CLOSED MAPS AND PARACOMPACT SPACES

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Let f be a map from a topological space X into a topological space Y. We say that f is *proper* in case f is closed continuous and $f^{-1}(y)$ is compact for all $y \in Y$. Proper maps have been extensively studied, see for example **(3**, Chapter I, §10**)** or **(6)**. (The definition of a proper map given above is different from, but equivalent to, that given by Bourbaki in **(3)**. In **(6)** only surjective proper maps are considered and these maps are called *fitting maps*.) It is known that if f is a proper map, then X is compact if and only if $f(X)$ is compact, and X is paracompact if and only if $f(X)$ is paracompact. In this paper we introduce a new kind of map strictly weaker than a proper map, with the property that it preserves paracompactness. We do this using the concept of P-embedding that we defined and studied in **(9)**.

The notation and terminology of this note will follow that of **(5)**. We say that X is *paracompact* if X is regular and if every open cover of X has a locally finite open refinement. In the same spirit, we do not require a regular space or a normal space to be T_1. However, a completely regular space is necessarily Hausdorff.

Let X and Y be topological spaces, let $S \subset X$, and let $f: X \to Y$ be a map. We say that S is *P-embedded* in X in case every continuous pseudometric on S can be extended to a continuous pseudometric on X. We say that f is *paraproper* in case f is closed continuous and $f^{-1}(y)$ is paracompact and P-embedded in X for every $y \in Y$. Our main result is the following.

THEOREM 1. *Suppose that X is a regular topological space, that Y is a topological space, and that $f: X \to Y$ is a paraproper map. Then X is paracompact if and only if $f(X)$ is paracompact.*

By Examples 1 through 3 we shall show that if any of the conditions in the definition of a paraproper map is eliminated, then Theorem 1 does not remain valid.

Clearly a paraproper map need not be a proper map. For, if X is a paracompact, non-compact space and if f is a map from X onto a one-point space Y, then f is paraproper but not proper. However, Proposition 1 shows that the converse is valid for completely regular spaces:

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PROPOSITION 1. *If X is completely regular, if Y is a topological space, and if $f: X \to Y$ is a proper map, then f is paraproper.*

Proof. Let $y \in Y$. In (**9**, Theorem 3.10) we saw that if $f^{-1}(y)$ is compact, then $f^{-1}(y)$ is P-embedded in X. The result now follows.

The following proposition shows an analogy between paraproper maps and proper maps:

PROPOSITION 2. *Suppose that X is a topological space, that Y is a T_1-space, and that $f: X \to Y$ is a closed continuous map. If X is paracompact, then f is a paraproper map.*

Proof. For each $y \in Y$, $f^{-1}(y)$ is closed in X and is therefore paracompact. Moreover, since X is collectionwise normal (**2**, Theorem 12), $f^{-1}(y)$ is P-embedded in X (**9**, Theorem 5.2), and it follows that f is paraproper.

We now state and prove a result more general than Theorem 1, which will then follow as a corollary. But first we need a lemma concerning P-embedding. Note that the proof of Theorem 2 requires notable modifications of the proof of (**6**, Theorem 2.2).

LEMMA 1. *Suppose that X is a topological space, that $S \subset X$, and that S is completely separated from every zero-set disjoint from it. Then the following statements are equivalent:*

(1) *S is P-embedded in X.*

(2) *Every locally finite cozero-set cover of S has a refinement that can be extended to a locally finite family of cozero-sets of X.*

(*In* (2), *the cover of S is understood to be locally finite in S, and to consist of cozero-sets of S.*)

Proof. Clearly (1) *implies* (2) since a locally finite cozero-set cover must be normal. Conversely, let $\mathscr{U}$ be a locally finite cozero-set cover of S. By (2), $\mathscr{U}$ has a refinement that can be extended to a locally finite family $\mathscr{V} = (V_\beta)_{\beta \in J}$ of cozero-sets of X. Let $V = \bigcup_{\beta \in J} V_\beta$ and note that V is a cozero-set of X and hence $X - V$ is a zero-set of X disjoint from S. Therefore, by hypothesis, there exists a cozero-set G such that $G \cap S = \emptyset$ and $X - V \subset G$. Choose $\lambda \in J$ arbitrary and let $\mathscr{W} = (W_\beta)_{\beta \in J}$ be defined as follows: set $W_\lambda = V_\lambda \cup G$; and if $\beta \neq \lambda$, let $W_\beta = V_\beta$. Then $\mathscr{W}$ is a locally finite cozero-set cover of X such that $\mathscr{W} \mid S$ refines $\mathscr{U}$. Therefore by (**9**, Theorems 2.1 and 2.8), S is P-embedded in X.

THEOREM 2. *Suppose that X is a regular space and that $f: X \to Y$ is a paraproper map. If L is a paracompact P-embedded subset of Y and if $S = f^{-1}(L)$ is C-embedded in X, then S is paracompact and P-embedded in X.*

Proof. We shall first show that S is paracompact. Let $\mathscr{U} = (U_\gamma)_{\gamma \in T}$ be an open cover of S. Then for each $\gamma \in T$ there exists an open set U'_γ in X such that $U'_\gamma \cap S = U_\gamma$. For each $y \in L$, let $\mathscr{U}_y = (U_\gamma \cap f^{-1}(y))_{\gamma \in T}$.

Consider any $y \in L$. Since $f^{-1}(y) \subset S$, $\mathscr{U}_y$ is an open cover of $f^{-1}(y)$; and since $f^{-1}(y)$ is paracompact and P-embedded in X, there exist, by **(9**, Theorems 2.1 and 2.8**)**, a locally finite cozero-set cover $(A'_y(\beta))_{\beta \in J_y}$ of X and a map $\sigma_y: J_y \to T$ such that $A'_y(\beta) \cap f^{-1}(y) \subset U_{\sigma_y(\beta)}$ for each $\beta \in J_y$. For each $\beta \in J_y$, let

$$A_y(\beta) = A'_y(\beta) \cap U'_{\sigma_y(\beta)}$$

and let

$$A_y = \bigcup_{\beta \in J_y} A_y(\beta).$$

Then A_y is a neighbourhood of $f^{-1}(y)$ and therefore, since f is a closed map, there exists an open neighbourhood V_y of y in Y such that $f^{-1}(V_y) \subset A_y$. Note that the family $(V_y \cap L)_{y \in L}$ is an open cover of L. Since L is paracompact and P-embedded in Y, there exist, by **(9**, Theorems 2.1 and 2.8**)**, a locally finite cozero-set cover $(W_\alpha)_{\alpha \in I}$ of Y and a map $\pi: I \to L$ such that $W_\alpha \cap L \subset V_{\pi(\alpha)}$ for each $\alpha \in I$. Now let

$$M = \{(\alpha, \beta): \alpha \in I \text{ and } \beta \in J_{\pi(\alpha)}\}$$

and, for each $(\alpha, \beta) \in M$, let

$$B_{\alpha\beta} = f^{-1}(W_\alpha) \cap A_{\pi(\alpha)}(\beta).$$

We assert that $\mathscr{B} = (B_{\alpha\beta})_{(\alpha,\beta) \in M}$ is locally finite in X. To see this, let $x \in X$. Then there exist a neighbourhood G of $f(x)$ and a finite subset K of I such that $G \cap W_\alpha = \emptyset$ if $\alpha \notin K$. Moreover, if $\alpha \in K$, then, since the family $(A_{\pi(\alpha)}(\beta))_{\beta \in J_{\pi(\alpha)}}$ is clearly locally finite in X, there exist a neighbourhood G_α of x and a finite subset K_α of $J_{\pi(\alpha)}$ such that $G_\alpha \cap A_{\pi(\alpha)}(\beta) = \emptyset$ if $\beta \notin K_\alpha$. Then $H = f^{-1}(G) \cap (\bigcap_{\alpha \in K} G_\alpha)$ is a neighbourhood of x and $N = \{(\alpha, \beta): \alpha \in K \text{ and } \beta \in K_\alpha\}$ is a finite subset of M. Suppose that $(\alpha, \beta) \in M$ and $H \cap B_{\alpha\beta} \neq \emptyset$. Then $f^{-1}(G) \cap f^{-1}(W_\alpha) \neq \emptyset$ and therefore $\alpha \in K$. But then $G_\alpha \cap A_{\pi(\alpha)}(\beta) \neq \emptyset$, so $\beta \in K_\alpha$. Thus $(\alpha, \beta) \in N$ and we conclude that $\mathscr{B}$ is locally finite in X.

Now suppose that $x \in S$. Then $x \in f^{-1}(W_\alpha \cap L) \subset f^{-1}(V_{\pi(\alpha)}) \subset A_{\pi(\alpha)}$ for some $\alpha \in I$, and hence $x \in A_{\pi(\alpha)}(\beta)$ for some $\beta \in J_{\pi(\alpha)}$. Thus $(\alpha, \beta) \in M$ and $x \in B_{\alpha\beta}$. On the other hand, for each $(\alpha, \beta) \in M$ we have

$$B_{\alpha\beta} \cap S \subset U'_{\sigma_{\pi(\alpha)}(\beta)} \cap S = U_{\sigma_{\pi(\alpha)}(\beta)},$$

and we conclude that $(B_{\alpha\beta} \cap S)_{(\alpha,\beta) \in M}$ is a locally finite open refinement of $\mathscr{U}$. Since S is regular, it follows that S is paracompact.

To see that S is P-embedded in X, suppose now that $(U_\gamma)_{\gamma \in T}$ is a locally finite cozero-set cover of S. Since S is C-embedded in X, the sets U'_γ above can be taken to be cozero-sets in X. Then the preceding argument shows that $\mathscr{B}$ is a locally finite family of cozero-sets of X such that $(B_{\alpha\beta} \cap S)_{(\alpha,\beta) \in M}$

refines $\mathscr{U}$. By Lemma 1, it follows that S is P-embedded in X. The proof is now complete.

Proof of Theorem 1. If X is paracompact, then $f(X)$ is paracompact since f is closed (**8**, Corollary 1). The converse follows from the fact that the map $X \to f(X)$ induced by f is paraproper.

As an immediate result of Theorem 1 and Proposition 1 we have the known result for proper maps:

COROLLARY (Henriksen-Isbell **5**). *Suppose that X is completely regular, that Y is a topological space, and that $f: X \to Y$ is a proper map. Then X is paracompact if and only if $f(X)$ is paracompact.*

The following three examples show that Theorem 1 does not remain valid if from the definition of a paraproper map we eliminate either "$f^{-1}(y)$ is paracompact for each $y \in Y$," "$f^{-1}(y)$ is P-embedded in X for each $y \in Y$," or "f is closed." We are indebted to Professor E. Michael for suggesting Example 2 below.

EXAMPLE 1. *A closed continuous map f from a regular space X into a space Y such that $f^{-1}(y)$ is P-embedded in X for each $y \in Y$ and such that $f(X)$ is paracompact but X is not paracompact.*

Let X be a regular topological space that is not paracompact and let f be a map from X onto a one-point space Y.

EXAMPLE 2. *A closed continuous map $f: X \to Y$ such that $f^{-1}(y)$ is paracompact for each $y \in Y$ and such that $f(X)$ is paracompact but X is a completely regular space that is not paracompact.*

Let Γ be the Niemytzki space as defined for example in (**5**, 3K); see also (**1**, § 1.6.2°). Thus Γ denotes the subset $\{(x, y): y \geqslant 0\}$ of $\mathbf{R} \times \mathbf{R}$ provided with the following enlargement of the product topology: for each $\epsilon > 0$, the set

$$V_\epsilon(x, 0) = \{(x, 0)\} \cup \{(u, v) \in \Gamma: (u - x)^2 + (v - \epsilon)^2 < \epsilon^2\}$$

is also a neighbourhood of the point $(x, 0)$. For each $n \in \mathbf{N}$, set

$$A_n = \{(m/n, 1/n): m + 1 \in \mathbf{N}\},$$

and let $D = \{(x, 0): x \in \mathbf{R}\}$. Set $X = (\bigcup_{n \in \mathbf{N}} A_n) \cup D$ and let X have the relative topology of Γ. Then X is completely regular since Γ is completely regular. Note that X is separable but not normal.

Let Y be the quotient space obtained from X by identifying the points of D and let $f: X \to Y$ be the canonical map. Then f is a closed continuous map such that $f^{-1}(y)$ is paracompact for each $y \in Y$. Moreover, since X is not normal, X is not paracompact. It remains to show that Y is paracompact. From (**7**) we know that a regular topological space is paracompact if

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and only if every open cover has an open σ-locally finite refinement. Since Y is a countable topological space, every open cover has a countable subcover (i.e., Y is Lindelöf). It is therefore sufficient to show that Y is regular. Let $p \in Y$ and let U be a neighbourhood of p. If $p \neq D$, then $\{p\}$ is a closed neighbourhood of p such that $\{p\} \subset U$. If $p = D$, then one easily verifies that U is a closed neighbourhood of p.

EXAMPLE 3. *A continuous map f from a regular space X into a space Y such that $f^{-1}(y)$ is paracompact and P-embedded in X for each $y \in Y$ and such that $f(X)$ is paracompact but X is not paracompact.*

Let X be the Niemytzki space Γ (see Example 2). Let

$$Y = \{(x, y) \in \mathbf{R} \times \mathbf{R}: y \geqslant 0\},$$

where Y has the relative topology inherited from the usual topology of $\mathbf{R} \times \mathbf{R}$ and let $f: X \to Y$ be the identity map. Then f is continuous and $f^{-1}(y)$ is paracompact and P-embedded in X for each $y \in Y$. Moreover, $f(X) = Y$ is paracompact; but X is not paracompact since X is not normal.

Let us note that if X is completely regular, then the requirement that f be closed in the definition of a paraproper map may be weakened to the requirement that f be Z-closed. (A map f from a topological space X to a topological space Y is *Z-closed* in case $f(Z)$ is closed in Y for each zero-set Z of X.) This remark is an immediate consequence of the following lemma.

LEMMA 2. *Suppose that X is completely regular, that Y is a topological space, and that $f: X \to Y$ is a Z-closed continuous map such that $f^{-1}(y)$ is paracompact and P-embedded in X for each $y \in Y$. Then f is a closed map.*

Proof. To show that f is closed, it is sufficient to show that for each $y \in Y$ and each neighbourhood U of $f^{-1}(y)$ in X, there exists a neighbourhood V of y in Y such that $f^{-1}(V) \subset U$. Thus, let $y \in Y$ and suppose that U is a neighbourhood of $f^{-1}(y)$ in X. For each $x \in f^{-1}(y)$, let V_x be a cozero-set of X such that $x \in V_x \subset U$. Then $\mathscr{V} = (V_x \cap f^{-1}(y))_{x \in f^{-1}(y)}$ is an open cover of the paracompact P-embedded set $f^{-1}(y)$. Therefore, by (**9**, Theorems 2.1 and 2.8), there exists a locally finite cozero-set cover $\mathscr{A} = (A_\alpha)_{\alpha \in I}$ of X such that $(A_\alpha \cap f^{-1}(y))_{\alpha \in I}$ refines $\mathscr{V}$. Hence there exists a map $\pi: I \to f^{-1}(y)$ such that $A_\alpha \cap f^{-1}(y) \subset V_{\pi(\alpha)}$. Let $B_\alpha = A_\alpha \cap V_{\pi(\alpha)}$. Then $\mathscr{B} = (B_\alpha)_{\alpha \in I}$ is a locally finite family of cozero-sets of X and it follows that $W = \bigcup \mathscr{B}$ is a cozero-set of X such that $f^{-1}(y) \subset W \subset U$. Therefore $X - W$ is a zero-set in X and, since f is Z-closed, $f(X - W)$ is closed in Y, whence

$$Y - f(X - W) = V$$

is open in Y. One easily verifies that $y \in V$ and $f^{-1}(V) \subset U$. The proof is now complete.

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Finally, we turn to a study of products of paraproper maps.

THEOREM 3. *Suppose that* X_1, X_2 *and* Y_1, Y_2 *are topological spaces and that* $f_i: X_i \to Y_i$ *is a map for* $i = 1, 2$. *If each* X_i *is non-empty and if the product map*

$$f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$$

is paraproper, then f_i *is paraproper for* $i = 1, 2$.

Proof. Let $f = f_1 \times f_2$. Note that each f_i is continuous since each X_i is non-empty. Suppose that F is a closed subset of X. Then $F \times X_2$ is a closed subset of $X_1 \times X_2$, therefore $f(F \times X_2) = f_1(F) \times f_2(X_2)$ is closed in $Y_1 \times Y_2$ and it follows (since $X_2 \neq \emptyset$) that $f_1(F)$ is closed in Y_1.

Now suppose that $y_1 \in Y_1$, choose $a \in X_2$, let $y_2 = f_2(a)$ and let $A = f_1^{-1}(y_1)$ and $B = f_2^{-1}(y_2)$ Then, since $f^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$, $A \times B$ is paracompact and it follows that A is paracompact. (If $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ is an open cover of A, then $\mathscr{U}' = (U_\alpha \times B)_{\alpha \in I}$ is an open cover of $A \times B$, therefore there exists a locally finite open refinement $(V_\beta)_{\beta \in J}$ of $\mathscr{U}'$. Then

$$\mathscr{V} = (V_\beta \cap (A \times \{a\}))_{\beta \in J}$$

is a locally finite open cover of $A \times \{a\}$ and the projection of the elements of $\mathscr{V}$ onto A is a locally finite open refinement of $\mathscr{U}$.) Finally, $A \times \{a\}$ is P-embedded in $A \times B$ (since a continuous pseudometric d on $A \times \{a\}$ can be extended to a continuous pseudometric d^* on $A \times B$ by $d^*((x_1, x_2), (x'_1, x'_2)) = d((x_1, a), (x'_1, a)))$; hence $A \times \{a\}$ is P-embedded in $X_1 \times X_2$ and so in $X_1 \times \{a\}$. It follows that $f_1^{-1}(y_1)$ is P-embedded in X_1. Therefore f_1 is paraproper. By a similar proof, f_2 can be shown to be paraproper. The proof is now complete.

COROLLARY. *Suppose that* $(X_\alpha)_{\alpha \in I}$ *and* $(Y_\alpha)_{\alpha \in I}$ *are two families of topological spaces and that* $f_\alpha: X_\alpha \to Y_\alpha$ *is a map for each* $\alpha \in I$. *If each* X_α *is non-empty and if the product map*

$$\pi_{\alpha \in I} f_\alpha: \pi_{\alpha \in I} X_\alpha \to \pi_{\alpha \in I} Y_\alpha$$

is paraproper, then f_α *is paraproper for each* $\alpha \in I$.

Since the topological product of a paracompact space and a compact space is paracompact, it is reasonable to conjecture that if $f_1: X_1 \to Y_2$ is a paraproper map and if $f_2: X_1 \to Y_2$ is a proper map, then the map

$$f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$$

is paraproper. However, this is not the case, as is shown below.

Let $X_1 = \mathbf{R}$ and let f_1 be a map from X_1 onto a one-point space Y_1. Let X_2 and Y_2 be the closed interval $[-1, 1]$ and let f_2 be the identity map. Then f_1 is a paraproper map and f_2 is a proper map but $f_1 \times f_2$ is not even a closed map.

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