ON THE ORDER OF AUTOMORPHISM GROUPS OF KLEIN SURFACES

by J. J. ETAYO GORDEJUELA

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1. Introduction. A problem of special interest in the study of automorphism groups of surfaces are the bounds of the orders of the groups as a function of the genus of the surface.

May has proved that a Klein surface with boundary of algebraic genus p has at most $12(p-1)$ automorphisms [**9**].

In this paper we study the highest possible prime order for a group of automorphisms of a Klein surface. This problem was solved for Riemann surfaces by Moore in [**10**]. We shall use his results for studying the Klein surfaces that are not Riemann surfaces. The more general result that we obtain is the following: if X is a Klein surface of algebraic genus p, and G is a group of automorphisms of X, of prime order n, then $n \leqslant p+1$.

2. Preliminaries. A Klein surface X is a surface with or without boundary, with an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ that fulfills the following two conditions:

(i) For each $U_i \in \mathcal{U}$ there exists a homeomorphism ϕ_i from U_i onto an open subset of $\mathbb{C}$.

(ii) If U_i, $U_j \in \mathcal{U}$, $U_i \cap U_j \neq \varnothing$, then $\phi_i\phi_j^{-1}$ is an analytic or anti-analytic application defined in $\phi_j(U_i \cap U_j)$.

An automorphism of the surface is a homeomorphism $f: X \to X$ such that $\phi_i f \phi_j^{-1}$ is analytic or anti-analytic.

Orientable Klein surfaces without boundary are Riemann surfaces.

A non-orientable Klein surface X with topological genus g and k boundary components has algebraic genus $p = g+k-1$; if X is orientable with boundary, its algebraic genus is $p = 2g+k-1$.

Klein surfaces and their automorphisms may be studied by means of non-Euclidean crystallographic groups (NEC groups). An NEC group is a discrete subgroup of isometries of the non-Euclidean plane with compact quotient space. NEC groups include reversing orientation isometries, reflections and glide-reflections.

NEC groups are classified according to their signatures. The signature of an NEC group is of the form

$$(*) \qquad (g, \pm, [m_1, \ldots, m_r], \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

The number g is the genus, the m_i are the proper periods, and the brackets $(n_{i1}, \ldots, n_{is_i})$ are the period-cycles.

The group Γ with signature $(*)$ has a presentation given by generators

(i) x_i, $i = 1, \ldots, r$

(ii) e_i, $i = 1, \ldots, k$

(iii) c_{ij}, $i=1,\ldots,k$, $j=0,\ldots,s_i$
(iv) (with sign '+') a_j, b_j, $j=1,\ldots,g$
(with sign '−') d_j, $j=1,\ldots,g$

and relations

(i) $x_i^{m_i}=1$, $i=1,\ldots,r$
(ii) $e_i^{-1}c_{i0}e_ic_{is_i}=1$, $i=1,\ldots,k$
(iii) $c_{i,j-1}^2=c_{ij}^2=(c_{i,j-1}c_{ij})^{n_{ij}}=1$, $i=1,\ldots,k$, $j=1,\ldots,s_i$
(iv) (with sign '+') $x_1\ldots x_re_1\ldots e_ka_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}=1$
(with sign '−') $x_1\ldots x_re_1\ldots e_kd_1^2\ldots d_g^2=1$.

The area of a fundamental region for an NEC group Γ is given by

$$|\Gamma|=2\pi\left(\alpha g+k-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)\right),$$

where $\alpha=1$ if the sign is '−' and $\alpha=2$ if the sign is '+'.

The relation between Klein surfaces and NEC groups comes from the following two results:

THEOREM A. [**12**]. *Let X be a Klein surface of topological genus g, k boundary components, and algebraic genus $\geqslant 2$. Then X may be represented as D/K, where $D=\{z\in\mathbb{C},\ \mathrm{im}\, z>0\}$ and K is an* NEC *group with signature* $(g,\pm,[-],\{(-),\overset{k}{\ldots},(-)\})$ *with sign '+' if X is orientable, and '−' if X is non-orientable.*

THEOREM B. [**8**]. *A finite group G is a group of automorphisms of the Klein surface D/K if and only if $G=\Gamma/K$, where Γ is an* NEC *group from which K is a normal subgroup.*

We shall establish first of all a result about normal subgroups of NEC groups.

LEMMA 1. *Let Γ be an* NEC *group, with signature* $(g,\pm,[m_1,\ldots,m_r],\{(-),\overset{k}{\ldots},(-)\})$, *and let Γ_0 be a normal subgroup of Γ, such that $|\Gamma:\Gamma_0|=N$. If q_i is the least integer such that $e_i^{q_i}\in\Gamma_0$, $i=1,\ldots,k$, and if $c_{10},\ldots,c_{k0}$ belong to Γ_0, then the signature of Γ_0 has $(N/q_1)+\ldots+(N/q_k)$ period-cycles, all of them empty.*

Proof. If Δ is an NEC group with empty period-cycles, then the number of them is equal to the number of conjugacy classes of reflections in Δ, as both equal the number of holes in D/Δ. In our case Γ_0 has empty period-cycles, and we need only find the number of conjugacy classes of reflections in Γ_0. We will show that there are N/q_i conjugacy classes of reflections in Γ_0 which are conjugate to c_{i0}, for $i=1,\ldots,k$.

The centralizer of c_{i0} in Γ is the abelian group A_i generated by e_i and c_{i0} [**13**]. If $\theta:\Gamma\to\Gamma/\Gamma_0$ is the canonical homomorphism, then as $c_{i0}\in\Gamma_0$, $\theta(A_i)=\{1,e_i,\ldots,e_i^{q_i-1}\}$ which has index N/q_i in Γ/Γ_0. To prove the result we need only show that if $g,h\in\Gamma$ then $gc_{i0}g^{-1}$ and $hc_{i0}h^{-1}$ are conjugate in Γ_0 if and only if $\theta(g)$ and $\theta(h)$ lie in the same coset of $\theta(A_i)$ in Γ/Γ_0, i.e., $\theta(h^{-1}g)\in\theta(A_i)$. This result holds for $\theta(h^{-1}g)\in\theta(A_i)$ is equivalent to $h^{-1}g=\lambda_0b$ where $\lambda_0\in\Gamma_0$, $b\in A_i$, and then $gc_{i0}g^{-1}=h\lambda_0bc_{i0}b^{-1}\lambda_0^{-1}h^{-1}=h\lambda_0c_{i0}\lambda_0^{-1}h^{-1}=\gamma_0hc_{i0}b^{-1}\gamma_0^{-1}h^{-1}=h\lambda_0c_{i0}\lambda_0^{-1}h^{-1}=\gamma_0hc_{i0}h^{-1}\gamma_0^{-1}$, where $h\lambda_0h^{-1}=\gamma_0\in\Gamma_0$.

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Thus we have shown that the number of conjugacy classes of reflections in Γ_0 conjugate to c_{i0} is N/q_i and so the number of conjugacy classes of reflections in Γ_0 is $(N/q_1)+\ldots+(N/q_k)$.

3. Prime order groups of automorphisms. Let X be a Klein surface that is not a Riemann surface. Then X may be represented as D/K, where K is an NEC group with signature (i) $(g,-,[—],\{—\})$, (ii) $(g,+,[—],\{(—),\overset{k}{\ldots},(—)\})$ or (iii) $(g,-,[—],\{(—),\overset{k}{\ldots},(—)\})$ if X is (i) without boundary, (ii) orientable with boundary, or (iii) non-orientable with boundary.

If G is a group of automorphisms of X, with prime order $\neq 2$, then $G=\Gamma/K$, where Γ is an NEC group with signature (i) $(\gamma,-,[\mu_1,\ldots,\mu_r],\{—\})$, (ii) $(\gamma,+,[\mu_1,\ldots,\mu_r],\{(—),\overset{k'}{\ldots},(—)\})$ or (iii) $(\gamma,-,[\mu_1,\ldots,\mu_r],\{(—),\overset{k'}{\ldots},(—)\})$ **[3]**.

Let Γ^+ and K^+ be the canonical fuchsian subgroups associated to Γ and K, i.e., the subgroups formed by the elements which preserve orientation, **[13]**. Then, by **[7**, Cor. 1**]**, if $t\in\Gamma/K$ has N fixed points, $t\in\Gamma^+/K^+$ has $2N$ fixed points. We shall denote by $N(t)$ the number of fixed points of t.

We shall indicate now the main result obtained by Moore for Riemann surfaces, that will be used throughout this paper.

LEMMA 2 **[10]**. *Let S be a Riemann surface of genus* g. *Let K be the fuchsian group with signature* $(g,+,[—],\{—\})$, *and G a group of automorphisms of S, with prime order* $n\neq 2$. *Then* $G=\Gamma/K$, *where* Γ *has signature* $(\gamma,+,[\mu_1,\ldots,\mu_r],\{—\})$; *and* $n\geq g$ *only in the next four cases:*

(i) $n=2g+1,\ N(t)=3,\ \gamma=0,$
(ii) $n=g+1,\ N(t)=4,\ \gamma=0,$
(iii) $n=g=3,\ N(t)=5,\ \gamma=0,$ *or*
(iv) $n=g,\ N(t)=2,\ \gamma=1,$

where t is a generator of the group G.

THEOREM 1. *Let X be a Klein surface of algebraic genus* $p\geq 2$. *Let G be a group of automorphisms of X, of prime order n. Then if X is without boundary, or orientable with boundary,* $n\leq p+1$; *and if X is non-orientable with boundary,* $n\leq p$.

Proof. 1. Let X be non-orientable, without boundary, of genus g. Then $X=D/K$, $G=\Gamma/K$, and the signatures of K and Γ are $(g,-,[—],\{—\})$ and $(\gamma,-,[\mu_1,\ldots,\mu_r],\{—\})$. From **[13]**, the subgroups Γ^+ and K^+ have signatures, respectively, $(\gamma-1,+,[\mu_1,\mu_1,\ldots,\mu_r,\mu_r],\{—\})$ and $(g-1,+,[—],\{—\})$. Also $|\Gamma^+:K^+|=n$, and so by Lemma 2, if $n>g-1$, we must have

(i) $n=2(g-1)+1,\ N(t)=3,\ \gamma-1=0,$ or
(ii) $n=(g-1)+1,\ N(t)=4,\ \gamma-1=0.$

If case (i) holds, by **[7**, Corollary 1**]** $t\in\Gamma/K$ would have 3/2 fixed points, impossible. So the highest possible prime n, is $n=(g-1)+1=g=p+1$, and then $N(t)=4/2=2$, $\gamma=1$.

This bound is attained for every $p+1$ prime: The group Γ with signature

$(1, -, [p+1, p+1], \{—\})$ fulfills the conditions of [**4**, Theorem 3.7] and hence there is an epimorphism from Γ onto $Z/p+1$ whose kernel has signature $(g, -, [—], \{—\})$. By the relation of areas [**13**],

$$1+p=\frac{g-2}{1-2+2\left(1-\frac{1}{p+1}\right)}=\frac{g-2}{1-\frac{2}{p+1}}=\frac{(g-2)(p+1)}{p-1};$$

so, $g=p+1$.

2. Let X be orientable, with boundary, of algebraic genus p. Then $X=D/K$, $G=\Gamma/K$, and the signatures of Γ and K are $(\gamma, +, [\mu_1, \ldots, \mu_r], \{(—), \overset{k'}{\ldots}, (—)\})$ and $(g, +, [—], \{(—), \overset{k}{\ldots}, (—)\})$. The canonical fuchsian subgroups Γ^+ and K^+ have signature $(2\gamma+k'-1, +, [\mu_1, \mu_1, \ldots, \mu_r, \mu_r], \{—\})$ and $(2g+k-1, +, [—], \{—\})$, i.e., the signature of K^+ is $(p, +, [—], \{—\})$.

By Lemma 2, if $n>p$, we must have

(i) $n=2p+1$, $N(t)=3$, $2\gamma+k'-1=0$, or

(ii) $n=p+1$, $N(t)=4$, $2\gamma+k'-1=0$.

As before, the case (i) is impossible, and the highest possible prime n is $n=p+1$, and then $N(t)=4/2=2$, $2\gamma+k'-1=0$. As $k'\neq 0$, it is $\gamma=0$, $k'=1$.

3. Let X be non-orientable, with boundary, of algebraic genus p. Then $X=D/K$, $G=\Gamma/K$, and the signatures of Γ and K are $(\gamma, -, [\mu_1, \ldots, \mu_r], \{(—), \overset{k'}{\ldots}, (—)\})$ and $(g, -, [—], \{(—), \overset{k}{\ldots}, (—)\})$. The subgroups Γ^+ and K^+ have signatures $(\gamma+k'-1, +, [\mu_1, \mu_1, \ldots, \mu_r, \mu_r], \{—\})$ and $(p, +, [—], \{—\})$. By Lemma 2, if $n>p$, it must be

(i) $n=2p+1$, $N(t)=3$, $\gamma+k'-1=0$, or

(ii) $n=p+1$, $N(t)=4$, $\gamma+k'-1=0$.

As before, the case (i) is impossible; but as $\gamma\geqslant 1$, $k'\geqslant 1$, $\gamma+k'-1$ is different from 0, and so the case (ii) is also impossible. Hence in no case is $n>p$.

Let us see now when $n=p$. By Lemma 2, we must have

(i) $n=p=3$, $N(t)=5$, $\gamma+k'-1=0$, or

(ii) $n=p$, $N(t)=2$, $\gamma+k'-1=1$.

The case (i) is again impossible, and thus the only possible case is $n=p$, $N(t)=2/2=1$, $\gamma=k'=1$.

We have obtained the bounds of the order as a function of the algebraic genus of the surface. We shall calculate now which are the values of the topological genus and boundary components that attain these bounds.

In the non-orientable surfaces without boundary the algebraic and topological genera are mutually determined. So we have seen that every prime g attains the bound.

Let us now study the surfaces with boundary.

PROPOSITION 1. *Let X be an orientable Klein surface with boundary of algebraic genus p. (1) If $p+1$ is prime, there is a group of automorphisms of X, of order $p+1$, if and only if X has* 1 *or $p+1$ boundary components, and topological genus $p/2$ and* 0, *respectively.* (2) *If p is prime, there is a group of automorphisms of X, or order p, if and only if X has* 2 *or $p+1$*

boundary components, and topological genus $(p-1)/2$ *and* 0, *respectively.* (3) *Otherwise, any automorphisms group of* X *with prime order has order smaller than* p.

Proof. Let n be the order of the group, say G. By Theorem 1, if $n>p$, we have $n=p+1$. Also $X=D/K$, $G=\Gamma/K$, and Γ and K have signatures $(0,+,[\mu_1,\ldots,\mu_r],\{(-)\})$ and $(g,+,[-],\{(-),\overset{k}{\ldots},(-)\})$, with $2g+k-1=p$. Hence by Lemma 1, $k\mid p+1$, and so $k=1$ or $k=p+1$.

If $k=1$, K has signature $(p/2,+,[-],\{(-)\})$. The group Γ with signature $(0,+,[p+1,p+1],\{(-)\})$ fulfills the conditions of [**5**, Theorem 3.5] and hence there is an epimorphism θ from Γ onto $Z/p+1$ whose kernel has signature $(g,+,[-],\{(-)\})$. By the relation of areas,

$$p+1=\frac{2g-1}{-1+2(1-(1/p+1))}=\frac{(2g-1)(p+1)}{p-1};$$

thus $g=p/2$ and so $\ker\theta=K$.

If $k=p+1$, K has signature $(0,+,[-],\{(-),\overset{p+1}{\ldots},(-)\})$. Let Γ be the group with signature $(0,+,[p+1,p+1],\{(-)\})$. The epimorphism θ from Γ onto $Z/p+1$ given by $\theta(x_1)=\bar{1}$, $\theta(x_2)=\bar{p}$, $\theta(e_1)=\theta(c_{10})=\bar{0}$, verifies that its kernel has signature $(g,+,[-],\{(-),\overset{p+1}{\ldots},(-)\})$, and

$$p+1=\frac{2g+p-1}{-1+2(1-(1/p+1))}=\frac{(2g+p-1)(p+1)}{p-1};$$

thus $g=0$, and so $\ker\theta=K$.

Let us see now when $n=p$, prime. By Lemma 2, we have $2\gamma+k'-1=1$, and so $\gamma=0$, $k'=2$. Thus Γ has signature $(0,+,[\mu_1,\ldots,\mu_r],\{(-)(-)\})$. By Lemma 1, $k=k_1+k_2$, where $k_i\mid p$, and hence $k=2$, $k=p+1$, or $k=2p$. As $2g+k-1=p$, $k=2p$ is impossible.

If $k=2$, K has signature $((p-1)/2,+,[-],\{(-)(-)\})$. Let Γ be the group with signature $(0,+,[p],\{(-)(-)\})$. The epimorphism θ form Γ onto Z/p given by $\theta(x_1)=\bar{1}$, $\theta(e_1)=\theta(e_2)=\overline{(p-1)/2}$, $\theta(c_{10})=\theta(c_{20})=\bar{0}$, verifies that its kernel has signature $(g,+,[-],\{(-)(-)\})$, and

$$p=\frac{2g}{1-(1/p)}=\frac{2gp}{p-1};$$

thus $g=(p-1)/2$ and so $\ker\theta=K$.

If $k=p+1$, K has signature $(0,+,[-],\{(-),\overset{p+1}{\ldots},(-)\})$. Let Γ be the group with signature $(0,+,[p],\{(-)(-)\})$. The epimorphism θ from Γ onto Z/p given by $\theta(x_1)=\bar{1}$, $\theta(e_1)=\overline{p-1}$, $\theta(e_2)=\theta(c_{10})=\theta(c_{20})=\bar{0}$, verifies that its kernel has signature $(g,+,[-],\{(-),\overset{p+1}{\ldots},(-)\})$, and

$$p=\frac{2g+p-1}{1-(1/p)}=\frac{(2g+p-1)p}{p-1};$$

thus $g=0$, and so $\ker\theta=K$.

PROPOSITION 2. *Let X be a non-orientable Klein surface with boundary, of algebraic genus p. (1). If p is prime, there is a group of automorphisms of X, of order p, if and only if X has 1 or p boundary components, and topological genus p and 1, respectively. (2) Otherwise, any automorphisms group of X with prime order has order smaller than p.*

Proof. Let n be the order of the group, say G. By Theorem 1, if $n=p$, $X=D/K$, $G=\Gamma/K$, and Γ and K have signatures $(1,-,[\mu_1,\ldots,\mu_r],\{(—)\})$ and $(g,-,[—],\{(—),\overset{k}{\ldots},(—)\})$, with $g+k-1=p$. Hence by Lemma 1, $k \mid p$, and so $k=1$ or $k=p$.

If $k=1$, K has signature $(p,-,[—],\{(—)\})$. The group Γ with signature $(1,-,[p],\{(—)\})$ fulfills the conditions of [**5**, Theorem 3.6] and hence there is an epimorphism θ from Γ onto Z/p whose kernel has signature $(g,-,[—],\{(—)\})$. By the relation of areas,

$$p=\frac{g-1}{1-(1/p)}=\frac{(g-1)p}{p-1};$$

thus, $g=p$, and so $\ker\theta=K$.

If $k=p$, K has signature $(1,-,[—],\{(—),\overset{p}{\ldots},(—)\})$. Let Γ be the group with signature $(1,-,[p],\{(—)\})$. The epimorphism θ from Γ onto Z/p given by $\theta(x_1)=\bar{1}$, $\theta(d_1)=\overline{(p-1)/2}$, $\theta(e_1)=\theta(c_{10})=\bar{0}$, verifies that its kernel has signature $(g,-,[—],\{(—),\overset{p}{\ldots},(—)\})$, and

$$p=\frac{g+p-2}{1-(1/p)}=\frac{(g+p-2)p}{p-1};$$

thus $g=1$, and so $\ker\theta=K$.

4. Real algebraic curves.

These results may be rewritten in terms of real algebraic curves, as follows:

COROLLARY 1. *Let V be an irreducible real algebraic curve of genus $g\geq 2$, and let $V_{\mathbb{C}}$ be its complexification. If $V_{\mathbb{C}}\setminus V$ is not connected, then,*

1. *If $g+1$ is prime, there is a group of automorphisms of V, of order $g+1$, if and only if V is connected or has $g+1$ connected components.*
2. *If g is prime, there is a group of automorphisms of V, of order g, if and only if V has 2 or $g+1$ connected components.*
3. *Otherwise, any automorphisms group of V with prime order has order smaller than g.*

COROLLARY 2. *Let V be an irreducible real algebraic curve of genus $g\geq 2$, and let $V_{\mathbb{C}}$ be its complexification. If $V_{\mathbb{C}}\setminus V$ is connected, then*

1. *If g is prime, there is a group of automorphisms of V, of order g, if and only if V is connected or has g connected components.*
2. *Otherwise, any automorphisms group of V with prime order has order smaller than g.*

Proof of Both Corollaries. By [**1, 2**] there is a functorial equivalence between the category of compact Klein surfaces with boundary, and that of irreducible real algebraic curves. So, each compact Klein surface with k boundary components has associated an irreducible real algebraic curve that admits a bounded smooth model with k connected components, and conversely.

From [**11**], the surface is orientable if and only if the curve disconnects its complexification.

Further, the groups of automorphisms of the curve and of the surface are isomorphic [**6**].

Hence, it suffices to rewrite Theorem 1 and Propositions 1 and 2, in this language.

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Departamento de Geometría y Topología
Facultad de Ciencias Matemáticas
Universidad Complutense
Madrid, Spain