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ON PERMUTATION GROUPS WITH REGULAR SUBGROUP

BY
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I. **Introduction.** W. Burnside [3, p. 343] showed that a cyclic group of order p^m (p prime, $m>1$) cannot occur as a regular subgroup of a simply transitive primitive group. (For definitions and notation see [9].) Groups which are contained regularly in a primitive group G only when G is doubly transitive are therefore called B-groups [9, p. 64]. Burnside [3, p. 343] conjectured that every abelian group is a B-group. A class of counterexamples which can be deduced from a 1906 paper of W. A. Manning [6] was given in 1936 by D. Manning [5] and generalized by H. Wielandt [9, p. 67]. The Burnside conjecture has been partially restored by I. Schur [7], H. Wielandt [8], R. Kochendorffer [4], and R. Bercov [1] by means of a method of Schur which associates with a group G with regular subgroup H a subring of the group ring of H, now called a Schur ring, which characterizes the action of the stabilizer in G of a point and hence the action of G on pairs of points. In [2] it is shown (apart from a minor exception associated with exponent 4) that if H is an abelian group which is not the direct product of two subgroups of the same exponent, then either H is a B-group or it is in the Wielandt class of counterexamples. It is the purpose of this note to generalize the Wielandt class of simply transitive group G (using the same regular subgroups H) and to compute the associated Schur-rings. We conjecture that we obtain in this way every non-trivial primitive Schur-ring (for definitions see [9]) over an abelian H which satisfies the hypothesis of [2]. This would mean that any simply transitive primitive group with such a regular subgroup H must move pairs of points in the same way as one of the group given here.

II. **The construction.** For $d\geq 2$, let $H_1, \dots, H_d$ be groups of the same order $a\geq 3$, and let T be a transitive group on $\{1, \dots, d\}$.

Let Φ be a set of size ad partitioned into subsets Φ_j, $j=1, \dots, d$, with $\Phi_j=\{\Phi_{ij} \mid i=1, \dots, a\}$.

Denote by S_j the symmetric group on Φ_j regarded as acting trivially on the Φ_i with $i\neq j$, and let S_j^* be the stabilizer in S_j of Φ_{1j}. Put $S=\langle S_j \mid j=1, \dots, d\rangle$ and $S^*=\langle S_j^* \mid j=1, \dots, d\rangle$.

We regard H_j as a subgroup of S_j by letting H_j act regularly on Φ_j and trivially on the Φ_i, $i\neq j$, and let T act on Φ by permuting the Φ_j; $\Phi_{ij}^t=\Phi_{ij^t}$.

We see easily that T normalizes both S and S^*, and we put $G=ST$ and $G^*=S^*T$.

Setting $\Delta=\{\Phi_{1j} \mid j=1, \dots, d\}$ and $\Omega=\{\Delta^x \mid x\in G\}$ we have that G^* is the stabilizer of Δ as a set and that the action of G on Ω is therefore equivalent to the action of G on the cosets of G^*. This action yields the desired permutation group. The proof that we give below was given by Wielandt [9] with T the symmetric group.

THEOREM *G acts faithfully and primitively on Ω but not two-transitively.*

$$H=H_1x.\ldots.xH_d$$

acts regularly on Ω.

Proof. Since every element of G permutes the Φ_j and Δ contains one element from each Φ_j, we have for any $x\in G$ that $|\Delta^x\cap\Phi_j|=1$ for $j=1,\ldots,d$. Since H_j acts regularly on Φ_j and trivially on the other Φ_i we have that Ω consists of all subsets of Φ which meet each Φ_j in a singleton. Clearly there is a unique $h\in H$ taking Δ to each such set and H therefore acts regularly.

Since every singleton from Φ is the intersection of two appropriately chosen sets in Ω, the kernel of the action of G on Ω must act trivially on Φ, and the action of G on Ω is therefore faithful.

Primitivity follows from the maximality of G^*. For $x=st\in G-G^*$, $s\in S$, $t\in T$, we have $s\in S-S^*$ and therefore $\Phi_{1j}^s\neq\Phi_{1j}$ for some j. Then $\langle G^*,x\rangle\geq\langle S_j^*,(S_j^*)^s\rangle^t=S_j^t$ for all $t\in T$ and since T is transitive we have $\langle G^*,x\rangle\geq ST=G$.

Finally we see that G has order $(a!)^d\,|T|$, G^* has order $((a-1)!)^d\,|T|$ and for $h\in H_1$, the stabilizer G^{**} of Δ and Δ^h has order $(a-2)![(a-1)!]^{d-1}\,|T_1|$ where T_1 is the stabilizer of Φ_1 in T. Thus G cannot act two-transitively, since the index of G^{**} in G^* is $(a-1)d$ which is not equal to $|\Omega|-1=a^d-1$.

III. **The Schur-rings.** To find the orbits of G^* on Ω we remark that since G^* contains S^*, for any $\Gamma_1,\Gamma_2\in\Omega$, $\Gamma_1-\Delta$ can be taken to $\Gamma_2-\Delta$ by an element of s of G^* which fixes Δ pointwise. The points of $\Gamma_1\cap\Delta$ can be taken to the points of $\Gamma_2\cap\Delta$ within G^* only by an element of T. Moreover if $t\in T$ takes $\Gamma_1\cap\Delta$ to $\Gamma_2\cap\Delta$ it is easy to see that $s\in S^*$ can be chosen so that st takes Γ_1 to Γ_2. Thus if for $h=\Pi_{i=1}^d h_i$, we put $\sigma(h)=\{j\mid h_j\neq 1\}$ we have

LEMMA *For $h,k\in H$, Δ^h and Δ^k are in the same G^*-orbit if and only if $\sigma(h)^t=\sigma(k)$ for some $t\in T$.*

Since h and k are in the same basis element of the Schur-ring if and only if Δ^h and Δ^k are in the same G^* orbit, this means that the Schur-ring of G has as its basis the sets

$$\bigcup_{t\in T}\prod_{i\in I^t}H_i^\#$$

where $H_i^\#=H_i-1$ and I is a fixed subset of $1,\ldots,d$.

EXAMPLE: For $d=4$ there are five choices for T, namely $T_1=\langle(12)(34),(13)(24)\rangle$, $T_2=\langle(1234)\rangle$, $T_3=\langle T_1,T_2\rangle$, $T_4=A_4$, $T_5=S_4$. If G_i is the group on Ω obtained by the above construction with $T=T_i$ we have for *all i* that the Schur-ring of G_i has basis sets

$$H_1^\# \cup H_2^\# \cup H_3^\# \cup H_4^\#, \qquad H_1^\# H_2^\# H_3^\# \cup H_1^\# H_2^\# H_4^\# \cup H_1^\# H_3^\# H_4^\# \cup H_2^\# H_3^\# H_4^\#,$$

and $H_1^\# H_2^\# H_3^\# H_4^\#$.

However the basis sets of length two are different.

For G_1 we have three more basis sets namely

$$H_1^\# H_2^\# \cup H_3^\# H_4^\#, \qquad H_1^\# H_3^\# \cup H_2^\# H_4^\#,$$

and

$$H_1^\# H_4^\# \cup H_2^\# H_3^\#.$$

For G_2 and G_3 we have $H_1^\# H_2^\# \cup H_2^\# H_3^\# \cup H_3^\# H_4^\# \cup H_1^\# H_4^\#$ and $H_1^\# H_3^\# \cup H_2^\# H_4^\#$. For G_4 and G_5 we have only $H_1^\# H_2^\# \cup H_2^\# H_3^\# \cup H_2^\# H_4^\# \cup H_1^\# H_4^\# \cup H_1^\# H_3^\# \cup H_2^\# H_4^\#$.

IV. **Conclusion.** It can be shown in general under the hypotheses of [2] that all elements of H of length 1, $d-1$, and d correspond to the same G^*-orbit. It can also be verified by direct computation that for $d \leq 7$ and H as in [2], every non-trivial primitive Schur-ring over H has a basis of the above type for some T. This means that for $d \leq 7$ every simply transitive primitive group with such an H as regular subgroup moves pairs of points in the same way as one of the groups constructed here. We conjecture that this is the case for all d. A counterexample would be of degree a^d where a and d are at least eight and hence would permute more than sixteen million points.

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