2}$$

for $x > 0$. The following lemma holds (see [T-3] §4).

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

Lemma 1. *For* $k \in \mathbf{N}$, $\widetilde{\zeta}_q(1-k,x) = -\dfrac{\widetilde{\beta}_k(x,q)}{k}$.

It follows from (1.2) that $\widetilde{\zeta}_q(s,x)$ is meromorphic in the whole complex plane and has a simple pole at $s = 1$ with residue $(q-1)/\log q$, since $\lim_{n\to\infty}[n] = 1/(1-q)$ if $0 < q < 1$. It is obvious that if $Re(s) > 1$ then $\lim_{q\to 1-0} \widetilde{\zeta}_q(s,x) = \zeta(s,x)$ which is the ordinary Hurwitz ζ-function. More strongly we can prove the following.

Lemma 2. $\lim_{q\to 1-0} \widetilde{\zeta}_q(s,x) = \zeta(s,x)$ *and* $\lim_{q\to 1-0} (\partial/\partial s)\widetilde{\zeta}_q(s,x) = (\partial/\partial s)\cdot \zeta(s,x)$ *for any* $s \in \mathbf{C}$ *except for* $s = 1$.

Proof. According to the well-known method(e.g. [W, Theorem 4.2]), we consider the function

$$H(s,q) = (e^{2\pi\sqrt{-1}s} - 1)\int_0^\infty t^{s-2} F_q(-q^x t)e^{-[x]t}\, dt,$$

for any $s \in \mathbf{C}$ and $q \in (0,1]$. Then it follows from (1.1) that $H(s,q) = (e^{2\pi\sqrt{-1}s} - 1)\Gamma(s)\widetilde{\zeta}_q(s,x)$, and $H(s,q)$ is holomorphic for any $s \in \mathbf{C}$ if $0 < q \le 1$. We can verify that $\lim_{q\to 1-0} H(s,q) = H(s,1)$ and $\lim_{q\to 1-0} (\partial/\partial s)H(s,q) = (\partial/\partial s)H(s,1)$. Thus we have the assertion.

If $0 < q < 1$ then, by (1.2), we have

$$\frac{\partial}{\partial s}\,\widetilde{\zeta}_q(s,x) = \frac{(1-q)^s\{\log(1-q)+1\}}{(1-s)^2\log q} - \sum_{n=0}^{\infty}\frac{q^{n+x}\log[n+x]}{[n+x]^s}. \tag{1.3}$$

Let

$$a(q) = \frac{\partial}{\partial s}\widetilde{\zeta}_q(0,1) = \frac{\log(1-q)+1}{\log q} - \sum_{m=1}^{\infty} q^m\log[m]. \tag{1.4}$$

By Lemma 2, we have

$$\lim_{q\to 1-0} a(q) = \lim_{q\to 1-0}\frac{\partial}{\partial s}\,\widetilde{\zeta}_q(0,1) = \frac{\partial}{\partial s}\,\zeta(0,1) = -\frac{1}{2}\log(2\pi). \tag{1.5}$$

Let $b(q) = \exp(-a(q))$. Then $\lim_{q\to 1-0} b(q) = \sqrt{2\pi}$. By combining (1.4) and (1.5), we get the following relation which can be regarded as a q-representation for the divergent formula $\prod_{m\ge 1} m = \infty! = \sqrt{2\pi}$ given by Riemann.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

Proposition 1. $\lim_{q\to 1-0} e^{-\frac{\log(1-q)+1}{\log q}} \prod_{m=1}^{\infty} [m]^{q^m} = \sqrt{2\pi}.$

§2. q-L-series

For a primitive Dirichlet character χ with conductor f, we define the modified q-L-series by

$$\widetilde{L}_q(s,\chi) = \sum_{a=1}^{f} \chi(a)[f]^{-s}\widetilde{\zeta}_{q^f}\left(s, \frac{a}{f}\right). \tag{2.1}$$

We can verify that

$$\begin{aligned}\widetilde{L}_q(s,\chi) &= \sum_{a=1}^{f} \chi(a)[f]^{-s}\left\{\frac{(1-q^f)^s}{(1-s)\log q^f} + \sum_{n=0}^{\infty} \frac{q^{f(n+a/f)}}{[n+a/f, q^f]^s}\right\}\\ &= \frac{(1-q)^s}{f(1-s)\log q}\sum_{a=1}^{f}\chi(a) + \sum_{n=1}^{\infty}\frac{\chi(n)q^n}{[n]^s}.\end{aligned}$$

So we have

$$\widetilde{L}_q(s,\chi) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^s} & (\chi \neq 1) \\ \displaystyle\frac{(1-q)^s}{(1-s)\log q} + \sum_{n=1}^{\infty}\frac{q^n}{[n]^s} & (\chi = 1) \end{cases} \tag{2.2}$$

In fact, $\widetilde{L}_q(s,1)$ coincides with the q-series $\widetilde{Z}_q(s)$ defined in [T-3], which can be regarded as a q-analogue of the Riemann ζ-function. Note that if $\chi \neq 1$ then $\widetilde{L}_q(s,\chi)$ is holomorphic in the whole complex plane.

Now we define the generalized q-Bernoulli numbers by

$$\widetilde{\beta}_{k,\chi}(q) = [f]^{k-1}\sum_{a=1}^{f} \chi(a)\widetilde{\beta}_k\left(\frac{a}{f}, q^f\right), \tag{2.3}$$

for $k \geq 0$. Note that $\lim_{q\to 1} \widetilde{\beta}_{k,\chi}(q) = B_{k,\chi}$ which is the generalized Bernoulli number defined by

$$\sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!}. \tag{2.4}$$

By (2.1),(2.3) and Lemma 1, we have the following.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

LEMMA 3. *For* $k \in \mathbf{N}$, $\widetilde{L}_q(1-k,\chi) = -\dfrac{\widetilde{\beta}_{k,\chi}(q)}{k}$.

From now on, we assume that $\chi \neq 1$. Let

$$H_q(t,\chi) = -t\sum_{n=1}^{\infty} \chi(n) q^n e^{[n]t},$$

for $q \in \mathbf{R}$ with $0 < q < 1$. It follows from the definition of $F_q(t)$ and (2.2) that $H_q(t,\chi)$ is the generating function of $\{\widetilde{\beta}_{k,\chi}(q)\}$, and is holomorphic in the whole complex plane. For the sake of convenience, let $H_1(t,\chi)$ be the function in the left-hand side of (2.4).

We can see that poles of $H_1(t,\chi)$ are $\{2\pi\sqrt{-1}l/f + 2n\pi\sqrt{-1} \mid n \in \mathbf{Z},\ l = 0,1,\cdots,f-1\}$. So we let

$$\begin{aligned} h(t,f) &= \prod_{l=1}^{f}(t - 2\pi\sqrt{-1}l/f)(t + 2\pi\sqrt{-1}l/f) \\ &= \prod_{l=1}^{f}(t^2 + 4\pi^2 l^2/f^2) = \sum_{l=0}^{f} C_l(f) t^{2l}, \end{aligned} \tag{2.5}$$

and let $I_q(t,\chi) = H_q(t,\chi)h(t,f)$ for any q with $0 < q \leq 1$. Then we see that $I_q(t,\chi)$ is holomorphic on $|t| \leq 2\pi$. Let

$$I_q(t,\chi) = \sum_{n=0}^{\infty} A_n(q,\chi)\frac{t^n}{n!}. \tag{2.6}$$

Then we have the following.

LEMMA 4. *Let r and d be real numbers with $0 < r < 2\pi$ and $0 < d < 1$. Then there exists a constant $R(r,d) > 0$ such that $|A_k(q,\chi)/k!| \leq R(r,d)/r^k$ for $k \geq 0$, if $d \leq q \leq 1$.*

Proof. Let C_r be a circle around O of radius r in the complex plane. By the consideration in §1, we can see that $I_q(t,\chi)$ is continuous as a function of (q,t) on the compact set $[d,1] \times C_r$. So we let $R(r,d) = \mathrm{Max}|I_q(t,\chi)|$ on $[d,1] \times C_r$. By the fact that

$$\frac{A_k(q,\chi)}{k!} = \frac{1}{2\pi\sqrt{-1}}\int_{C_r} I_q(t,\chi)t^{-k-1}\,dt,$$

we get the proof of Lemma.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

Now we consider the following permutation and combination function:

$$P(X,k)=\prod_{j=0}^{k-1}(X-j),\quad \binom{X}{k}=\frac{P(X,k)}{k!},$$

for any $k\in\mathbf{Z}$ with $k\geq 0$. Formally we let $P(0,0)=1$. If $m\in\mathbf{Z}$ with $0\leq m<k$, then $P(m,k)=0$. By considering the binomial expansions of both sides of $(1+t)^{X+Y}=(1+t)^X(1+t)^Y$, we get the following.

Lemma 5. $\binom{X+Y}{k}=\sum_{j=0}^{k}\binom{X}{k-j}\binom{Y}{j}$, *namely* $P(X+Y,k)=\sum_{j=0}^{k}\binom{k}{j}P(X,k-j)P(Y,j)$.

By Lemma 3 and using the above notations, we have

$$\begin{aligned}I_q(t,\chi)&=\sum_{l=0}^{f}C_l(f)\sum_{n=0}^{\infty}\widetilde{\beta}_{n,\chi}(q)\frac{t^{n+2l}}{n!}\\&=\sum_{l=0}^{f}C_l(f)\sum_{N\geq 2l}P(N,2l)\widetilde{\beta}_{N-2l,\chi}(q)\,\frac{t^N}{N!}\\&=-\sum_{N=0}^{\infty}\left(\sum_{l=0}^{f}C_l(f)P(N,2l+1)\widetilde{L}_q(1-N+2l,\chi)\right)\,\frac{t^N}{N!}.\end{aligned}$$

Thus we have the following.

Lemma 6. *For* $N\in\mathbf{Z}$ *with* $N\geq 0$,

$$\begin{aligned}A_N(q,\chi)&=\sum_{l=0}^{f}C_l(f)P(N,2l)\widetilde{\beta}_{N-2l,\chi}(q)\\&=-\sum_{l=0}^{f}C_l(f)P(N,2l+1)\widetilde{L}_q(1-N+2l,\chi).\end{aligned}$$

Remark. Since $B_{2k+1,\chi}=0$ if $\chi(-1)=1$ and $B_{2k,\chi}=0$ if $\chi(-1)=-1$ (e.g. [W] Chap.4), we have $\lim_{q\to 1}A_{2k+1}(q,\chi)=0$ if $\chi(-1)=1$, and $\lim_{q\to 1}A_{2k}(q,\chi)=0$ if $\chi(-1)=-1$, for $k\geq 0$.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

LEMMA 7. *For* $m \in \mathbf{N}$ *and* $\theta \in \mathbf{R}$ *with* $|\theta| \leq 2\pi$,

$$(1)\quad \sum_{d=0}^{f} P(2m+2d-1,2d) \sum_{l=d}^{f} \binom{2l+1}{2d} C_l(f)(-1)^{l-d}\theta^{2(l-d)+1} \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \cos([n]\theta)$$
$$-\sum_{d=0}^{f} P(2m+2d,2d+1) \sum_{l=d}^{f} \binom{2l+1}{2d+1} C_l(f)(-1)^{l-d}\theta^{2(l-d)} \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \sin([n]\theta)$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^{k+1}\theta^{2k+1}}{(2k+1)!} \times \sum_{l=0}^{f} C_l(f) P(2m-2k+2l-1,2l+1) \widetilde{L}_q(2m-2k+2l,\chi)$$
$$+ (-1)^m \theta^{2m} \sum_{n=0}^{\infty} \frac{1}{P(2n+2m+1,2m)} \frac{(-1)^{n+1}\theta^{2n+1}}{(2n+1)!} A_{2n+1}(q,\chi).$$

$$(2)\quad \sum_{d=0}^{f} P(2m+2d-1,2d) \sum_{l=d}^{f} \binom{2l+1}{2d} C_l(f)(-1)^{l-d+1}\theta^{2(l-d)+1} \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \sin([n]\theta)$$
$$+\sum_{d=0}^{f} P(2m+2d,2d+1) \sum_{l=d}^{f} \binom{2l+1}{2d+1} C_l(f)(-1)^{l-d+1}\theta^{2(l-d)} \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \cos([n]\theta)$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^{k+1}\theta^{2k}}{(2k)!} \times \sum_{l=0}^{f} C_l(f) P(2m-2k+2l,2l+1) \widetilde{L}_q(2m-2k+2l+1,\chi)$$
$$+(-1)^{m-1} \theta^{2m} \sum_{n=0}^{\infty} \frac{1}{P(2n+2m,2m+1)} \frac{(-1)^{n}\theta^{2n}}{(2n)!} A_{2n}(q,\chi).$$

Proof. We only give the proof of (1). The proof of (2) is given in just

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

the same manner as that of (1). For simplicity, we denote C_l instead of $C_l(f)$. Let

$$J_q(\theta,\chi,m)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \times\left\{\sum_{l=0}^{f}C_lP(2k-2m+1,2l+1)\ \widetilde{L}_q(1-(2k-2m+1)+2l,\chi)\right\}.$$

By Lemma 5, we have

$$\begin{aligned}
&J_q(\theta,\chi,m)\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\ \sum_{l=0}^{f}C_l\\
&\quad\times\sum_{u=0}^{2l+1}\binom{2l+1}{u}P(2k+1,2l+1-u)P(-2m,u)\ \widetilde{L}_q(-2k+2m+2l,\chi)\\
&=\sum_{l=0}^{f}C_l\sum_{u=0}^{2l+1}\binom{2l+1}{u}P(-2m,u)\\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\ P(2k+1,2l+1-u)\ \widetilde{L}_q(-2k+2m+2l,\chi)\\
&=\sum_{l=0}^{f}C_l\sum_{d=0}^{l}\binom{2l+1}{2d}P(-2m,2d)\\
&\quad\times\sum_{k=l-d}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\ P(2k+1,2l+1-2d)\ \widetilde{L}_q(-2k+2m+2l,\chi)\\
&\qquad\qquad+\sum_{l=0}^{f}C_l\sum_{d=0}^{l}\binom{2l+1}{2d+1}P(-2m,2d+1)\\
&\quad\times\sum_{k=l-d}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\ P(2k+1,2l-2d)\ \widetilde{L}_q(-2k+2m+2l,\chi).
\end{aligned}$$

Since $0<q<1$, we can easily verify that

$$\sum_{n=1}^{\infty}\frac{\chi(n)q^n}{[n]^s}\cos([n]\theta)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}\widetilde{L}_q(s-2k,\chi), \tag{2.7}$$

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

and

$$\sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^s} \sin([n]\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \widetilde{L}_q(s-2k-1,\chi). \tag{2.8}$$

By noticing that $P(-N,e)=(-1)^e P(N+e-1,e)$, and letting $n=k-l+d$, we have

$$\begin{aligned} J_q(\theta,\chi,m) &\\ = \sum_{d=0}^{f} P(2m+2d-1,2d) \sum_{l=0}^{f} \binom{2l+1}{2d} C_l (-1)^{l-d} \theta^{2(l-d)+1} & \\ \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \cos([n]\theta) & \\ + \sum_{d=0}^{f} P(2m+2d,2d+1) \sum_{l=0}^{f} \binom{2l+1}{2d+1} C_l (-1)^{l-d} \theta^{2(l-d)} & \\ \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \sin([n]\theta). & \end{aligned}$$

On the other hand, by Lemma 6, we have

$$\begin{aligned} J_q(\theta,\chi,m) = & \\ \sum_{k=0}^{m-1} \frac{(-1)^{k+1}\theta^{2k+1}}{(2k+1)!} \sum_{l=0}^{f} C_l P(2k-2m+1,2l+1)\ \widetilde{L}_q(2m-2k+2l,\chi) & \\ + \sum_{k=m}^{\infty} \frac{(-1)^{k+1}\theta^{2k+1}}{(2k+1)!}\ A_{2k-2m+1}(q,\chi). & \end{aligned}$$

Thus we have the proof of (1).

By letting $q \to 1$ with respect to the equations in Lemma 7, we get some relations for the values of ordinary Dirichlet L-series at positive integers.

Proposition 2. *Let* $m \in \mathbf{N}$ *and* $C_l(f) \in \mathbf{R}$ *defined by* (2.5).
(1) *If* $\chi(-1)=1$ *and* $\chi \neq 1$, *then*

$$\sum_{d=0}^{f} P(2m+2d-1,2d)$$

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

$$\times \sum_{l=d}^{f} \binom{2l+1}{2d} C_l(f)(-1)^{l-d}(2\pi)^{2(l-d)+1} L(2m+2d,\chi)$$

$$= \sum_{k=0}^{m-1} \frac{(-1)^{k+1}(2\pi)^{2k+1}}{(2k+1)!}$$

$$\times \sum_{l=0}^{f} C_l(f) P(2m-2k+2l-1, 2l+1) L(2m-2k+2l,\chi).$$

(2) *If* $\chi(-1) = -1$, *then*

$$\sum_{d=0}^{f} P(2m+2d, 2d+1)$$

$$\times \sum_{l=d}^{f} \binom{2l+1}{2d+1} C_l(f)(-1)^{l-d+1}(2\pi)^{2(l-d)} L(2m+2d+1,\chi)$$

$$= \sum_{k=0}^{m-1} \frac{(-1)^{k+1}(2\pi)^{2k}}{(2k)!}$$

$$\times \sum_{l=0}^{f} C_l(f) P(2m-2k+2l, 2l+1) L(2m-2k+2l+1,\chi).$$

Proof. By Lemma 4, we can see that both sides of the equations in (1) and (2) of Lemma 7 are uniformly convergent with respect to $q \in (0,1]$, if $\theta = 2\pi$. So we can let $q \to 1$. By Remark after Lemma 6, we get the proof.

In [Ka], Katsurada recently proved the following series representations for the values of $L(s,\chi)$ at positive integers by using the Mellin transformation technique. In the rest of this section, we give another proof of Katsurada's result by using the same method as above.

Proposition 3. ([Ka, Theorem 3]) *Let* n *be a positive integer,* x *be a real number with* $|x| \le 1$ *and* $\tau(\chi) = \sum_{a=1}^{f} \chi(a) \exp(2\pi\sqrt{-1}a/f)$ *be the Gauss sum.*

(1) *If* $\chi(-1) = 1$ *and* $\chi \neq 1$, *then*

$$nL(2n+1,\chi) - n\sum_{l=1}^{\infty} \frac{\chi(l)\cos(2\pi lx/f)}{l^{2n+1}} - \frac{\pi x}{f} \sum_{l=1}^{\infty} \frac{\chi(l)\sin(2\pi lx/f)}{l^{2n}}$$

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

$$= (-1)^n \left(\frac{2\pi x}{f}\right)^{2n} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k L(2k+1,\chi)}{(2n-2k)!(2\pi x/f)^{2k}} + \frac{\tau(\chi)}{f} \sum_{k=1}^{\infty} \frac{(2k)! L(2k,\overline{\chi})}{(2n+2k)!} x^{2k} \right\};$$

(2) *If* $\chi(-1) = -1$, *then*

$$L(2n,\chi) - \sum_{l=1}^{\infty} \frac{\chi(l)\cos(2\pi l x/f)}{l^{2n}}$$
$$= (-1)^n \left(\frac{2\pi x}{f}\right)^{2n-1} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} L(2k,\chi)}{(2n-2k)!(2\pi x/f)^{2k-1}} + \frac{2\sqrt{-1}\tau(\chi)}{f} \sum_{k=0}^{\infty} \frac{(2k)! L(2k+1,\overline{\chi})}{(2n+2k)!} x^{2k+1} \right\}.$$

Proof. Suppose that $\chi(-1) = 1$ and $\chi \neq 1$, $q \in \mathbf{R}$ with $0 < q < 1$, and $\theta \in \mathbf{R}$ with $|\theta| < 2\pi/f$. By (2.7), (2.8) and Lemma 3, we have

$$\text{(3.1)} \quad n \sum_{l=1}^{\infty} \frac{\chi(l) q^l \cos([l]\theta)}{[l]^{2n+1}} - \frac{\theta}{2} \sum_{l=1}^{\infty} \frac{\chi(l) q^l \sin([l]\theta)}{[l]^{2n}}$$
$$= n\widetilde{L}_q(2n+1,\chi) + \sum_{j=1}^{n-1} \frac{(-1)^j \theta^{2j}}{(2j)!} (n-j) \widetilde{L}_q(2n+1-2j,\chi) + \frac{1}{2} \sum_{j=n+1}^{\infty} \frac{(-1)^j \theta^{2j}}{(2j)!} \widetilde{\beta}_{2j-2n,\chi}(q).$$

By the definition of $\widetilde{\beta}_{n,\chi}(q)$ and the same reason as that in the proof of Proposition 2, we can see that both sides of (3.1) are uniformly convergent with respect to $q \in (0,1]$ if $|\theta| < 2\pi/f$. Hence we can let $q \to 1$ in both sides of (3.1). By using the well-known relation

$$B_{2j,\chi} = \frac{2(-1)^{j+1}\tau(\chi)}{f} \left(\frac{f}{2\pi}\right)^{2j} (2j)! L(2j,\overline{\chi}),$$

we have

$$n \sum_{l=1}^{\infty} \frac{\chi(l)\cos(l\theta)}{l^{2n+1}} - \frac{\theta}{2} \sum_{l=1}^{\infty} \frac{\chi(l)\sin(l\theta)}{l^{2n}}$$

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

$$= nL(2n+1,\chi) + (-1)^n\theta^{2n}\Bigg\{\sum_{k=1}^{n-1}\frac{(-1)^k kL(2k+1,\chi)}{(2n-2k)!\theta^{2k}} - \frac{\tau(\chi)}{f}\sum_{m=1}^{\infty}\frac{(-1)^m\theta^{2m}(2m)!}{(2m+2n)!}\left(\frac{f}{2\pi}\right)^{2m}L(2m,\overline{\chi})\Bigg\}.$$

By putting $\theta = (2\pi x/f)$, we get the proof of (1).

Suppose that $\chi(-1) = -1$. By (2.7) and Lemma 3, we have

$$\sum_{l=1}^{\infty}\frac{\chi(l)q^l\cos([l]\theta)}{[l]^{2n}} = \widetilde{L}_q(2n,\chi) + \sum_{j=1}^{n-1}\frac{(-1)^j\theta^{2j}}{(2j)!}\widetilde{L}_q(2n-2j,\chi) + \sum_{j=n}^{\infty}\frac{(-1)^j\theta^{2j}}{(2j)!}\left(-\frac{\widetilde{\beta}_{2j-2n+1,\chi}(q)}{2j-2n+1}\right).$$

By letting $q \to 1$, putting $\theta = 2\pi x/f$ and by using the relation

$$B_{2j+1,\chi} = \frac{2(-1)^j\sqrt{-1}\tau(\chi)}{f}\left(\frac{f}{2\pi}\right)^{2j+1}(2j)!L(2j+1,\overline{\chi}),$$

we get the proof of (2). Thus we have the assertion.

References

[C-K] D. Cvijović and J. Klinowski, *New rapidly convergent series representations for* $\zeta(2n+1)$, Proc. Amer. Math. Soc., **125** (1997), 1263–1271.

[Ka] M. Katsurada, *Rapidly convergent series representations for* $\zeta(2n+1)$ *and their* χ*-analogue*, Acta Arith., **40** (1999), 79–89.

[Ko] N. Koblitz, *On Carlitz's q-Bernoulli numbers*, J. Number Theory, **14** (1982), 332–339.

[S-1] J. Satoh, *q-analogue of Riemann's* ζ*-function and q-Euler numbers*, J. Number Theory, **31** (1989), 346–362.

[S-2] ———, *Another look at the q-analogue from the viewpoint of formal groups*, Preprint Ser. in Math. Sciences, Nagoya Univ. (1999-4).

[T-1] H. Tsumura, *A note on q-analogues of the Dirichlet series and q-Bernoulli numbers*, J. Number Theory, **39** (1991), 251–256.

[T-2] ———, *On evaluation of the Dirichlet series at positive integers by q-calculation*, J. Number Theory, **48** (1994), 383–391.

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press

[T-3] ———, *A note on q-analogues of Dirichlet series*, Proc. Japan Acad., **75** ser.A (1999), 23–25.

[W] R.C. Washington, Introduction to Cyclotomic Fields, 2nd ed., Springer-Verlag, 1997.

Department of Management
Tokyo Metropolitan College
Azuma-cho, Akishima-shi
Tokyo 196-8540, Japan
tsumura@tmca.ac.jp

https://doi.org/10.1017/S0027763000008096 Published online by Cambridge University Press