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SATURATION ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract

Let G be a locally compact abelian group, (μ_ρ) a net of bounded Radon measures on G. In this paper we consider conditions under which (μ_ρ) is saturated in $L^p(G)$ and apply these results to the Fejér and Picard approximation processes.

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Throughout G will denote a locally compact abelian group, Γ its character group. Haar measures λ, θ on G, Γ respectively will be chosen so that Plancherel's theorem holds. For each $p \in [1, \infty]$ we denote by $L^p(G)$ the usual Lebesgue space of pth-integrable functions with respect to the Haar measure λ. The characteristic function of the set E will be denoted by ξ_E. The symbols **T**, **N**, **Z**, **R** will be reserved for the circle group, the set of natural numbers, the group of integers and the real line respectively. We take Hewitt and Ross [7] as our standard reference for harmonic analysis on G; any unexplained notation will be found there.

Take (μ_ρ) to be a bounded net in $M_b(G)$, the space of bounded Radon measures on G. The family (μ_ρ) is said to be a bounded approximate unit on G if $\lim_\rho \|\mu_\rho * f - f\|_1 = 0$ for each $f \in L^1(G)$. It is of fundamental interest in approximation theory to examine the rate of convergence of bounded approximate units. In many cases it happens that there is essentially a limit to the rate of convergence; for example if (μ_n) is a sequence of even probability measures on the circle group $\mathbf{T}$ then the optimal rate of convergence is given by

$$\|\mu_n * f - f\|_\infty \leqslant C\beta_n^2,$$

where $\beta_n = (1 - \hat{\mu}_n(\gamma_1))^{1/2}$, γ_1 is the character of $\mathbf{T}$ given by $\gamma_1(x) = x$, and f has a derivative belonging to the Lipschitz class of order 1. This rate of convergence cannot in general be improved, as is indicated by the fact that even the infinitely differentiable function $t \to \cos t$ has rate of convergence given by (β_n^2).

Saturation theory is concerned with determining this optimal rate of convergence, called the saturation order, and the space V of functions for which this rate is attained. In this case V is called the saturation class (or Favard space) for (μ_ρ).

We are concerned with determining the saturation class for certain bounded approximate units in $L^2(G)$. We begin in Section 1 with some preliminary results in the theory of saturation. Section 2 will be concerned with results specific to saturation in $L^2(G)$, and in the third section we present some examples to support the theory.

1. General results in saturation theory

Let (μ_ρ) be a bounded approximate unit on G. The trivial class of (μ_ρ) is defined to be $T_p(\mu_\rho) = \{f \in L^p(G)$: there exists ρ_0 such that $\mu_\rho * f = f$ for all $\rho \geqslant \rho_0\}$. Let (ϕ_ρ) be a net of positive real numbers (with the same index set as (μ_ρ)) that converges to zero. We say that (μ_ρ) is *saturated* in $L^p(G)$ with order (ϕ_ρ) if the following are satisfied:

(i) for $f \in L^p(G)$, $\|\mu_\rho * f - f\|_p = o(\phi_\rho)$ if and only if $f \in T_p(\mu_\rho)$;

(ii) there exists $g \in L^p(G) \setminus T_p(\mu_\rho)$ for which $\|\mu_\rho * g - g\|_p = O(\phi_\rho)$.

(By $\psi_\rho = o(\phi_\rho)$ we mean $\liminf_\rho \phi_\rho^{-1}\psi_\rho = 0$, and by $\psi_\rho = O(\phi_\rho)$, $\limsup_\rho \phi_\rho^{-1}\psi_\rho < \infty$.) If (μ_ρ) is saturated in $L^p(G)$ with order (ϕ_ρ) then its saturation class is defined to be the non-empty set

$$S_p(\mu_\rho) = \left\{f \in L^p(G): \|\mu_\rho * f - f\|_p = O(\phi_\rho)\right\}.$$

Also if $E \subset L^p(G)$ we write $S_E(\mu_\rho)$ for the space $S_p(\mu_\rho) \cap E$.

It is usual to take the trivial class to consist of only the constant functions in $L^p(G)$; see DeVore [4], 3.1.5, for example. Nishishiraho [8] allowed for a possibly larger trivial class by requiring $\mu_\rho * f = f$ for all ρ. We feel that our slightly more general definition is better suited to approximation processes.

If (μ_ρ) is saturated in $L^p(G)$ with two saturation orders, (ϕ_ρ) and (ϕ'_ρ), then $\phi_\rho = O(\phi'_\rho)$. For if $\phi_\rho \neq O(\phi'_\rho)$ then $\phi'_\rho = o(\phi_\rho)$ and so for $g \in S_p(\mu_\rho)$, $\|\mu_\rho * g - g\|_p = O(\phi'_\rho) = o(\phi_\rho)$ implies $g \in T_p(\mu_\rho)$, which contradicts the definition of saturation class. Thus we speak of "the" saturation order of (μ_ρ) and observe that it defines a unique saturation class in $L^p(G)$.

Write $\Gamma_T(\mu_\rho) = \{\gamma \in \Gamma$: there exists ρ_0 such that $\hat{\mu}_\rho(\gamma) = 1$ for all $\rho \geqslant \rho_0\}$ and, if (μ_ρ) is saturated in $L^p(G)$ with order (ϕ_ρ), write

$$\Gamma_S(\mu_\rho) = \left\{\gamma \in \Gamma : \left|\hat{\mu}_\rho(\gamma) - 1\right| = O(\phi_\rho)\right\}.$$

In general we can say little about the structure of these sets, except that in practice $\Gamma_S(\mu_\rho) = \Gamma$ and $\Gamma_T(\mu_\rho) = \{1\}$. (Here 1 denotes the identity character.) However we do have the following result:

THEOREM 1. *Suppose that each μ_ρ is a probability measure. Then $\Gamma_T(\mu_\rho)$ and $\Gamma_S(\mu_\rho)$ are subgroups of Γ.*

PROOF. Write $C_u(G)$ for the space of bounded uniformly continuous functions on G and, for each ρ, write

$$\mu_\rho(f) = \int_G f\, d\mu_\rho,$$

so that μ_ρ can be regarded as a positive linear functional on $C_u(G)$.

If $\gamma_1, \gamma_2 \in \Gamma$ then

$$1 - \gamma_1\gamma_2 = 1 - \operatorname{Re}\gamma_1 \operatorname{Re}\gamma_2 + \operatorname{Im}\gamma_1 \operatorname{Im}\gamma_2 - i(\operatorname{Re}\gamma_1 \operatorname{Im}\gamma_2 + \operatorname{Im}\gamma_1 \operatorname{Re}\gamma_2).$$

Since

$$\begin{aligned} 0 &\leqslant 1 - \operatorname{Re}\gamma_1 \operatorname{Re}\gamma_2 \\ &= (1 - \operatorname{Re}\gamma_1)\operatorname{Re}\gamma_2 + (1 - \operatorname{Re}\gamma_2) \leqslant (1 - \operatorname{Re}\gamma_1) + (1 - \operatorname{Re}\gamma_2) \end{aligned}$$

we deduce

$$0 \leqslant 1 - \mu_\rho(\operatorname{Re}\gamma_1 \operatorname{Re}\gamma_2) \leqslant \big(1 - \mu_\rho(\operatorname{Re}\gamma_1)\big) + \big(1 - \mu_\rho(\operatorname{Re}\gamma_2)\big).$$

To estimate $\mu_\rho(\operatorname{Im}\gamma_1 \operatorname{Im}\gamma_2)$ we use the Cauchy-Schwarz inequality for positive linear functionals to obtain

$$\begin{aligned} \left|\mu_\rho(\operatorname{Im}\gamma_1 \operatorname{Im}\gamma_2)\right| &\leqslant \left|\mu_\rho\big((\operatorname{Im}\gamma_1)^2\big)\right|^{1/2} \left|\mu_\rho\big((\operatorname{Im}\gamma_2)^2\big)\right|^{1/2} \\ &= \left|\mu_\rho\big(1 - (\operatorname{Re}\gamma_1)^2\big)\right|^{1/2} \left|\mu_\rho\big(1 - (\operatorname{Re}\gamma_2)^2\big)\right|^{1/2} \\ &\leqslant \left|2\big(1 - \mu_\rho(\operatorname{Re}\gamma_1)\big)\right|^{1/2} \left|2\big(1 - \mu_\rho(\operatorname{Re}\gamma_2)\big)\right|^{1/2}. \end{aligned}$$

Also we note that

$$-(1 - \operatorname{Re}\gamma_1) \leqslant (1 - \operatorname{Re}\gamma_1)\operatorname{Im}\gamma_2 \leqslant 1 - \operatorname{Re}\gamma_1$$

implies

$$\left|\mu_\rho(\operatorname{Im}\gamma_2) - \mu_\rho(\operatorname{Re}\gamma_1 \operatorname{Im}\gamma_2)\right| \leqslant \left|1 - \mu_\rho(\operatorname{Re}\gamma_1)\right|,$$

which implies

$$\begin{aligned} \left|\mu_\rho(\operatorname{Re}\gamma_1 \operatorname{Im}\gamma_2)\right| &\leqslant \left|1 - \mu_\rho(\operatorname{Re}\gamma_1)\right| + \left|\mu_\rho(\operatorname{Im}\gamma_2)\right| \\ &\leqslant \left|1 - \mu_\rho(\gamma_1)\right| + \left|1 - \mu_\rho(\gamma_2)\right|, \end{aligned}$$

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since $\mu_\rho(\operatorname{Re}\gamma_1) = \operatorname{Re}\mu_\rho(\gamma_1)$ and $\mu_\rho(\operatorname{Im}\gamma_2) = \operatorname{Im}\mu_\rho(\gamma_2)$. Similarly

$$|\mu_\rho(\operatorname{Re}\gamma_2 \operatorname{Im}\gamma_1)| \leqslant |1 - \mu_\rho(\gamma_1)| + |1 - \mu_\rho(\gamma_2)|.$$

Then putting all these inequalities together we obtain

$$|1 - \mu_\rho(\gamma_1\gamma_2)| \leqslant 3\big(|1 - \mu_\rho(\gamma_1)| + |1 - \mu_\rho(\gamma_2)|\big) + 2|1 - \mu_\rho(\gamma_1)|^{1/2}|1 - \mu_\rho(\gamma_2)|^{1/2}.$$

We also have

$$\mu_\rho(\gamma^{-1}) = \hat{\mu}_\rho(\gamma).$$

From this and the preceding inequality the result follows.

The space $\Gamma_T(\mu_\rho)$ plays an important role in the saturation theory, as the following result shows.

THEOREM 2. *Suppose that G is a compact abelian group. If (μ_ρ) is totally ordered and saturated then*

$$T_p(\mu_\rho) = \{f \in L^p(G) \colon \operatorname{supp}(\hat{f}) \subset \Gamma_T(\mu_\rho)\}.$$

PROOF. Suppose $\mu_\rho * f = f$. Then $\hat{\mu}_\rho \hat{f} = \hat{f}$ (Hewitt and Ross [7], (31.5)) and so $\hat{f}(\gamma) \neq 0$ implies $\hat{\mu}_\rho(\gamma) = 1$; and hence if $f \in T_p(\mu_\rho)$, $\{\gamma \in \Gamma\colon \hat{f}(\gamma) \neq 0\} \subset \Gamma_T(\mu_\rho)$. Since Γ is discrete, this just says that $\operatorname{supp}\hat{f} \subset \Gamma_T(\mu_\rho)$.

Conversely, suppose $\operatorname{supp}\hat{f} \subset \Gamma_T(\mu_\rho)$. Since $f \in L^p(G) \subset L^1(G)$ and Γ is discrete, the Riemann-Lebesgue lemma (Hewitt and Ross [7], (28.40)) gives that $\operatorname{supp}\hat{f}$ is countable; write $\operatorname{supp}\hat{f} = \{\gamma_n\}_{n=1}^\infty$ (we suppose that $\operatorname{supp}\hat{f}$ is infinite, otherwise it is obvious that $f \in T_p(\mu_\rho)$). For each $n \in \mathbf{N}$ choose ρ_n increasing such that $\rho \geqslant \rho_n$ implies $\hat{\mu}_\rho(\gamma_n) = 1$. We may assume that (μ_{ρ_n}) is a subnet of (μ_ρ), since if there exists ρ_0 such that $\rho_n \leqslant \rho_0$ for all $n \in \mathbf{N}$ then we immediately deduce that $f \in T_p(\mu_\rho)$. Then choose a sequence (α_n) of positive real numbers such that $\sum_{n=1}^\infty \alpha_n$ is convergent and $\sum_{n=k+1}^\infty \alpha_n \leqslant \phi_{\rho_k}^2$ for each $k \in \mathbf{N}$. Let $g = \sum_{n=1}^\infty \alpha_n \gamma_n$; clearly $g \in L^p(G)$. Then

$$\|\mu_{\rho_k} * g - g\|_p = \left\| \sum_{n=k+1}^{\infty} \alpha_n(\mu_{\rho_k} * \gamma_n - \gamma_n)\right\|_p \leqslant \phi_{\rho_k}^2\Big(\sup_\rho \|\mu_\rho\| + 1\Big).$$

Hence

$$\liminf_\rho \phi_\rho^{-1}\|\mu_\rho * g - g\|_p \leqslant \liminf_k \phi_{\rho_k}^{-1}\|\mu_{\rho_k} * g - g\|_p \leqslant \liminf_k \phi_{\rho_k}\Big(\sup_\rho \|\mu_\rho\| + 1\Big) = 0.$$

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Since (μ_ρ) is saturated, we deduce that $g \in T_p(\mu_\rho)$. That is, there exists ρ_0 such that $\mu_\rho * g = g$ for all $\rho \geqslant \rho_0$. Then $\hat{\mu}_\rho \hat{g} = \hat{g}$ for all $\rho \geqslant \rho_0$ and so $\hat{\mu}_\rho \hat{f} = \hat{f}$ for all $\rho \geqslant \rho_0$ (since $\operatorname{supp} \hat{f} = \operatorname{supp} \hat{g} = \{\gamma_n\}_{n=1}^{\infty}$); thus $f \in T_p(\mu_\rho)$.

If (μ_ρ) is saturated in $L^1(G)$ with saturation order (ϕ_ρ), then $f \in S_1(\mu_\rho)$ implies $\operatorname{supp} \hat{f} \subset \Gamma_S(\mu_\rho)^-$. For if $f \in S_1(\mu_\rho)$ and $\gamma \in \Gamma$ then $|\hat{\mu}_\rho(\gamma)\hat{f}(\gamma) - \hat{f}(\gamma)| \leqslant \|\mu_\rho * f - f\|_1 = O(\phi_\rho)$ and so $\hat{f}(\gamma) \neq 0$ implies $|\hat{\mu}_\rho(\gamma) - 1| = O(\phi_\rho)$; that is, $\gamma \in \Gamma_S(\mu_\rho)$.

If $g \in T_1(\mu_\rho)$ then there exists ρ_0 such that $\mu_\rho * g = g$ for all $\rho \geqslant \rho_0$ and so

$$\{\gamma \in \Gamma: \hat{g}(\gamma) \neq 0\} \subset \Gamma \setminus \bigcup_{\rho \geqslant \rho_0} \{\gamma \in \Gamma: \hat{\mu}_\rho(\gamma) \neq 1\},$$

which is a closed set contained in $\Gamma_T(\mu_\rho)$. Hence $\operatorname{supp} \hat{g} \subset \Gamma_T(\mu_\rho)$ (regardless of whether (μ_ρ) is saturated or not).

Finally note that even when G is a compact abelian group we do not in general have $S_p(\mu_\rho) = \{f \in L^p(G): \operatorname{supp} \hat{f} \subset \Gamma_S(\mu_\rho)\}$, as is illustrated by Example A below.

THEOREM 3. *Let G be a compact abelian group and suppose that (μ_ρ) is totally ordered and saturated in $L^p(G)$ with saturation order (ϕ_ρ). Then $\Gamma_S(\mu_\rho) \setminus \Gamma_T(\mu_\rho) \neq \varnothing$ and for each $\gamma \in \Gamma_S(\mu_\rho) \setminus \Gamma_T(\mu_\rho)$ there exists ρ_0 and positive constants c_1 and c_2 such that $c_1 \leqslant \phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - 1| \leqslant c_2$ for all $\rho \geqslant \rho_0$. Conversely suppose $T_p(\mu_\rho) = \{f \in L^p(G): \operatorname{supp} \hat{f} \subset \Gamma_T(\mu_\rho)\}$, (ϕ_ρ) is a net of positive numbers converging to zero and the following conditions are satisfied*:

(i) *for each $\gamma \in \Gamma$, $|\hat{\mu}_\rho(\gamma) - 1| = o(\phi_\rho)$ implies $\gamma \in \Gamma_T(\mu_\rho)$*;

(ii) *there exists $\gamma \in \Gamma \setminus \Gamma_T(\mu_\rho)$ with $|\hat{\mu}_\rho(\gamma) - 1| = O(\phi_\rho)$.*

Then (μ_ρ) is saturated in $L^p(G)$ with order (ϕ_ρ).

PROOF. Suppose (μ_ρ) is saturated in $L^p(G)$ with saturation order (ϕ_ρ). Choose $g \in S_p(\mu_\rho) \setminus T_p(\mu_\rho)$. For $\gamma \in \Gamma$,

$$|\hat{\mu}_\rho(\gamma)\hat{g}(\gamma) - \hat{g}(\gamma)| \leqslant \|\mu_\rho * g - g\|_1 \leqslant \|\mu_\rho * g - g\|_p = O(\phi_\rho),$$

so that $\operatorname{supp} \hat{g} \subset \Gamma_S(\mu_\rho)$ (Γ is discrete). In view of Theorem 2, $\operatorname{supp} \hat{g} \not\subset \Gamma_T(\mu_\rho)$, so there exists $\gamma \in \Gamma_S(\mu_\rho) \setminus \Gamma_T(\mu_\rho)$. Now $\Gamma \subset L^p(G)$ and, by Theorem 2, $\gamma \notin \Gamma_T(\mu_\rho)$ implies $\gamma \notin T_p(\mu_\rho)$; which says that $\liminf_\rho \phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - 1| > 0$. Hence there exist ρ_1 and $c_1 > 0$ such that $\phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - 1| \geqslant c_1$ for all $\rho \geqslant \rho_1$. Similarly, $\gamma \in \Gamma_S(\mu_\rho)$ implies that $|\hat{\mu}_\rho(\gamma) - 1| = O(\phi_\rho)$; that is, there exist ρ_2 and $c_2 > 0$ such that $\phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - 1| \leqslant c_2$ for all $\rho \geqslant \rho_2$.

To prove the converse part of the theorem we note that if (ϕ_ρ) is a net of positive numbers converging to zero such that, for each $\gamma \in \Gamma$, $|\hat{\mu}_\rho(\gamma) - 1| = o(\phi_\rho)$ implies $\gamma \in \Gamma_T(\mu_\rho)$ then, for any $g \in L^p(G)$ such that $\|\mu_\rho * g - g\|_p = o(\phi_\rho)$, it

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is the case that $\operatorname{supp}\hat{g} \subset \Gamma_T(\mu_\rho)$ (since $|\hat{\mu}_\rho(\gamma)\hat{g}(\gamma) - \hat{g}(\gamma)| \leqslant \|\mu_\rho * g - g\|_p$ for all $\gamma \in \Gamma$) and so $g \in T_p(\mu_\rho)$. Also if $\gamma \in \Gamma \setminus \Gamma_T(\mu_\rho)$ with $|\hat{\mu}_\rho(\gamma) - 1| = O(\phi_\rho)$ then $\gamma \in S_p(\mu_\rho) \setminus T_p(\mu_\rho)$, and this finishes the proof.

Under the conditions of Theorem 3 we have the saturation order of (μ_ρ) given by $|\hat{\mu}_\rho(\gamma) - 1|$. This result should be compared with DeVore [4], Theorem 3.1.

2. Description of some saturation classes

Let (μ_t) be a bounded approximate unit on G, where the index set is $(0, \infty)$ with ordering $t \preccurlyeq t'$ if and only if $t \geqslant t'$. We say that (μ_t) is of saturation type (ϕ, ψ) on $L^p(G)$ if the following are satisfied:

(i) The mapping $(t, \gamma) \to \hat{\mu}_t(\gamma)$, from $(0, t_0] \times \Gamma$ into $\mathbf{C}$, is continuous for some $t_0 \in (0, \infty)$.

(ii) There exists a continuous mapping $\phi: (0, \infty) \to (0, \infty)$ such that $\lim_{t \to 0^+} \phi(t) = 0$, and a continuous mapping $\psi: \Gamma \to \mathbf{C}$ that does not vanish in $\Gamma \setminus \{1\}$ satisfying $\lim_{t \to 0^+} \phi(t)^{-1}(\hat{\mu}_t(\gamma) - 1) = \psi(\gamma)$ for all $\gamma \in \Gamma$.

(iii) There is a bounded family $(\omega_t)_{t>0} \subset M_b(G)$ such that $\phi(t)^{-1}(\hat{\mu}_t - 1) = \psi\hat{\omega}_t$ for all $t \in (0, \infty)$.

(iv) $f \in L^p(G)$ and $\|\mu_t * f - f\|_p = o(\phi_\rho)$ imply

$$f = \begin{cases} \text{constant} & \text{if } G \text{ is compact,} \\ 0 & \text{if } G \text{ is non-compact.} \end{cases}$$

If $(\mu_t)_{t>0}$ is of saturation type (ϕ, ψ) on $L^p(G)$ then its saturation class (or Favard space) is the set

$$S_p(\mu_t) = \left\{ f \in L^p(G) \colon \|\mu_t * f - f\|_p = O(\phi(t)) \right\}.$$

Dreseler and Schempp [5] (see also Buchwalter [2]) have shown that

$$S_1(\mu_t) = \left\{ f \in L^1(G) \colon \psi\hat{f} \in M_b(G)\hat{\ } \right\}$$

and, for $p \in (1, 2]$,

$$S_p(\mu_t) = \left\{ f \in L^p(G) \colon \psi\hat{f} \in L^p(G)\hat{\ } \right\};$$

in the above if E is a set of functions or measures then $E\hat{\ }$ denotes the set of Fourier transforms of members of E.

In practice it is difficult to verify condition (iii) above, since it involves deciding whether a given net of functions on Γ is a net of Fourier transforms. It has been pointed out by the referee that the results of Dreseler and Schempp continue to hold with this condition replaced by

(iii)′ There is a bounded family $(\omega_t)_{t>0}$ of multipliers on $L^p(G)$ such that $\phi(t)^{-1}(\hat{\mu}_t - 1) = \psi\hat{\omega}_t$ for all $t \in (0, \infty)$;

(see Dreseler and Schempp [6], Section 3). In the case $p = 2$, condition (iii)′ just says that the family (ω_t) is a bounded set of functions in $L^\infty(G)$. Also Dreseler and Schempp implicitly assume that $\Gamma_S(\mu_t) = \Gamma$ and $\Gamma_T(\mu_t) = \{1\}$ or $\varnothing$.

In this section we consider the saturation problem for $p = 2$ without the restriction that the net (μ_ρ) be defined on $(0, \infty)$. We require two preliminary results.

THEOREM 4. *Let (μ_ρ) be saturated in $L^2(G)$ with order (ϕ_ρ). Suppose that the net $(\phi_\rho^{-1}(\hat{\mu}_\rho - 1))$ is equicontinuous and that $\phi_\rho^{-1}(\hat{\mu}_\rho - 1) \to \psi$ pointwise on $\Gamma_S(\mu_\rho)$. Then ψ is continuous on $\Gamma_S(\mu_\rho)$, the convergence is uniform on compact subsets of $\Gamma_S(\mu_\rho)$, and $\Gamma_S(\mu_\rho)$ is both open and closed in Γ.*

PROOF. Let $\gamma \in \Gamma_S(\mu_\rho)$ and choose ρ_0 such that $\phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - 1| \leqslant K$ for all $\rho \geqslant \rho_0$, where K is a constant. Using the equicontinuity of $(\phi_\rho^{-1}(\hat{\mu}_\rho - 1))$, choose an open neighbourhood Ω of γ such that $\phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - \hat{\mu}_\rho(\chi)| \leqslant 1$ for all ρ and for all $\chi \in \Omega$. Then $\phi_\rho^{-1}|\hat{\mu}_\rho(\chi) - 1| \leqslant K + 1$ for all $\rho \geqslant \rho_0$ and $\chi \in \Omega$, so that $\Omega \subset \Gamma_S(\mu_\rho)$. This shows that $\Gamma_S(\mu_\rho)$ is open.

Similarly let $\gamma \in \Gamma \setminus \Gamma_S(\mu_\rho)$, so that given $n \in \mathbf{N}$ and ρ there exists $\rho_n \geqslant \rho$ with $\phi_{\rho_n}^{-1}|\hat{\mu}_{\rho_n}(\gamma) - 1| \geqslant n$. With Ω chosen as above we have $\phi_{\rho_n}^{-1}|\hat{\mu}_{\rho_n}(\chi) - 1| \geqslant n - 1$ for all $n \in \mathbf{N}$ and $\chi \in \Omega$. This shows that $\Omega \subset \Gamma \setminus \Gamma_S(\mu_\rho)$, so that $\Gamma_S(\mu_\rho)$ is closed.

The other assertions of the theorem are standard consequences of the assumption of equicontinuity.

THEOREM 5. *Let (μ_ρ) satisfy the conditions of Theorem 4 and let $f \in S_2(\mu_\rho)$. Then $\hat{f} = 0$ almost everywhere on $\Gamma \setminus \Gamma_S(\mu_\rho)$.*

PROOF. Suppose that there exists compact $\Lambda \subset \Gamma \setminus \Gamma_S(\mu_\rho)$ with $\|\xi_\Lambda \hat{f}\|_2 \neq 0$. For each $\gamma \in \Lambda$ choose an open neighbourhood $\Omega_\gamma \subset \Gamma \setminus \Gamma_S(\mu_\rho)$ such that $\phi_\rho^{-1}|\hat{\mu}_\rho(\gamma) - \hat{\mu}_\rho(\chi)| \leqslant 1$ for all ρ and for all $\chi \in \Omega_\gamma$, and then an open cover $\Omega_{\gamma_1}, \Omega_{\gamma_2}, \ldots, \Omega_{\gamma_m}$ of Λ. We see immediately that $\|\xi_{\Omega_{\gamma_i}} \hat{f}\|_2 \neq 0$ for some i. Arguing as in Theorem 4 we have that for any $n \in \mathbf{N}$ and ρ there exists $\rho_n \geqslant \rho$ such that $\phi_{\rho_n}^{-1}|\hat{\mu}_{\rho_n}(\gamma) - 1| \geqslant n - 1$ for all $\gamma \in \Omega_{\gamma_i}$, so that

$$(n-1)\left\|\xi_{\Omega_{\gamma_i}} \hat{f}\right\|_2 \leqslant \phi_{\rho_n}^{-1}\left\|\mu_{\rho_n} * f - f\right\|_2,$$

and hence $f \notin S_2(\mu_\rho)$.

We can now state our main result for this section.

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THEOREM 6. *Let* (μ_ρ) *satisfy the conditions of Theorem* 4 *and write* $\Omega = \{\gamma \in \Gamma_S(\mu_\rho): |\psi(\gamma)| < 1\}$. *Suppose that the net*

$$\left(\sup\left\{\phi_\rho^{-1}\left(\hat{\mu}_\rho(\gamma) - 1\right)\omega(\gamma): \gamma \in \Gamma\right\}\right)$$

is eventually bounded, where ω *is a bounded function satisfying*

$$\omega = \begin{cases} \psi^{-1} & \text{on } \Gamma_S(\mu_\rho) \setminus \Omega, \\ 0 & \text{on } \Gamma \setminus \Gamma_S(\mu_\rho), \end{cases}$$

and $|\omega|$ *is bounded away from zero on* Ω. *Then*

$$S_2(\mu_\rho) = \left(\omega L^2(\Gamma)\right)^{\vee}.$$

If furthermore $\omega = \hat{\mu}$ *for some* $\mu \in M_b(G)$ *then*

$$S_2(\mu_\rho) = \mu * L^2(G).$$

PROOF. Suppose $f \in S_2(\mu_\rho)$ so that, by Theorem 3, $\hat{f} = 0$ almost everywhere on $\Gamma \setminus \Gamma_S(\mu_\rho)$. Let $f_1 \in L^2(G)$ be such that $\hat{f}_1 = \xi_{\Gamma\setminus\Omega}\hat{f}$, and write

$$f_\rho = \phi_\rho^{-1}(\mu_\rho * f_1 - f_1).$$

Then (f_ρ) is eventually bounded in $L^2(G)$, since

$$\|f_\rho\|_2 = \left\|\phi_\rho^{-1}(\mu_\rho * f_1 - f_1)\right\|_2 = \phi_\rho^{-1}\left\|(\hat{\mu}_\rho - 1)\hat{f}_1\right\|_2$$
$$\leqslant \phi_\rho^{-1}\left\|(\hat{\mu}_\rho - 1)\hat{f}\right\|_2 = O(1),$$

and so has a weak*-convergent subnet, $f_{\rho_\alpha} \to g \in L^2(G)$ say. Using Parseval's identity (Hewitt and Ross [7], (31.19)) this gives that for each $h \in L^2(G)$,

$$\text{(1)} \qquad \int_\Gamma (\hat{f}_{\rho_\alpha} - \hat{g})\bar{\hat{h}} \to 0.$$

Now

$$\text{(2)} \qquad \int_\Gamma (\hat{f}_1 - \omega\hat{g})\bar{\hat{h}} = \int_\Gamma (\hat{f}_{\rho_\alpha} - \hat{g})\omega\bar{\hat{h}} + \int_\Gamma \hat{f}_1\left(1 - \phi_{\rho_\alpha}^{-1}(\hat{\mu}_{\rho_\alpha} - 1)\omega\right)\bar{\hat{h}}$$

and, if $\hat{h}$ vanishes off a compact subset of Γ,

$$\int_\Gamma \hat{f}_1\left(1 - \phi_{\rho_\alpha}^{-1}(\hat{\mu}_{\rho_\alpha} - 1)\omega\right)\bar{\hat{h}} \to 0,$$

since $\hat{f}_1 = 0$ almost everywhere on $(\Gamma \setminus \Gamma_S(\mu_\rho)) \cup \Omega$, $(\Gamma_S(\mu_\rho) \setminus \Omega) \cap \operatorname{supp}(\bar{\hat{h}})$ is a compact subset of $\Gamma_S(\mu_\rho)\setminus\Omega$, and $\phi_{\rho_\alpha}^{-1}(\hat{\mu}_{\rho_\alpha} - 1)\omega \to 1$ uniformly on compact subsets of $\Gamma_S(\mu_\rho)\setminus\Omega$. Also, for the same h, the first integral on the right-hand side of (2) converges to zero using (1) since ω is bounded. Hence, for such h,

$$\int_\Gamma (\hat{f}_1 - \omega\hat{g})\bar{\hat{h}} = 0.$$

This implies that $\hat{f}_1 = \omega\hat{g}$ locally almost everywhere, which entails that they agree as elements of $L^2(\Gamma)$ (Hewitt and Ross [7], (12.2)). Then, using the assumption that $|\omega|$ is bounded away from zero on Ω,

$$\hat{f} = \hat{f}_1 + (f - f_1)\hat{} = \omega\big(g + \omega^{-1}(f - f_1)\big)\hat{} \in \omega L^2(\Gamma).$$

To prove the reverse inclusion, consider $f = (\omega g)\check{}$ for some $g \in L^2(\Gamma)$. Then

$$\begin{aligned}\phi_\rho^{-1}\|\mu_\rho * f - f\|_2 &= \big\|\phi_\rho^{-1}(\hat{\mu}_\rho - 1)\omega g\big\|_2\\ &\leqslant \sup_{\gamma\in\Gamma}\big|\phi_\rho^{-1}(\hat{\mu}_\rho(\gamma) - 1)\omega(\gamma)\big|\,\|g\|_2\end{aligned}$$

which, by assumption, gives $f \in S_2(\mu_\rho)$.

In the case that $\omega = \hat{\mu}$ for some $\mu \in M_b(G)$,

$$\big(\mu * L^2(G)\big)\hat{} = \hat{\mu}L^2(G)\hat{} = \omega L^2(\Gamma),$$

and this gives the final statement of the theorem.

3. Examples

We apply the results in the previous section to describe the saturation classes in $L^2(G)$ for some of the standard approximate units on the circle and real line.

A. *Saturation of the Fejér approximate unit on the circle group*

The Fejér kernel (F_n) on the circle group (see Hewitt and Ross [7], (31.7)(j)) is defined by

$$F_n = \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)\gamma_k, \qquad n \in \mathbf{N},$$

where γ_k is the character that takes x to x^k for all $x \in \mathbf{T}$. Our sequence (μ_n) is then given by $d\mu_n = F_n\,dx$. Clearly $\Gamma_T(\mu_n) = \{\gamma_0\}$. The trivial class of (μ_n) in each of the spaces $L^p(\mathbf{T})$, $p \in [1, \infty]$, and in $C(\mathbf{T})$ is the space of constant functions: $\mu_n * f = f$ implies $\hat{\mu}_n\hat{f} = \hat{f}$, which implies $\hat{f}(\gamma_k) = 0$ for $k \neq 0$ and so f is a constant. Thus Theorem 3 gives that (μ_n) is saturated in each of these spaces with order (n^{-1}). $\Gamma_S(\mu_n) = \{\gamma_k\colon k \in \mathbf{Z}\}$, the entire dual of $\mathbf{T}$.

The conditions of Theorem 6 are satisfied with $\Omega = \{\gamma_0\}$ and ω defined by $\omega(\gamma_k) = -|k|^{-1}$ for $k \neq 0$ and $\omega(\gamma_0) = 1$; for we have

$$n\big(\hat{\mu}_n(\gamma_k) - 1\big)\omega(\gamma_k) = \begin{cases}0 & \text{for } k = 0,\\ n/(n+1) & \text{for } 0 < |k| \leqslant n,\\ n|k|^{-1} & \text{for } |k| > n.\end{cases}$$

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Note also that $\omega \in L^2(\mathbf{Z})$ so that $\omega = \hat{\mu}$ for some $\mu \in L^2(\mathbf{T}) \subset M_b(G)$ and $S_2(\mu_n) = \mu * L^2(\mathbf{T})$. As $\mu \in L^2(\mathbf{T})$ it follows that $S_2(\mu_n) \subset C(\mathbf{T})$ (Hewitt and Ross [7], (20.19) (iii)).

With a little more effort we can show that

$$S_p(\mu_n) = \mu * L^p(\mathbf{T}), \quad p \in (1, \infty], \quad \text{and} \quad S_1(\mu_n) = \mu * M_b(\mathbf{T}).$$

We observe that $L^p(\mathbf{T})$ is the dual of $L^{p'}(\mathbf{T})$, where $p \in (1, \infty]$ and $p^{-1} + p'^{-1} = 1$, and $M_b(\mathbf{T})$ is the dual of $C(\mathbf{T})$ (Hewitt and Ross [7], (12.18) and (14.4)). If $f \in S_p(\mu_n)$ then $(n\|\mu_n * f - f\|_p)$ is bounded and so $(n(\mu_n * f - f))$ has a weak*-convergent subnet, $n_\alpha(\mu_{n_\alpha} * f - f) \to h$ (where $h \in L^p(\mathbf{T})$ for $p \in (1, \infty]$ and $h \in M_b(\mathbf{T})$ for $p = 1$). In particular $n_\alpha(\hat{\mu}_{n_\alpha} - 1)\hat{f} \to \hat{h}$ pointwise, so that $\hat{f}(\gamma_k) = \hat{\mu}(\gamma_k)\hat{h}(\gamma_k)$ for $k \neq 0$; that is, $f - \mu * h$ is constant. Hence $S_p(\mu_n) \subset \mu * L^p(\mathbf{T})$ for $p \in (1, \infty]$ and $S_1(\mu_n) \subset \mu * M_b(\mathbf{T})$.

The reverse inclusion is obvious once we show that $(n\|\mu_n * \mu - \mu\|_1)$ is bounded. DeVore [4], pages 9–11, shows that for an even measure $\nu \in M_b(\mathbf{T})$,

$$\|\nu\| \leqslant \left(\sum_{k=0}^{\infty} (k+1)\left|\Delta^2 \hat{\nu}(\gamma_k)\right| \right) + \sup_k |\hat{\nu}(\gamma_k)|$$

where $\Delta^2\hat{\nu}(\gamma_k) = \hat{\nu}(\gamma_k) - 2\hat{\nu}(\gamma_{k+1}) + \hat{\nu}(\gamma_{k+2})$. For $n > 2$,

$$\Delta^2\big(n(\hat{\mu}_n(\gamma_k) - 1)\hat{\mu}(\gamma_k)\big) = \begin{cases} \dfrac{-n}{n+1}, & k = 0, \\ 0, & 0 < k \leqslant n-1, \\ \dfrac{-n}{(n+1)(n+2)}, & k = n, \\ \dfrac{2n}{k(k+1)(k+2)}, & k > n. \end{cases}$$

Hence

$$n\|\mu_n * \mu - \mu\|_1 \leqslant 3 + 2n \sum_{k=n+1}^{\infty} \frac{1}{k(k+2)} \leqslant 5.$$

Thus $S_p(\mu_n) = \mu * L^p(\mathbf{T})$, $p \in (1, \infty]$, and $S_1(\mu_n) = \mu * M_b(\mathbf{T})$. Furthermore, since $\mu \in L^1(\mathbf{T})$, $S_\infty(\mu_n) = \mu * L^\infty(\mathbf{T}) \subset C(\mathbf{T})$ (the space of continuous functions on $\mathbf{T}$) and so $S_{C(\mathbf{T})}(\mu_n) = \mu * L^\infty(\mathbf{T})$. One can compare this to the description of the Fejér saturation class given by DeVore [4], Theorem 3.4, which states that

$$S_{C(\mathbf{T})}(\mu_n) = \{ f \in C(\mathbf{T}) : \tilde{f} \in \operatorname{Lip} 1 \},$$

where $\tilde{f}$ denotes the function conjugate to f and Lip 1 the Lipschitz class with exponent 1. It follows from DeVore [4], Theorem 1.9 that these two descriptions of $S_{C(\mathbf{T})}(\mu_n)$ agree.

Finally we note that

$$S_1(\mu_n) = \mu * M_b(\mathbf{T}) \subset L^2(\mathbf{T})$$

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is a proper subspace of $L^1(\mathbf{T})$ (refer to the comment immediately preceding Theorem 3 above).

B. *Saturation on the Cantor group*

Let $\mathbf{D}_2$ be the Cantor group, that is, the complete countable direct product $\Pi_{\mathbf{N}}\mathbf{Z}(2)$, where $\mathbf{Z}(2)$ is the cyclic group of order two (with its discrete topology). The dual $\hat{\mathbf{D}}_2$ is topologically isomorphic to $\mathbf{D}_2^* = \Pi_{\mathbf{N}}^*\mathbf{Z}(2)$. For $n \in \mathbf{N}$ let

$$G_n = \{(x_i) \in \mathbf{D}_2 : x_i = 0 \text{ for } i = 1, 2, \ldots, n\}$$

and put

$$k_n = 2^n \xi_{G_n} = \sum \{\gamma : \gamma \in A(\mathbf{D}_2^*, G_n)\}$$

($A(\mathbf{D}_2^*, G_n)$ denotes the annihilator of G_n in $\mathbf{D}_2^*$; see Hewitt and Ross [7], (23.23).). Let (μ_n) be the bounded sequence of measures on $\mathbf{D}_2$ defined by $d\mu_n = k_n\,dx$. Since $\hat{\mu}_n(\gamma) = 1$ for each $\gamma \in A(\mathbf{D}_2^*, G_n)$, it is obvious that $\Gamma_T(\mu_n) = \mathbf{D}_2^*$.

For each $i \in \mathbf{N}$ let γ_i be the continuous character of $\mathbf{D}_2$ given by $\gamma_i(x) = (-1)^{x_i}$ $(x = (x_i) \in \mathbf{D}_2)$ and put $f = \sum_{n=1}^{\infty} 2^{-n}\gamma_n$. Clearly $f \in C(\mathbf{D}_2)$ and, for each $n \in \mathbf{N}$, $\mu_n * f - f = -\sum_{i=n+1}^{\infty} 2^{-i}\gamma_i \neq 0$, so that $f \notin T_p(\mu_n)$ for any $p \in [1, \infty]$. In particular note that $T_p(\mu_n)$ is not closed (since $k_m * f \in T_p(\mu_n)$ for each $m \in \mathbf{N}$ and $\|k_m * f - f\|_p \to 0$) and that (μ_n) is not saturated in $L^p(G)$ for any $p \in [1, \infty]$ (by Theorem 3).

C. *Saturation of the Picard approximate unit in* $L^2(\mathbf{R})$

We take the Haar measure on $\mathbf{R}$ (and on its dual, which is topologically isomorphic to $\mathbf{R}$) to be $(2\pi)^{-1/2}$ times the Lebesgue measure. The Picard kernel (K_n) on $\mathbf{R}$, which arises from the Laplace distribution, is defined by

$$K_n(x) = \left(\frac{\pi}{2}\right)^{1/2} n \exp(-n|x|), \qquad x \in \mathbf{R}.$$

Thus $\hat{K}_n(x) = n^2/(n^2 + x^2)$ for all $x \in \mathbf{R}$; see Berg and Forst [1], 5.2. Our sequence of measures (μ_n) is given by $d\mu_n = K_n\,dx$ so that

$$n^2(\hat{\mu}_n(x) - 1) = \frac{-x^2}{1 + x^2/n^2} \to -x^2 \quad \text{for each } x \in \mathbf{R}.$$

Clearly $T_2(\mu_n) = \{0\}$.

Also (μ_n) is saturated in $L^2(\mathbf{R})$ with order (n^{-2}). Indeed let ψ be the function on $\mathbf{R}$ given by $\psi(x) = -x^2$. If $f \in L^2(\mathbf{R})$ and $(n^2\|\mu_n * f - f\|_2)$ is bounded then we can argue as in the first part of the proof of Theorem 6 to deduce that $\psi\hat{f} \in L^2(\mathbf{R})$. Since $|n^2(\hat{\mu}_n - 1)| \leqslant |\psi|$ for all $n \in \mathbf{N}$, Lebesgue's dominated convergence theorem (Hewitt and Ross [7], (14.23)) and Plancherel's theorem give

$$n^2\|\mu_n * f - f\|_2 = \|n^2(\hat{\mu}_n - 1)\hat{f}\|_2 \to \|\psi\hat{f}\|_2.$$

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Thus $\liminf_n n^2\|\mu_n * f - f\|_2 = 0$ implies $\|\psi\hat{f}\|_2 = 0$; which implies $\hat{f} = 0$, and so $f = 0$. That is, $f \in T_2(\mu_n)$. Also if $g \in L^2(\mathbf{R})$ is such that $\hat{g}$ vanishes outside some compact set then $\psi\hat{g} \in L^2(\mathbf{R})$ and $n^2\|\mu_n * g - g\|_2 = \|n^2(\hat{\mu}_n - 1)\hat{g}\|_2 \leqslant \|\psi\hat{g}\|_2$, so that there are non-trivial functions in $S_2(\mu_n)$. Hence (μ_n) is saturated in $L^2(\mathbf{R})$.

Now $\Gamma_T(\mu_n) = \{0\}$ and $\Gamma_S(\mu_n) = \mathbf{R}$. The conditions of Theorem 6 are satisfied with $\Omega = (-1,1)$ and

$$\omega(x) = \begin{cases} -1 & \text{for } x \in (-1,1), \\ -x^{-2} & \text{elsewhere on } \mathbf{R}. \end{cases}$$

Referring to Butzer and Nessel [3], Proposition 6.3.10, we see that there exists $\mu \in L^1(\mathbf{R})$ such that $\hat{\mu} = \omega$. Thus Theorem 6 gives $S_2(\mu_n) = \mu * L^2(\mathbf{R})$. In particular $S_2(\mu_n) \subset C_0(\mathbf{R})$ (the space of continuous functions on $\mathbf{R}$ vanishing at ∞), since μ is also an element of $L^2(\mathbf{R})$ (Hewitt and Ross [7], (20.19) (iii)).

Alternative characterizations of the saturation class of the Picard approximate identity are given in Butzer and Nessel [3], Proposition 12.4.2, for the space $L^p(\mathbf{R}), p \in [1,2]$.

D. *Saturation of the Fejér approximate unit in* $L^2(\mathbf{R})$

The Fejér kernel (F_ρ) on $\mathbf{R}$ is defined for $\rho > 0$ by

$$F_\rho(x) = \frac{1}{\sqrt{2\pi}}\rho\left(\frac{\sin\frac{1}{2}\rho x}{\frac{1}{2}\rho x}\right)^2, \qquad x \in \mathbf{R},$$

with Fourier transform

$$\hat{F}_\rho(x) = \left(1 - \frac{|x|}{\rho}\right)\xi_{[-\rho,\rho]}(x), \qquad x \in \mathbf{R},$$

(Hewitt and Ross [7], (31.7) (h)). Our net of measures (μ_ρ) is given by $d\mu_\rho = F_\rho\, dx$. Clearly $T_2(\mu_\rho) = \{0\}$. Also it is easily seen that (μ_ρ) is saturated in $L^2(\mathbf{R})$ with saturation order (ρ^{-1}). For if $f \in L^2(\mathbf{R})$ then

$$\begin{aligned}\rho\|\mu_\rho * f - f\|_2 &= \rho\|(\hat{\mu}_\rho - 1)\hat{f}\|_2 \\ &= \left[\frac{1}{\sqrt{2\pi}}\left(\int_{-\rho}^{\rho} |x|^2|\hat{f}(x)|^2\, dx \right.\right. \\ &\qquad \left.\left. + \int_{-\infty}^{-\rho} \rho^2|\hat{f}(x)|^2\, dx + \int_{\rho}^{\infty} \rho^2|\hat{f}(x)|^2\, dx\right)\right]^{1/2}.\end{aligned}$$

Hence $\liminf_\rho \rho\|\mu_\rho * f - f\|_2$ implies $f = 0$; and if $\hat{f}$ vanishes outside a compact set, $f \in S_2(\mu_\rho)$. Also $\Gamma_T(\mu_\rho) = \{0\}$ and $\Gamma_S(\mu_\rho) = \mathbf{R}$. The conditions of Theorem 6 are satisfied with $\psi(x) = -|x|$, $\Omega = (-1,1)$ and

$$\omega(x) = \begin{cases} -1 & \text{for } x \in (-1,1), \\ -|x|^{-1} & \text{elsewhere on } \mathbf{R}. \end{cases}$$

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Referring to Butzer and Nessel [3], Proposition 6.3.10, we see that there exists $\mu \in L^1(\mathbf{R})$ such that $\hat{\mu} = \omega$. Thus, by Theorem 6, we deduce that

$$S_2(\mu_\rho) = \mu * L^2(\mathbf{R})$$

(and hence $S_2(\mu_\rho) \subset C_0(\mathbf{R})$). Butzer and Nessel [3], 12.4.1, give alternative characterizations for the saturation class in $L^p(\mathbf{R})$, $p \in [1, 2]$.

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