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REPRESENTING REGULAR PSEUDOCOMPLEMENTED KLEENE ALGEBRAS BY TOLERANCE-BASED ROUGH SETS

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This paper is dedicated to Professor T. Katriňák on the occasion of his 80th birthday

Abstract

We show that any regular pseudocomplemented Kleene algebra defined on an algebraic lattice is isomorphic to a rough set Kleene algebra determined by a tolerance induced by an irredundant covering.

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1. Introduction

Kleene algebras were introduced by Brignole and Monteiro in [BM67]. Earlier, Kalman [Kal58] called such distributive lattices with an involution $\sim$ satisfying $x \wedge {\sim}x \leq y \vee {\sim}y$ as 'normal i-lattices'. Kleene algebras can be seen as generalisations of, for instance, Boolean, Łukasiewicz, Nelson and Post algebras; see [BD74]. The notion used here should not to be confused with the other Kleene algebra notion generalising regular expressions.

Clearly, any pseudocomplemented Kleene algebra is defined on a distributive double pseudocomplemented lattice. According to [San86], a pseudocomplemented Kleene algebra $(L, \vee, \wedge, \sim, {}^*, 0, 1)$ is congruence-regular if and only if the distributive double p-algebra $(L, \vee, \wedge, {}^*, {}^+, 0, 1)$ is congruence-regular. Varlet [Var72] has shown that any double p-algebra is congruence-regular if and only it is determination-trivial, that is, $x^* = y^*$ and $x^+ = y^+$ imply that $x = y$. Therefore, a pseudocomplemented Kleene algebra is congruence-regular if and only if it is determination-trivial. In this paper, we study the representation of (congruence-)regular pseudocomplemented Kleene algebras whose underlying lattice is algebraic.

It is well known that any Boolean algebra defined on an algebraic lattice is isomorphic to the powerset algebra $\wp(U)$ of some set U. In this paper, we prove an analogous result for a regular pseudocomplemented Kleene algebra and the algebra of rough sets defined by a tolerance induced by an irredundant covering of a set.

Rough sets were introduced by Pawlak in [Paw82]. In rough set theory it is assumed that our knowledge about a universe of discourse U is given in terms of a binary relation reflecting the distinguishability or indistinguishability of the elements of U. According to Pawlak's original definition, the knowledge is given by an equivalence E on U interpreted so that two elements of U are E-related if they cannot be distinguished by their properties known by us. Nowadays, in the literature, numerous studies can be found in which approximations are determined by other types of relations.

If R is a given binary relation on U, then, for any subset $X \subseteq U$, the *lower approximation* of X is defined as

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}$$

and the *upper approximation of* X is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid x\,R\,y\}$. The set $X^{\blacktriangledown}$ may be interpreted as the set of objects that certainly are in X in view of the knowledge R, because if $x \in X^{\blacktriangledown}$, then all elements to which x is R-related are in X. Similarly, the set $X^{\blacktriangle}$ may be considered as the set of elements that are possibly in X, since $x \in X^{\blacktriangle}$ means that there exists at least one element in X to which x is R-related. Note that the maps $^{\blacktriangledown}$ and $^{\blacktriangle}$ are dual, that is, $X^{\blacktriangledown c} = X^{c\blacktriangle}$ and $X^{\blacktriangle c} = X^{\blacktriangledown c}$ for all $X \subseteq U$, where X^c denotes the complement $U \backslash X$ of X.

The *rough set* of X is the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$, and the set of all rough sets is

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}.$$

The set RS may be canonically ordered by the coordinatewise order

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

The structure of RS is well studied in the case when R is an equivalence; see [Com93, Dün97, GW92, Itu99, Jär07, Pag98, PP88]. In particular, J. Pomykała and J. A. Pomykała showed in [PP88] that RS is a Stone lattice. Later this result was improved by Comer [Com93] by showing that RS is a regular double Stone algebra. In [GW92], Gehrke and Walker proved that RS is isomorphic to $\mathbf{2}^I \times \mathbf{3}^K$, where I is the set of singleton R-classes and K is the set of nonsingleton equivalence classes of R.

If R is a quasiorder (that is, a reflexive and transitive binary relation), then RS is a completely distributive and algebraic lattice [JRV09]. We showed in [JR11] how one can define a Nelson algebra $\mathbb{RS}$ on this algebraic lattice. In addition, we proved that if $\mathbb{L} = (L, \vee, \wedge, \sim, \to, 0, 1)$ is a Nelson algebra defined on an algebraic lattice, then there exists a set U and a quasiorder R on U such that $\mathbb{L}$ is isomorphic to the rough set Nelson algebra $\mathbb{RS}$ determined by R.

Let R be a tolerance on U: that is, R is a reflexive and symmetric binary relation on U. The pair $(^{\blacktriangledown}, ^{\blacktriangle})$ of rough approximation operations forms an order-preserving Galois connection on the powerset lattice $(\wp(U), \subseteq)$ of U: that is, $X^{\blacktriangle} \subseteq Y \Leftrightarrow X \subseteq Y^{\blacktriangledown}$ for any $X, Y \subseteq U$. The essential facts about Galois connections can be found in [EKMS93], for instance. Because R is reflexive, also $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$ for all $X \subseteq U$. Properties of rough approximations defined by tolerances are given in [Jär99, JR14], and they are not recalled here.

It is known that if R is a tolerance, then RS is not necessarily even a lattice [Jär99]. However, we proved in [JR14] that if R is a tolerance induced by an irredundant covering of U, then $(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$ is a Kleene algebra such that RS is algebraic and completely distributive. This means that RS forms a double pseudocomplemented lattice. Our main result shows that if $\mathbb{L} = (L, \vee, \wedge, \sim, ^*, 0, 1)$ is a regular pseudocomplemented Kleene algebra defined on an algebraic lattice, then there exists a set U and a tolerance R induced by an irredundant covering of U such that $\mathbb{L}$ is isomorphic to the rough set pseudocomplemented Kleene algebra $\mathbb{RS} = (RS, \vee, \wedge, \sim, ^*, (\emptyset, \emptyset), (U, U))$ determined by R.

The paper is structured as follows. In the next section, we recall some notions and facts related to De Morgan, Kleene and Heyting algebras. In particular, we are interested in De Morgan and Kleene algebras enriched by pseudocomplementation. In Section 3, we study rough sets defined by tolerances induced by irredundant coverings. In particular, the structure of their completely join-irreducible elements is given. Varlet [Var72] has proved that any distributive double pseudocomplemented lattice is regular if and only if any chain of its prime filters has at most two elements. In Section 4, we first show that if $\mathbb{L} = (L, \vee, \wedge, \sim, ^*, 0, 1)$ is a pseudocomplemented De Morgan algebra defined on an algebraic lattice, then $\mathbb{L}$ is regular if and only if the set of completely join-irreducible elements of L has at most two levels. At the end of the section, we consider irredundant coverings and their tolerances determined by regular pseudocomplemented Kleene algebras defined on algebraic lattices. Section 5 contains our representation theorem and its proof. The construction is also illustrated by an example. Finally, Section 6 contains some concluding remarks.

2. Preliminaries

In this section, we recall some general lattice-theoretical notions and notation which can be found, for instance, in the books [BD74, DP02, Grä98]. For more specific results, a reference will be given.

An element j of a complete lattice L is called *completely join-irreducible* if $j = \bigvee S$ implies that $j \in S$ for every subset S of L. Note that the least element, 0, of L is not completely join-irreducible. The set of completely join-irreducible elements of L is denoted by $\mathcal{J}(L)$, or simply by $\mathcal{J}$ if there is no danger of confusion. A complete lattice L is *spatial* if, for each $x \in L$,

$$x = \bigvee \{j \in \mathcal{J} \mid j \leq x\}.$$

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An element x of a complete lattice L is said to be *compact* if, for every $S \subseteq L$,

$$x \leq \bigvee S \Longrightarrow x \leq \bigvee F \quad \text{for some finite subset } F \text{ of } S.$$

Let us denote by $\mathcal{K}(L)$ the set of compact elements of L. A complete lattice L is said to be *algebraic* if, for each $a \in L$,

$$a = \bigvee \{x \in \mathcal{K}(L) \mid x \leq a\}.$$

Note that if L is an algebraic lattice, then its completely join-irreducible elements are compact. Let the lattice L be both algebraic and spatial. Since any compact element can be written as a finite join and any finite join of compact elements is compact, the compact elements of L are exactly those that can be written as a finite join of completely join-irreducible elements.

A complete lattice L is *completely distributive* if, for any doubly indexed subset $\{x_{i,j}\}_{i \in I,\, j \in J}$ of L,

$$\bigwedge_{i \in I} \Big(\bigvee_{j \in J} x_{i,j} \Big) = \bigvee_{f\colon I \to J} \Big(\bigwedge_{i \in I} x_{i, f(i)} \Big),$$

that is, any meet of joins may be converted into the join of all possible elements obtained by taking the meet over $i \in I$ of elements $x_{i,k}$, where k depends on i.

A *complete ring of sets* is a family $\mathcal{F}$ of sets such that $\bigcup \mathcal{S}$ and $\bigcap \mathcal{S}$ belong to $\mathcal{F}$ for any subfamily $\mathcal{S} \subseteq \mathcal{F}$.

Remark 2.1. Let L be a lattice. Then the following are equivalent.

(a) L is isomorphic to a complete ring of sets.
(b) L is algebraic and completely distributive.
(c) L is distributive and doubly algebraic (that is, both L and the dual L^{∂} of L are algebraic).
(d) L is algebraic, distributive and spatial.

A *De Morgan algebra* is an algebra $\mathbb{L} = (L, \vee, \wedge, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\sim$ satisfies, for all $x, y \in L$,

$$\sim\sim x = x \quad \text{and} \quad x \leq y \iff \sim y \leq \sim x.$$

This definition means that $\sim$ is an isomorphism between the lattice L and its dual L^{∂}. Thus, for a De Morgan algebra $\mathbb{L}$, the underlying lattice L is self-dual and, for each $x, y \in L$,

$$\sim(x \vee y) = \sim x \wedge \sim y \quad \text{and} \quad \sim(x \wedge y) = \sim x \vee \sim y.$$

We say that a De Morgan algebra is *completely distributive* if its underlying lattice is completely distributive. Let $\mathbb{L}$ be a completely distributive De Morgan algebra. We define, for any $j \in \mathcal{J}$, the element

$$g(j) = \bigwedge \{x \in L \mid x \nleq \sim j\}. \tag{2.1}$$

This $g(j) \in \mathcal{J}$ is the least element which is not below $\sim j$. The function $g\colon \mathcal{J} \to \mathcal{J}$ satisfies that:

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(J1) if $x \leq y$, then $g(x) \geq g(y)$; and
(J2) $g(g(x)) = x$.

In fact, $(\mathcal{J}, \leq)$ is self-dual by the map g.

Let $\mathbb{L}$ be a De Morgan algebra defined on an algebraic lattice. The underlying lattice L is doubly algebraic, because it is self-dual. Therefore, the lattice L has all equivalent properties (a)–(d) of Remark 2.1. Also, the operation $\sim$ is expressed in terms of g by

$$\sim x = \bigvee \{ j \in \mathcal{J} \mid g(j) \nleq x \}. \tag{2.2}$$

For studies on the properties of the map g, see, for example [Cig86, JR11, Mon63].

A *Kleene algebra* is a De Morgan algebra $\mathbb{L}$ satisfying

$$x \wedge \sim x \leq y \vee \sim y \tag{K}$$

for each $x, y \in L$. It is proved in [CdG81] that, for any Kleene algebra $\mathbb{L}$ and $x, y \in L$,

$$x \wedge y = 0 \quad \text{implies} \quad y \leq \sim x. \tag{2.3}$$

If $\mathbb{L}$ is a completely distributive Kleene algebra, then j and $g(j)$ are comparable for any $j \in \mathcal{J}$: that is,

(J3) $g(j) \leq j$ or $j \leq g(j)$.

A *Heyting algebra* is a bounded lattice L such that, for all $a, b \in L$, there is a greatest element x of L satisfying $a \wedge x \leq b$. This element x is called the *relative pseudocomplement* of a with respect to b, and it is denoted by $a \Rightarrow b$. Heyting algebras are not only distributive, but they satisfy the *join-infinite distributive law* (JID): that is, if $\bigvee S$ exists for some $S \subseteq L$, then, for each $x \in L$, $\bigvee \{x \wedge y \mid a \in S\}$ exists and $x \wedge (\bigvee S) = \bigvee \{x \wedge y \mid y \in S\}$. Conversely, any complete lattice satisfying the JID is a Heyting algebra, with $a \Rightarrow b = \bigvee \{c \mid a \wedge c \leq b\}$.

A *double Heyting algebra* L is such that both L and its dual L^∂ are Heyting algebras; see, for instance [Kat73]. This means that in L there are two implications, $\Rightarrow$ and $\Leftarrow$, which are fully determined by $\leq$, and $\Leftarrow$ satisfies $a \vee x \geq b$ if and only if $x \geq a \Leftarrow b$ for all $a, b, x \in L$. These structures are also called *Heyting–Brouwer algebras*.

A Heyting algebra L such that $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra is called a *symmetric Heyting algebra*; see [Mon80]. Each symmetric Heyting algebra defines a double Heyting algebra such that $a \Leftarrow b$ equals $\sim(\sim a \Rightarrow \sim b)$.

EXAMPLE 2.2. (a) Every De Morgan algebra defined on an algebraic lattice determines a symmetric Heyting algebra, by Remark 2.1.

(b) A complete lattice is a double Heyting algebra if and only if it satisfies the JID and the MID, the dual condition of the JID. In particular, every finite distributive lattice is a double Heyting algebra. Of course, not every finite and distributive lattice is self-dual, that is, a symmetric Heyting algebra.

(c) One double Heyting algebra may define several symmetric Heyting algebras. For instance, the Boolean algebra $\mathbf{2}^2$ with $0 < a, b < 1$ is a double Heyting algebra and we can define a De Morgan operation $\sim$ in $\mathbf{2}^2$ by two ways: either (i) $a \mapsto a$ and $b \mapsto b$; or (ii) $a \mapsto b$ and $b \mapsto a$. These mappings are De Morgan operations when, for both cases, we set $\sim 0 = 1$ and $\sim 1 = 0$.

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In a lattice L with the least element 0, an element $x \in L$ is said to have a *pseudocomplement* if there exists an element x^* in L having the property that for any $z \in L$, $x \wedge z = 0 \Leftrightarrow z \leq x^*$. The lattice L itself is called *pseudocomplemented*, if every element of L has a pseudocomplement. Every pseudocomplemented lattice is necessarily bounded, having 0^* as the greatest element. The algebra $(L, \vee, \wedge, {}^*, 0, 1)$ is called also a *p-algebra* for short. The following hold for every $a, b \in L$.

(i) $a \leq b$ implies that $b^* \leq a^*$.
(ii) The map $a \mapsto a^{**}$ is a closure operator.
(iii) $a^* = a^{***}$.
(iv) $(a \vee b)^* = a^* \wedge b^*$.
(v) $(a \wedge b)^* \geq a^* \vee b^*$.

An algebra $(L, \vee, \wedge, {}^*, {}^+, 0, 1)$ is called a *double p-algebra* if $(L, \vee, \wedge, {}^*, 0, 1)$ is a p-algebra and $(L, \vee, \wedge, {}^+, 0, 1)$ is a dual p-algebra (that is, $z \geq x^+ \Leftrightarrow x \vee z = 1$ for all $x, y \in L$). In the literature, the term *double pseudocomplemented lattice* is often used instead of double p-algebra. Each Heyting algebra L defines a distributive p-algebra by setting $x^* := x \Rightarrow 0$, and if L is also a double Heyting algebra, it determines a distributive double p-algebra, where $x^+ := x \Leftarrow 1$.

For a double p-algebra $(L, \vee, \wedge, {}^*, {}^+, 0, 1)$, the *determination congruence* Φ is defined by

$$\Phi := \{(x, y) \mid x^* = y^* \text{ and } x^+ = y^+\}.$$

A double p-algebra is called *determination-trivial* if $\Phi = \{(x, x) \mid x \in L\}$. This is obviously equivalent to the fact that the double p-algebra satisfies that

$$x^* = y^* \quad \text{and} \quad x^+ = y^+ \quad \text{imply} \quad x = y. \tag{M}$$

An algebra is called *congruence-regular* if every congruence is determined by any class of it: two congruences are necessarily equal when they have a class in common. Varlet has proved in [Var72] that double p-algebras satisfying (M) are exactly the congruence-regular ones. In addition, Katriňák [Kat73] has shown that any congruence-regular double pseudocomplemented lattice forms a double Heyting algebra such that

$$a \Rightarrow b = (a^* \vee b^{**})^{**} \wedge [(a \vee a^*)^+ \vee a^* \vee b \vee b^*], \tag{2.4}$$

$$a \Leftarrow b = (a^+ \wedge b^{++})^{++} \vee [(a \wedge a^+)^* \wedge a^+ \wedge b \wedge b^+]. \tag{2.5}$$

A *pseudocomplemented De Morgan algebra* is an algebra $(L, \vee, \wedge, \sim, {}^*, 0, 1)$ such that $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra and $(L, \vee, \wedge, {}^*, 0, 1)$ is a p-algebra. In fact, such an algebra forms a double p-algebra, where the pseudocomplement operations determine each other by

$$\sim x^* = (\sim x)^+ \quad \text{and} \quad \sim x^+ = (\sim x)^*. \tag{2.6}$$

Sankappanavar has proved in [San86] that a pseudocomplemented De Morgan algebra satisfying (M) truly is a congruence-regular pseudocomplemented De Morgan algebra.

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Therefore, in the subsequent work, we may call pseudocomplemented De Morgan and Kleene algebras *regular* when they satisfy (M). Note that regular pseudocomplemented De Morgan algebras define symmetric (double) Heyting algebras, where the operations $\Rightarrow$ and $\Leftarrow$ are given by (2.4) and (2.5).

A pseudocomplemented De Morgan algebra $\mathbb{L}$ is *normal* (see [Mon80]) if, for all $x \in L$,

$$x^* \leq \sim x. \tag{N}$$

Note that if $\mathbb{L}$ is normal, then, for every $x \in L$ and $y = \sim x$, $\sim(\sim y)^+ = y^* \leq \sim y$. Hence $(\sim y)^+ \geq y$ and so $x^+ \geq \sim x$. Thus

$$x^* \leq \sim x \leq x^+.$$

It is known (see, for example, [Kat73]) that in any distributive double p-algebra, the 'regularity condition' (M) is equivalent to the condition

$$x \wedge x^+ \leq y \vee y^*. \tag{D}$$

This means that if $\mathbb{L}$ is a normal and regular pseudocomplemented De Morgan algebra, then, for any $x, y \in L$,

$$x \wedge \sim x \leq x \wedge x^+ \leq y \vee y^* \leq y \vee \sim y.$$

Therefore, any normal and regular pseudocomplemented De Morgan algebra forms a Kleene algebra. On the other hand, any pseudocomplemented Kleene algebra is normal by (2.3). Hence we have shown the following result.

Remark 2.3. Any regular pseudocomplemented De Morgan algebra is a Kleene algebra if and only if it is normal.

A filter F of a lattice L is called *proper* if $F \neq L$. A proper filter F is a *prime filter* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$. The set of prime filters of L is denoted by $\mathcal{F}_p(L)$, or by $\mathcal{F}_p$ if there is no danger of confusion. Proper ideals and prime ideals are defined analogously. Clearly, F is a prime filter if and only if $L \backslash F$ is a prime ideal.

If L is a bounded distributive lattice, then any prime filter F is contained in a maximal prime filter. Moreover, any maximal prime filter is a maximal proper filter. If L is a distributive lattice, then the principal filter $[j) = \{x \in L \mid x \geq j\}$ of each $j \in \mathcal{J}$ is prime. For any $j \in \mathcal{J}$, the prime filter $[j)$ is maximal if and only if j is an atom.

Fact 2.4. For any bounded distributive lattice, the following are equivalent.

(a) There is no three-element chain of prime filters.
(b) For any $P, Q \in \mathcal{F}_p$, $P \subset Q$ implies that Q is a maximal filter.

Proposition 2.5 [Var72]. *Let* $(L, \vee, \wedge, ^*, ^+, 0, 1)$ *be a distributive double p-algebra. The following are equivalent.*

(a) *L is regular.*
(b) *Any chain of prime filters (or ideals) of L has at most two elements.*

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3. Rough sets defined by tolerances induced by irredundant coverings

Let R be a tolerance on U. A nonempty subset X of U is a *preblock* if $X^2 \subseteq R$. A *block* is a maximal preblock: that is, a preblock B is a block if $B \subseteq X$ implies that $B = X$ for any preblock X. Thus any subset $\emptyset \neq X \subseteq U$ is a preblock if and only if it is contained in some block. Each tolerance R is completely determined by its blocks: that is, $a \, R \, b$ if and only if there exists a block B such that $a, b \in B$.

A collection $\mathcal{H}$ of nonempty subsets of U is called a *covering* of U if $\bigcup \mathcal{H} = U$. A covering $\mathcal{H}$ is *irredundant* if $\mathcal{H} \setminus \{X\}$ is not a covering for any $X \in \mathcal{H}$. Each covering $\mathcal{H}$ defines a tolerance $R_{\mathcal{H}} = \bigcup \{X^2 \mid X \in \mathcal{H}\}$, called the *tolerance induced* by $\mathcal{H}$. Obviously, the sets in $\mathcal{H}$ are preblocks of $R_{\mathcal{H}}$ and $R_{\mathcal{H}}(x) = \bigcup \{B \in \mathcal{H} \mid x \in B\}$. Thus $x \in B$ implies that $B \subseteq R_{\mathcal{H}}(x)$ for any $B \in \mathcal{H}$.

We proved in [JR14] that $\mathcal{H}$ is irredundant if and only if $\mathcal{H} \subseteq \{R_{\mathcal{H}}(x) \mid x \in U\}$. In addition, if $\mathcal{H}$ is irredundant, then $\mathcal{H}$ consists of blocks of $R_{\mathcal{H}}$ [JR15]. We can now write the following lemma which states that each irredundant covering $\mathcal{H}$ consists of such $R_{\mathcal{H}}(x)$-sets that are blocks of $R_{\mathcal{H}}$. Therefore, we may simply speak about tolerances induced by an irredundant covering without specifying the covering in question.

Lemma 3.1. *Let R be a tolerance induced by an irredundant covering $\mathcal{H}$ of U. Then $\mathcal{H} = \{R(x) \mid R(x)$ is a block$\}$.*

Proof. By the above, $\mathcal{H} \subseteq \{R(x) \mid R(x)$ is a block$\}$. On the other hand, suppose that $R(x)$ is a block. Because $\mathcal{H}$ is a covering, there is $B \in \mathcal{H}$ such that $x \in B$. This gives that $B \subseteq R(x)$. Since $\mathcal{H}$ is irredundant, B is a block. Because both $R(x)$ and B are blocks, $B = R(x)$ and $R(x) \in \mathcal{H}$. □

Let L be a lattice with the least element 0. An element a is an *atom* of L if it covers 0, that is, $0 \prec a$. We denote by $\mathcal{A}(L)$ the set of atoms of L, and simply by $\mathcal{A}$ if there is no danger of confusion. The lattice L is *atomistic*, if $x = \bigvee \{a \in \mathcal{A} \mid a \leq x\}$ for all $x \in L$. It is well known that a Boolean lattice is atomistic if and only if it is completely distributive; see [Grä98], for example.

Let R be tolerance on U. In [Jär99] it is proved that $({}^{\blacktriangle}, {}^{\blacktriangledown})$ is an order-preserving Galois connection on the complete lattice $(\wp(U), \subseteq)$. This implies that $\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\}$ is a complete lattice such that

$$\bigwedge \mathcal{H} = \bigcap \mathcal{H} \quad \text{and} \quad \bigvee \mathcal{H} = \Big(\bigcup \mathcal{H}\Big)^{\blacktriangle\blacktriangledown}$$

for all $\mathcal{H} \subseteq \wp(U)^{\blacktriangledown}$. Similarly, $\wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}$ is a complete lattice in which, for all $\mathcal{H} \subseteq \wp(U)^{\blacktriangle}$,

$$\bigwedge \mathcal{H} = \Big(\bigcap \mathcal{H}\Big)^{\blacktriangledown\blacktriangle} \quad \text{and} \quad \bigvee \mathcal{H} = \bigcup \mathcal{H}.$$

Because $({}^{\blacktriangle}, {}^{\blacktriangledown})$ is a Galois connection, $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are isomorphic. In [JR14], we proved that if R is a tolerance induced by an irredundant covering, then $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are atomistic Boolean lattices such that $\{R(x)^{\blacktriangledown} \mid R(x)$ is a block$\}$ and $\{R(x) \mid R(x)$ is a block$\}$ are their sets of atoms, respectively. By Lemma 3.1,

$\{R(x) \mid R(x) \text{ is a block}\}$ is the unique irredundant covering inducing R. The Boolean complement operation in $\wp(U)^{\blacktriangledown}$ is $X^{\blacktriangledown} \mapsto X^{\blacktriangledown c \blacktriangledown}$ and the complement operation in $\wp(U)^{\blacktriangle}$ is $X^{\blacktriangle} \mapsto X^{\blacktriangle c \blacktriangle}$.

Lemma 3.2. *Let R be a tolerance induced by a covering $\mathcal{H}$ of U, let $B \in \mathcal{H}$ and let $X \subseteq U$. Then:*

(a) $X^{\blacktriangle} = \bigcup\{C \in \mathcal{H} \mid X \cap C \neq \emptyset\}$;
(b) $B^{\blacktriangledown} = \{x \in U \mid R(x) = B\}$; *and*
(c) *if $\mathcal{H}$ is irredundant, then $\emptyset \neq B^{\blacktriangledown} = B \setminus \bigcup(\mathcal{H} \setminus \{B\})$.*

Proof. (a) The proof can be found in [Jär99].

(b) Let $B \in \mathcal{H}$. If $x \in B^{\blacktriangledown}$, then $R(x) \subseteq B$. Since B is a block, $x \in B$ implies that $B \subseteq R(x)$. Thus $R(x) = B$. On the other hand, $R(x) = B$ gives $x \in B^{\blacktriangledown}$.

(c) Suppose that $\mathcal{H}$ is irredundant and $B \in \mathcal{H}$. Then $X := B \setminus \bigcup(\mathcal{H} \setminus \{B\})$ is nonempty. We prove that $X = B^{\blacktriangledown}$. Let $x \in X$ and $y \in R(x)$. Since $x\,R\,y$, there exists $C \in \mathcal{H}$ such that $x, y \in C$. If $C \neq B$, then $x \in \bigcup(\mathcal{H} \setminus \{B\})$ and $x \notin X$, which is a contradiction. Therefore $C = B$ and $y \in B$. Thus $R(x) \subseteq B$ and $x \in B^{\blacktriangledown}$. Conversely, let $x \in B^{\blacktriangledown}$. Suppose that $x \in \bigcup(\mathcal{H} \setminus \{B\})$. Then there exists $C \neq B$ in $\mathcal{H}$ such that $x \in C$. But $x \in C$ implies that $C \subseteq R(x) \subseteq B$, which is not possible because $\mathcal{H}$ is irredundant. Therefore $x \in X$ and the proof is complete. □

We studied, in [JR14], the lattice-theoretical properties of

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}.$$

Let us recall here some of these results. We showed that RS is a complete lattice if and only if RS is a complete sublattice of the direct product $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. This means that if RS is a complete lattice, then, for $\{(A_i, B_i)\}_{i \in I} \subseteq RS$,

$$\bigwedge_{i \in I} (A_i, B_i) = \Big(\bigcap_{i \in I} A_i, \Big(\bigcap_{i \in I} B_i \Big)^{\blacktriangledown \blacktriangle} \Big) \tag{3.1}$$

and

$$\bigvee_{i \in I} (A_i, B_i) = \Big(\Big(\bigcup_{i \in I} A_i \Big)^{\blacktriangle \blacktriangledown}, \bigcup_{i \in I} B_i \Big). \tag{3.2}$$

In addition, we proved that RS is an algebraic and completely distributive lattice if and only if R is induced by an irredundant covering. We also noted that if R is a tolerance induced by an irredundant covering of U, then the algebra

$$(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a Kleene algebra such that the operations $\wedge$ and $\vee$ are defined as in (3.1) and (3.2), and

$$\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{c\blacktriangledown}, X^{c\blacktriangle}) = (X^{\blacktriangle c}, X^{\blacktriangledown c}).$$

Because RS is a self-dual algebraic and distributive lattice, it is spatial by Remark 2.1. In addition, RS forms a double p-algebra and a symmetric (double) Heyting algebra. Our next lemma describes the pseudocomplements and the dual pseudocomplements in RS.

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LEMMA 3.3. *Let R be a tolerance induced by an irredundant covering. For any $(A, B) \in RS$,*

$$(A, B)^* = (B^{c\blacktriangledown}, B^{c\blacktriangle}) \quad \textit{and} \quad (A, B)^+ = (A^{c\blacktriangledown}, A^{c\blacktriangle}).$$

PROOF. The lattice operations of RS are described in (3.1) and (3.2). Recall also that $({}^\blacktriangle, {}^\blacktriangledown)$ is a Galois connection. First, we show that $(A, B) \wedge (B^{c\blacktriangledown}, B^{c\blacktriangle}) = (A \cap B^{c\blacktriangledown}, (B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle})$ equals $(\emptyset, \emptyset)$. It suffices to show that the right component $(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle}$ is $\emptyset$, because then, necessarily, the left component $A \cap B^{c\blacktriangledown}$ is empty. Indeed, $(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle} = (B^\blacktriangledown \cap B^{c\blacktriangle\blacktriangledown})^\blacktriangle = (B^\blacktriangledown \cap B^{\blacktriangledown\blacktriangle c})^\blacktriangle \subseteq (B^{\blacktriangledown\blacktriangle} \cap B^{\blacktriangledown\blacktriangle c})^\blacktriangle = \emptyset^\blacktriangle = \emptyset$.

On the other hand, if $(A, B) \wedge (X, Y) = \emptyset$ for some $(A, B) \in RS$, then $B \wedge Y = \emptyset$ in the corresponding Boolean lattice $\wp(U)^\blacktriangle$. This gives $Y \subseteq B^{c\blacktriangle}$. Because $X = Z^\blacktriangledown$ and $Y = Z^\blacktriangle$ for some $Z \subseteq U$, $X^\blacktriangle = Z^{\blacktriangledown\blacktriangle} \subseteq Z \subseteq Z^{\blacktriangle\blacktriangledown} = Y^\blacktriangledown$. This implies that $X^{\blacktriangle\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} \subseteq Y \subseteq B^{c\blacktriangle}$. We obtain $X^\blacktriangle \subseteq (X^\blacktriangle)^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangle\blacktriangledown}$. Now $B \in \wp(U)^\blacktriangle$ implies that $B^c \in \wp(U)^\blacktriangledown$. Hence $B^{c\blacktriangle\blacktriangledown} = B^c$ and we get $X \subseteq X^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangledown}$. We have now shown that $(X, Y) \leq (B^{c\blacktriangledown}, B^{c\blacktriangle})$, which completes the proof.

The other claim for $(A, B)^+$ can be proved similarly. □

Now the rough set algebra

$$\mathbb{RS} = (RS, \vee, \wedge, \sim, {}^*, (\emptyset, \emptyset), (U, U))$$

is a pseudocomplemented Kleene algebra.

PROPOSITION 3.4. *If R is a tolerance induced by an irredundant covering, then the pseudocomplemented Kleene algebra $\mathbb{RS}$ is regular.*

PROOF. We show that condition (M) holds. If $(A, B)^* = (C, D)^*$, then $B^{\blacktriangledown c} = B^{c\blacktriangle} = D^{c\blacktriangle} = D^{\blacktriangledown c}$. So $B^\blacktriangledown = D^\blacktriangledown$ and $B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle}$. Because $B, D \in \wp(U)^\blacktriangle$, $B = B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle} = D$. Similarly, $(A, B)^+ = (C, D)^+$ implies that $A = C$. We have proved that $(A, B) = (C, D)$. □

Let R be a tolerance on U induced by an irredundant covering. By Remark 2.3, the pseudocomplemented Kleene algebra $\mathbb{RS}$ is normal. This means that, for all $(A, B) \in RS$,

$$(A, B)^* \leq \sim(A, B) \leq (A, B)^+.$$

The elements $(X^\blacktriangledown, X^\blacktriangle) \Rightarrow (Y^\blacktriangledown, Y^\blacktriangle)$ and $(X^\blacktriangledown, X^\blacktriangle) \Leftarrow (Y^\blacktriangledown, Y^\blacktriangle)$ can be computed as in (2.4) and (2.5). It is well known that, for any distributive p-algebra L, the *skeleton* $S^*(L) = \{a^* \mid a \in L\}$ forms a Boolean algebra $(S^*(L), \sqcup, \wedge, {}^*, 0, 1)$, where $a \sqcup b = (a^* \wedge b^*)^*$. If L is a distributive double p-algebra, also the *dual skeleton* $S^+(L) = \{a^+ \mid a \in L\}$ forms a Boolean algebra $(S^+(L), \vee, \sqcap, {}^+, 0, 1)$, where $a \sqcap b = (a^+ \vee b^+)^+$. We may now define $S^*(RS) = \{(B^{c\blacktriangledown}, B^{c\blacktriangle}) \mid B \in \wp(U)^\blacktriangle\}$ and $S^+(RS) = \{(A^{c\blacktriangledown}, A^{c\blacktriangle}) \mid A \in \wp(U)^\blacktriangledown\}$.

LEMMA 3.5. *If R is a tolerance induced by an irredundant covering, then*

$$(\wp(U)^\blacktriangle, \supseteq) \cong (S^*(RS), \leq) \quad \textit{and} \quad (\wp(U)^\blacktriangledown, \supseteq) \cong (S^+(RS), \leq).$$

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Proof. We prove that $\varphi\colon B \mapsto (B^{c\blacktriangledown}, B^{c\blacktriangle})$ is an order-isomorphism from $(\wp(U)^\blacktriangle, \supseteq)$ to $(S^*(RS), \leq)$. If $B, C \in \wp(U)^\blacktriangle$ and $B \supseteq C$, then $B^c \subseteq C^c$. This gives $B^{c\blacktriangledown} \subseteq C^{c\blacktriangledown}$ and $B^{c\blacktriangle} \subseteq C^{c\blacktriangle}$, that is, $(B^{c\blacktriangledown}, B^{c\blacktriangle}) \leq (C^{c\blacktriangledown}, C^{c\blacktriangle})$ in RS. Conversely, $(B^{c\blacktriangledown}, B^{c\blacktriangle}) \leq (C^{c\blacktriangledown}, C^{c\blacktriangle})$ implies that $B^{\blacktriangledown c} = B^{c\blacktriangle} \subseteq C^{c\blacktriangledown} = C^{\blacktriangle c}$ and $B^\blacktriangledown \supseteq C^\blacktriangledown$. Therefore, also, $B^{\blacktriangledown\blacktriangle} \supseteq C^{\blacktriangledown\blacktriangle}$. Because $B, C \in \wp(U)^\blacktriangle$, $B = B^{\blacktriangledown\blacktriangle}$ and $C = C^{\blacktriangledown\blacktriangle}$. Thus $B \supseteq C$. The map φ is trivially onto.

Similarly, we can show that the map $A \mapsto (A^{c\blacktriangle}, A^{c\blacktriangle})$ is an order-isomorphism from $(\wp(U)^\blacktriangledown, \supseteq)$ to $(S^+(RS), \leq)$. □

Note that all lattices mentioned in Lemma 3.5 are as Boolean lattices also dually isomorphic with themselves. Our next proposition describes the set of completely join-irreducible elements of RS.

Proposition 3.6. *Let R be a tolerance induced by an irredundant covering. Then*

$$\mathcal{J}(RS) = \{(R(x)^\blacktriangledown, R(x)^\blacktriangle) \mid R(x) \text{ is a block}\} \\ \cup \{(\emptyset, R(x)) \mid R(x) \text{ is a block and } |R(x)| \geq 2\}.$$

Proof. Let x be an element such that $R(x)$ is an block. If $|R(x)| \geq 2$, then $(\{x\}^\blacktriangledown, \{x\}^\blacktriangle) = (\emptyset, R(x)) \in RS$. Since $R(x)$ is an atom of $\wp(U)^\blacktriangle$, $(\emptyset, R(x))$ is an atom of RS. Trivially, all atoms are completely join-irreducible.

Next, we show that if $R(x)$ is a block, then $(R(x)^\blacktriangledown, R(x)^\blacktriangle)$ is completely join-irreducible. Assume that $(R(x)^\blacktriangledown, R(x)^\blacktriangle) = \bigvee\{(X_i^\blacktriangledown, X_i^\blacktriangle) \mid i \in I\}$. This means that $X_i^\blacktriangledown \subseteq R(x)^\blacktriangledown$ for all $i \in I$. Because each $X_i^\blacktriangledown$ belongs to $\wp(U)^\blacktriangledown$ and $R(x)^\blacktriangledown$ is an atom of $\wp(U)^\blacktriangledown$, $\emptyset \subseteq X_i^\blacktriangledown \subseteq R(x)^\blacktriangledown$ implies that every $X_i^\blacktriangledown$ is equal either to $\emptyset$ or to $R(x)^\blacktriangledown$. But since $x \in R(x)^\blacktriangledown = \{x\}^{\blacktriangle\blacktriangledown}$, each $X_i^\blacktriangledown$ cannot be $\emptyset$. Therefore, there exists $k \in I$ such that $X_k^\blacktriangledown = R(x)^\blacktriangledown$. We know that $R(x) = \{x\}^\blacktriangle = \{x\}^{\blacktriangle\blacktriangledown\blacktriangle} = R(x)^{\blacktriangledown\blacktriangle} = X_k^{\blacktriangledown\blacktriangle} \subseteq X_k$. Thus $R(x)^\blacktriangle \subseteq X_k^\blacktriangle$. By assumption, $X_i^\blacktriangle \subseteq R(x)^\blacktriangle$ for all $i \in I$. Hence $R(x)^\blacktriangle = X_k^\blacktriangle$ and $(R(x)^\blacktriangledown, R(x)^\blacktriangle) = (X_k^\blacktriangledown, X_k^\blacktriangle)$.

On the other hand, suppose that $(X^\blacktriangledown, X^\blacktriangle)$ is a completely join-irreducible element of RS. In [JR14, Remark 4.6], we proved that each element of RS can be represented as the join of a subset of

$$\{(R(x)^\blacktriangledown, R(x)^\blacktriangle) \mid x \in U\} \cup \{(\emptyset, R(x)) \mid |R(x)| \geq 2\}.$$

But since $(X^\blacktriangledown, X^\blacktriangle)$ is itself a completely join-irreducible element, $(X^\blacktriangledown, X^\blacktriangle) = (R(x)^\blacktriangledown, R(x)^\blacktriangle)$ for some $x \in U$, or $(X^\blacktriangledown, X^\blacktriangle) = (\emptyset, R(x))$ for some x such that $|R(x)| \geq 2$.

Let us assume first that $(X^\blacktriangledown, X^\blacktriangle) = (R(x)^\blacktriangledown, R(x)^\blacktriangle)$ for some $x \in U$. Because $R(x) \in \wp(U)^\blacktriangle$, there is a set $\{x_i\}_{i\in I} \subseteq U$ such that $R(x) = \bigcup_{i\in I} R(x_i)$ and each $R(x_i)$ is a block. This gives $R(x)^\blacktriangle = (\bigcup_{i\in I} R(x_i))^\blacktriangle = \bigcup_{i\in I} R(x_i)^\blacktriangle$. Analogously,

$$\Big(\bigcup_{i\in I} R(x_i)^\blacktriangledown\Big)^{\blacktriangle\blacktriangledown} = \Big(\bigcup_{i\in I} R(x_i)^{\blacktriangledown\blacktriangle}\Big)^\blacktriangledown = \Big(\bigcup_{i\in I} R(x_i)\Big)^\blacktriangledown = R(x)^\blacktriangledown.$$

This means that

$$(R(x)^\blacktriangledown, R(x)^\blacktriangle) = \bigvee_{RS}\{(R(x_i)^\blacktriangledown, R(x_i)^\blacktriangle) \mid i \in I\}.$$

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Since $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$ is completely join-irreducible,

$$(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) = (R(x_k)^{\blacktriangledown}, R(x_k)^{\blacktriangle})$$

for some block $R(x_k)$.

Second, if $(X^{\blacktriangledown}, X^{\blacktriangle}) = (\emptyset, R(x))$ for some x such that $|R(x)| \geq 2$, then $R(x) = \bigcup_{i \in I} R(x_i)$ for some index set I such that each $R(x_i)$ is a block. Because $R(x_i) \subseteq R(x)$ for all $i \in I$, $x \, R \, x_i$ for all $i \in I$. If $x \neq x_i$, then $x, x_i \in R(x_i)$ means that $|R(x_i)| \geq 2$, and if $x = x_i$, the assumption that $|R(x)| \geq 2$ gives $|R(x_i)| \geq 2$. Therefore, each $(\emptyset, R(x_i))$ is in RS and

$$(\emptyset, R(x)) = \bigvee_{RS} \{(\emptyset, R(x_i)) \mid i \in I\}.$$

But, since $(\emptyset, R(x))$ is completely join-irreducible by assumption, $(\emptyset, R(x)) = (\emptyset, R(x_k))$ for some $k \in I$ such that $|R(x_k)| \geq 2$ and $R(x_k)$ is a block. □

For a tolerance induced by an irredundant covering $\mathcal{H}$, we can express the completely join-irreducible elements of RS also using elements of $\mathcal{H}$ as

$$\mathcal{J}(RS) = \{(B^{\blacktriangledown}, B^{\blacktriangle}) \mid B \in \mathcal{H}\} \cup \{(\emptyset, B) \mid B \in \mathcal{H} \text{ and } |B| \geq 2\}.$$

Recall that $B^{\blacktriangledown}$ and $B^{\blacktriangle}$ are given in terms of the irredundant covering $\mathcal{H}$ in Lemma 3.2. Note also that, since RS is spatial, its every element can be described as the join of some elements in $\mathcal{J}(RS)$.

Because $\mathbb{RS}$ is a completely distributive Kleene algebra and, for any (A, B), $\sim(A, B) = (B^c, A^c)$, we can by (2.1) define the map $g \colon \mathcal{J}(RS) \to \mathcal{J}(RS)$ by setting

$$g((C, D)) = \bigwedge \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid (X^{\blacktriangledown}, X^{\blacktriangle}) \nleq (D^c, C^c)\}$$

for any $(C, D) \in \mathcal{J}(RS)$.

LEMMA 3.7. *Let R be a tolerance induced by an irredundant covering.*

(a) *If $R(x)$ is a block such that $|R(x)| \geq 2$, then*

$$g((\emptyset, R(x))) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) \quad \text{and} \quad g((R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})) = (\emptyset, R(x)).$$

(b) *If $R(x) = \{x\}$, then $g((\{x\}, \{x\})) = (\{x\}, \{x\})$.*

PROOF. (a) Suppose that $R(x)$ is a block such that $|R(x)| \geq 2$. Now $(X^{\blacktriangledown}, X^{\blacktriangle}) \nleq (R(x)^c, U)$ is equivalent to $X^{\blacktriangledown} \nsubseteq R(x)^c$. This means that $X^{\blacktriangledown} \cap R(x) \neq \emptyset$. Thus there is $y \in X^{\blacktriangledown} \cap R(x)$. Because $y \in R(x)$, $R(x) \subseteq R(y)$, and $y \in X^{\blacktriangledown}$ yields $R(x) \subseteq R(y) \subseteq X$. Thus $R(x)^{\blacktriangledown} \subseteq X^{\blacktriangledown}$ and $R(x)^{\blacktriangle} \subseteq X^{\blacktriangle}$. Therefore $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) \leq \bigwedge\{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid (X^{\blacktriangledown}, X^{\blacktriangle}) \nleq (R(x)^c, U)\} = g((\emptyset, R(x)))$.

On the other hand, $x \in \{x\}^{\blacktriangle\blacktriangledown} = R(x)^{\blacktriangledown} \nsubseteq R(x)^c$ gives $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) \nleq (R(x)^c, U)$ and $g((\emptyset, R(x))) \leq (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. Thus $g((\emptyset, R(x))) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. Because $g(g(j)) = j$ for any $j \in \mathcal{J}(RS)$, $g((R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})) = (\emptyset, R(x))$.

(b) If $R(x) = \{x\}$, then $R(x)$ is a block and $R(x)^{\blacktriangledown} = R(x)^{\blacktriangle} = R(x)$, because x is R-related only to itself. Therefore $(\{x\}, \{x\}) \in \mathcal{J}(RS)$. Now $\sim(\{x\}, \{x\}) = (\{x\}^c, \{x\}^c)$ and $(X^{\blacktriangledown}, X^{\blacktriangle}) \nleq (\{x\}^c, \{x\}^c)$ holds if and only if $x \in X^{\blacktriangledown} \subseteq X^{\blacktriangle}$. This implies that $(\{x\}, \{x\}) \leq \bigwedge\{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid (X^{\blacktriangledown}, X^{\blacktriangle}) \nleq (\{x\}^c, \{x\}^c)\} = g((\{x\}, \{x\}))$. On the other hand, $(\{x\}, \{x\}) \nleq (\{x\}^c, \{x\}^c)$ implies that $g((\{x\}, \{x\})) \leq (\{x\}, \{x\})$. Thus $g((\{x\}, \{x\})) = (\{x\}, \{x\})$. □

In the next section (see Lemma 4.5), we will show that if $\mathbb{L}$ is a regular pseudocomplemented Kleene algebra defined on an algebraic lattice, then $x \in \mathcal{J}$ is an atom if and only if $x \leq g(x)$. Therefore, by Lemma 3.7,

$$\mathcal{A}(RS) = \{(\{x\}, \{x\}) \mid R(x) = \{x\}\} \\ \cup \{(\emptyset, R(x)) \mid R(x) \text{ is a block and } |R(x)| \geq 2\}. \tag{3.3}$$

Note that (3.3) can be seen also directly. Let $R(x)$ be a block. If $|R(x)| \geq 2$, then we have already seen in the proof of Proposition 3.6 that $(\emptyset, R(x))$ is an atom. Obviously, the completely join-irreducible element $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$ cannot now be an atom. If $R(x) = \{x\}$, then $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) = (\{x\}, \{x\})$ is an atom, because there is no element $(\emptyset, \{x\})$ in RS.

Each equivalence relation E on U is 'induced' by the irredundant covering U/E which consists of the equivalence classes of E. The covering U/E forms a partition of U, that is, the sets in U/E do not intersect. The following lemma presents equivalent conditions for such 'isolated blocks' in the case of tolerances induced by irredundant coverings.

Lemma 3.8. *Let R be a tolerance induced by an irredundant covering. For each $R(x)$ that is a block, the following are equivalent.*

(a) *$R(y) = R(x)$ for all $y \in R(x)$.*
(b) *$(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) = (R(x), R(x))$.*
(c) *Either $R(x) = \{x\}$ or $(\emptyset, R(x))$ is the only atom below $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$.*

Proof. (a) ⇒ (b). If $R(y) = R(x)$ for all $y \in R(x)$, then $y \in R(x)$ yields $y \in R(x)^{\blacktriangledown}$. Thus $R(x)^{\blacktriangledown} = R(x)$. This implies that $R(x) \subseteq R(x)^{\blacktriangle} = R(x)^{\blacktriangledown\blacktriangle} \subseteq R(x)$ and $R(x)^{\blacktriangle} = R(x)$.

(b) ⇒ (c). If $|R(x)| = 1$, then $R(x) = \{x\}$. Suppose that $|R(x)| \geq 2$. By (3.3), $(\emptyset, R(x))$ is an atom of RS. Clearly, $(\emptyset, R(x)) < (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. Assume that $(X^{\blacktriangledown}, X^{\blacktriangle})$ is an atom below $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. Then, by (3.3), either: (i) $X^{\blacktriangledown} = X^{\blacktriangle} = \{y\}$ for some y such that $R(y) = \{y\}$; or (ii) $X^{\blacktriangledown} = \emptyset$ and $X^{\blacktriangle} = R(y)$ for some y such that $R(y)$ is a block having at least two elements. (i) If $R(y) = \{y\}$, then $\{y\} \subseteq R(x)$ gives that $y\,R\,x$, and hence $y = x$. However, this is impossible, because $|R(x)| \geq 2$ and $|R(y)| = 1$. (ii) If $X^{\blacktriangledown} = \emptyset$ and $X^{\blacktriangle} = R(y)$ for some y such that $R(y)$ is a block having at least two elements, then $(\emptyset, R(y)) \leq (R(x), R(x))$ gives $R(y) \subseteq R(x)$. Because $R(x)$ and $R(y)$ are blocks, $R(x) = R(y)$.

(c) ⇒ (a). If $R(x) = \{x\}$, then obviously (a) is satisfied. On the other hand, suppose that $(\emptyset, R(x))$ is the only atom of RS below $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. If $y \in R(x)$, then $R(x) \subseteq R(y)$, because $R(x)$ is a block. We are going to prove that $R(x) = R(y)$. Assume, by contraction, that $R(x) \subset R(y)$. This means that there is $z \in R(y) \backslash R(x)$. Since $z\,R\,y$, there is $w \in U$ such that $R(w)$ is a block, $R(w) \neq R(x)$ and $z, y \in R(w)$. Therefore $|R(w)| \geq 2$. Because $y \in R(w)$, $R(w) \subseteq R(y)$. On the other hand, $x \in R(x) \subset R(y)$ gives $y \in R(x)$ and $R(y) = \{y\}^{\blacktriangle} \subseteq R(x)^{\blacktriangle}$. Therefore, $(\emptyset, R(w)) \leq (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$. Because $(\emptyset, R(w))$ is an atom, we obtain by assumption that $R(x) = R(w)$ and $z \in R(x)$, which is a contradiction. □

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4. Regularity in pseudocomplemented Kleene algebras

In this section, we study the structure of completely join-irreducible elements of Kleene algebras defined on algebraic lattices. Such algebras form pseudocomplemented Kleene algebras and we will prove that such an algebra is regular if and only if the set $\mathcal{J}$ of the completely join-irreducible elements has at most two levels. The obtained results are used in defining irredundant coverings and their tolerances.

A lattice L with 0 is called *atomic* if, for any $x \neq 0$, there exists an atom $a \leq x$. Clearly, every atomistic lattice is atomic.

Definition 4.1. The set of completely join-irreducible elements of a complete lattice *has at most two levels* if, for any completely join-irreducible elements j and k, $j < k$ implies that j is an atom.

Remark 4.2. Let L be a complete lattice. If $\mathcal{J}$ has at most two levels, then, clearly, $\mathcal{J}$ does not contain a chain of three (or more) elements.

If the lattice L is spatial, then these conditions are equivalent. Namely, assume that $\mathcal{J}$ does not contain a chain of three elements. Let $x < y$ be a maximal chain in $\mathcal{J}$. Suppose, by contradiction, that x is not an atom. Then there is $z \in L$ with $0 < z < x$. Since L is spatial, there is $j \in \mathcal{J}$ such that $j \leq z$. Now $j < x < y$ is a chain in $\mathcal{J}$ of three elements, which is a contradiction.

Let L be a complete lattice. It is well known that if j is a completely join-irreducible element, then j covers exactly one element, the *lower cover* of j. We denote this element by $j_\prec$. Obviously,

$$j_\prec = \bigvee \{x \in L \mid x < j\}.$$

It is clear that $j \in \mathcal{J}$ is an atom if and only if $j_\prec = 0$.

Lemma 4.3. *Let L be a spatial lattice such that $\mathcal{J}$ has at most two levels.*

(a) *If $j \in \mathcal{J} \setminus \mathcal{A}$, then $j_\prec$ is a join of atoms.*
(b) *The lattice L is atomic.*

Proof. (a) Let $j \in \mathcal{J} \setminus \mathcal{A}$. Since L is spatial, $j_\prec = \bigvee \{x \in \mathcal{J} \mid x < j\}$. Because $\mathcal{J}$ has at most two levels, each $x \in \mathcal{J}$ such that $x < j$ is an atom. Therefore $j_\prec$ is a join of atoms.

(b) Since L is spatial, we need to show only that there is an atom below each $j \in \mathcal{J}$. If j is an atom, we have nothing to prove. Now let $j \in \mathcal{J} \setminus \mathcal{A}$. Since $j_\prec \neq 0$ is a join of atoms by (i), there must be an atom below j. □

Proposition 4.4. *Let $(L, \vee, \wedge, \sim, ^*, 0, 1)$ be a pseudocomplemented De Morgan algebra defined on an algebraic lattice. The following are equivalent.*

(a) *L is regular.*
(b) *$\mathcal{J}$ has at most two levels.*

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Proof. (a) ⇒ (b). Any pseudocomplemented De Morgan algebra defines a distributive double p-algebra in which the dual pseudocomplement is defined as in (2.6). By Proposition 2.5, there is no three-element chain in $\mathcal{F}_p$. If $j, k \in \mathcal{J}$ with $j < k$, then $[j)$ and $[k)$ are prime filters such that $[k) \subset [j)$. By using 2.4, we get that $[j)$ is a maximal prime filter. Then j is an atom of L and $\mathcal{J}$ has at most two levels.

(b) ⇒ (a). We show that $x^* = y^*$ and $x^+ = y^+$ imply that $\sim x = \sim y$. Notice that $\sim x = \sim y$ is equivalent to $x = y$. Because L is spatial by Remark 2.1, $\sim x = \bigvee\{j \in \mathcal{J} \mid j \leq \sim x\}$ and $\sim y = \bigvee\{j \in \mathcal{J} \mid j \leq \sim y\}$. It suffices to show that, for any $j \in \mathcal{J}$, $j \leq \sim x$ if and only if $j \leq \sim y$. Suppose for this that $j \leq \sim x$.

If $j \in \mathcal{A}$, then it must be that $j \leq \sim y$. Otherwise, $j \wedge \sim y = 0$, which further implies that $j \leq (\sim y)^* = \sim y^+ = \sim x^+ = (\sim x)^*$ by (2.5). Hence, we would get $j = j \wedge \sim x = 0$, which is a contradiction.

If $j \in \mathcal{J} \backslash \mathcal{A}$, then there is $a \in \mathcal{A}$ such that $a < j$. This is because L is atomic by Lemma 4.3. Because L is, by Remark 2.1, completely distributive, we may define the map $g \colon \mathcal{J} \to \mathcal{J}$ as in (2.1). We have $g(j) < g(a)$. Since $\mathcal{J}$ has at most two levels, $g(j)$ is an atom.

Now $j \leq \sim x$ yields $g(j) \nleq x$ by (2.2). This implies that $g(j) \wedge x = 0$ since $g(j)$ is an atom. Thus $g(j) \leq x^* = y^*$. This gives $g(j) \wedge y = 0$ and $g(j) \nleq y$. By using (2.2) again, we obtain $j \leq \sim y$.

We have now proved that $j \leq \sim x$ implies that $j \leq \sim y$. The converse can be proved symmetrically. Hence, for any $j \in \mathcal{J}$, $j \leq \sim x$ if and only if $j \leq \sim y$, as required. □

Let $(L, \vee, \wedge, \sim, 0, 1)$ be a Kleene algebra defined on an algebraic lattice. By Proposition 4.4, the pseudocomplemented Kleene algebra $\mathbb{L} = (L, \vee, \wedge, \sim, {}^*, 0, 1)$ is regular if and only if $\mathcal{J}$ has at most two levels. Therefore, if $\mathbb{L}$ is regular, $\mathcal{J}$ can be trivially divided into two disjoint parts: the atoms $\mathcal{A}$ and the nonatoms $\mathcal{J} \backslash \mathcal{A}$. On the other hand, by (J1)–(J3), the map $g \colon \mathcal{J} \to \mathcal{J}$ is an order-isomorphism between $(\mathcal{J}, \leq)$ and $(\mathcal{J}, \geq)$ such that each element x in $\mathcal{J}$ is comparable with $g(x) \in \mathcal{J}$. This means that $\mathcal{J}$ can be divided into three disjoint parts in terms of g: $\{x \in \mathcal{J} \mid x < g(x)\}$, $\{x \in \mathcal{J} \mid x = g(x)\}$ and $\{x \in \mathcal{J} \mid x > g(x)\}$. We can write the following lemma connecting these two different ways of partitioning $\mathcal{J}$.

Lemma 4.5. *Let $(L, \vee, \wedge, \sim, {}^*, 0, 1)$ be a regular pseudocomplemented Kleene algebra defined on an algebraic lattice. Then:*

(a) $\mathcal{A} = \{x \in \mathcal{J} \mid x \leq g(x)\}$ *and* $\mathcal{J} \backslash \mathcal{A} = \{x \in \mathcal{J} \mid x > g(x)\}$*; and*

(b) *if $g(x) = x$, then x is incomparable with other elements of $\mathcal{J}$.*

Proof. (a) Let $x \in \mathcal{J}$. First, suppose that $x \nleq g(x)$. Because x and $g(x)$ are comparable, $x > g(x)$. The fact that $g(x) \in \mathcal{J}$ means that $g(x) \neq 0$. Then $0 < g(x) < x$ and $x \notin \mathcal{A}$. On the other hand, if $x < g(x)$, then, because $\mathcal{J}$ has at most two levels, $x \in \mathcal{A}$. Finally, if $x = g(x)$ and $x \notin \mathcal{A}$, there exists y such that $0 < y < x$. Because L is spatial, there exists $j \in \mathcal{J}$ such that $j \leq y < x$. Therefore $x = g(x) < g(j)$ and this yields that x is an atom, which is a contradiction.

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(b) Suppose that $g(x) = x$. By (a), x is an atom, so there cannot be $y < x$ in $\mathcal{J}$. If $x \le y$, then $g(y) \le g(x) = x$ gives $g(y) = x$, because x is an atom. Hence $x = g(x) = g(g(y)) = y$. So x is comparable only with itself. □

Let $(L, \vee, \wedge, \sim, ^*, 0, 1)$ be a regular pseudocomplemented Kleene algebra defined on an algebraic lattice. Because $\mathcal{J}$ has at most two levels, $\mathcal{A}$ is the 'lower level' and $\mathcal{J} \setminus \mathcal{A}$ is the 'upper level'. For each x in $\mathcal{J} \setminus \mathcal{A}$, the element $g(x)$ is an atom and $g(x) < x$. Obviously, $\mathcal{A}$ is an antichain, that is, any two elements in $\mathcal{A}$ are incomparable. This implies that also $\mathcal{J} \setminus \mathcal{A}$ is an antichain, because if x and y are elements of $\mathcal{J} \setminus \mathcal{A}$ such that $x < y$, then $g(x)$ and $g(y)$ are atoms and $g(x) > g(y)$, which is not possible.

We define a relation $\simeq$ on $\mathcal{A}$ by

$$x \simeq y \iff x \le g(y).$$

Because each atom x is such that $x \le g(x)$ and $x \le g(y)$ implies that $y = g(g(y)) \le g(x)$, the relation $\simeq$ is a tolerance. For any $x \in \mathcal{A}$, we denote

$$\langle x \rangle = \{x \vee y \mid y \simeq x\} \cup \{g(x)\}. \tag{4.1}$$

Lemma 4.6. *Let $x, y \in \mathcal{A}$.*

(a) $y \in \langle x \rangle$ *if and only if* $g(y) \in \langle x \rangle$ *if and only if* $x = y$.
(b) $\langle x \rangle = \{x\}$ *if and only if* $g(x) = x$.
(c) $\langle x \rangle \cap \langle y \rangle \neq \emptyset$ *if and only if* $x \simeq y$.

Proof. (a) The equivalences follow directly from the definition of $\langle x \rangle$.

(b) Because $g(x) \in \langle x \rangle$ by definition, $\langle x \rangle = \{x\}$ implies that $g(x) = x$. On the other hand, if $g(x) = x$, then $x \simeq y$ implies that $y \le g(x) = x$. Because x and y are atoms, $x = y$. Thus $\langle x \rangle = \{x\}$.

(c) If $x = y$, the claim is clear. Let $x \neq y$ and $x \simeq y$. Then $x \vee y \in \langle x \rangle \cap \langle y \rangle$. Conversely, assume that $z \in \langle x \rangle \cap \langle y \rangle$. It is clear that z is not an atom. Obviously, z cannot be of the form $g(a)$ for any atom a either, because $g(a)$ can belong only to $\langle a \rangle$. Thus $z \notin \mathcal{J}$. Now $z \in \langle x \rangle$ implies that $z = x \vee a$ for some $a \simeq x$ and $z \in \langle y \rangle$ gives $z = y \vee b$ for some $b \simeq y$. Then $x = (y \vee b) \wedge x = (x \wedge y) \vee (x \wedge b)$. Because $x \neq y$ are atoms, $x \wedge y = 0$. Thus $x = x \wedge b$, which gives $x \le b$. Because also b is an atom, $x = b$ and $x \simeq y$. □

Let us define $U = \bigcup\{\langle x \rangle \mid x \in \mathcal{A}\}$. It is clear that the family $\mathcal{H} = \{\langle x \rangle \mid x \in \mathcal{A}\}$ is an irredundant covering of U, because x and $g(x)$ belong only to $\langle x \rangle$ for any $x \in \mathcal{A}$. We denote by R the tolerance induced by $\mathcal{H}$. For each $x \in \mathcal{A}$, the set $\langle x \rangle$ is a block of R. Because R is induced by $\mathcal{H}$,

$$R(x) = \bigcup\{\langle a \rangle \mid x \in \langle a \rangle\} \tag{4.2}$$

for all $x \in U$. Since $\mathcal{A} \subseteq \mathcal{J} \subseteq U$, there are three kinds of element in U. The following corollary is obvious by equations (4.1) and (4.2).

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COROLLARY 4.7. *Let x, y, z be elements of $U = \bigcup\{\langle a\rangle \mid a \in \mathcal{A}\}$.*

(a) *If $x \in \mathcal{A}$, then $R(x) = \langle x\rangle$.*
(b) *If $y \in \mathcal{J}\backslash\mathcal{A}$, then $y = g(a)$ for some $a \in \mathcal{A}$ and $R(y) = R(a) = \langle a\rangle$.*
(c) *If $z \in U\backslash\mathcal{J}$, then $R(z) = \langle a\rangle \cup \langle b\rangle$ for some distinct $a, b \in \mathcal{A}$ such that $z = a \vee b$.*

By applying the conditions of Corollary 4.7 in Lemma 3.2, we can write, for any $x \in \mathcal{J}$,

$$R(x)^{\blacktriangledown} = \{x, g(x)\} \quad \text{and} \quad R(x)^{\blacktriangle} = \bigcup\{\langle y\rangle \mid R(x) \cap \langle y\rangle \neq \emptyset\}. \tag{4.3}$$

Lemma 4.6(c) gives that, for every $x \in \mathcal{A}$,

$$R(x)^{\blacktriangle} = R(g(x))^{\blacktriangle} = \bigcup\{\langle y\rangle \mid \langle x\rangle \cap \langle y\rangle \neq \emptyset\} = \bigcup\{\langle y\rangle \mid x \simeq y\}. \tag{4.4}$$

5. Representation theorem

Let $\mathbb{L} = (L, \vee, \wedge, \sim, ^*, 0, 1)$ be a regular pseudocomplemented Kleene algebra such that its underlying lattice is algebraic. As in Section 4, we denote $\mathcal{H} = \{\langle x\rangle \mid x \in \mathcal{A}\}$ and $U = \bigcup \mathcal{H}$. The tolerance R is induced by the irredundant covering $\mathcal{H}$ of U and the corresponding rough set lattice is denoted by RS. Let us agree that we denote $\mathcal{J}(L)$ simply by $\mathcal{J}$ and that $\mathcal{J}(RS)$ denotes the completely join-irreducible elements of RS.

For any $x \in \mathcal{J}$, we define

$$\varphi(x) = \begin{cases} (\emptyset, R(x)) & \text{if } x < g(x), \\ (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) & \text{otherwise.} \end{cases}$$

If $x \in \mathcal{J}$, then $R(x)$ is a block. Indeed, if $x \in \mathcal{A}$, then $R(x) = \langle x\rangle$ is a block, and if $x \in \mathcal{J}\backslash\mathcal{A}$, then $g(x)$ is an atom and $R(x) = R(g(x)) = \langle g(x)\rangle$ is a block. Thus $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) \in \mathcal{J}(RS)$ for every $x \in \mathcal{J}$ by Proposition 3.6. Furthermore, if $x < g(x)$, then $g(x) \in \langle x\rangle = R(x)$ and $|R(x)| \geq 2$. Therefore $(\emptyset, R(x)) \in \mathcal{J}(RS)$. This means that the map $\varphi\colon \mathcal{J} \to \mathcal{J}(RS)$ is well defined. Note also that if $x = g(x)$, then $x \in \mathcal{A}$ and $R(x) = \langle x\rangle = \{x\}$. This gives that $R(x)^{\blacktriangledown} = R(x)^{\blacktriangle} = \{x\}$ and $\varphi(x) = (\{x\}, \{x\})$.

LEMMA 5.1. *The map $\varphi\colon \mathcal{J} \to \mathcal{J}(RS)$ is an order-isomorphism.*

PROOF. First, we show that $x \leq y$ implies that $\varphi(x) \leq \varphi(y)$. If $x = y$, then, trivially, $\varphi(x) = \varphi(y)$. If $x < y$, then $x \in \mathcal{A}$, $y \in \mathcal{J}\backslash\mathcal{A}$ and $g(y) \in \mathcal{A}$. Therefore $x < g(x)$ and $g(y) < y$ which imply that $\varphi(x) = (\emptyset, R(x))$ and $\varphi(y) = (R(y)^{\blacktriangledown}, R(y)^{\blacktriangle})$. By (4.4), $R(y)^{\blacktriangle} = R(g(y))^{\blacktriangle} = \bigcup\{\langle z\rangle \mid z \simeq g(y)\}$. Since $x \leq y = g(g(y))$, we know that $x \simeq g(y)$ and $R(x) = \langle x\rangle \subseteq \bigcup\{\langle z\rangle \mid z \simeq g(y)\} = R(y)^{\blacktriangle}$, which implies that $\varphi(x) \leq \varphi(y)$.

Second, we show that $\varphi(x) \leq \varphi(y)$ implies that $x \leq y$. We begin by noting that if $\varphi(x) \leq \varphi(y)$, then $x = g(x)$ and $y = g(y)$ are equivalent, and they imply that $x = y$. To see this, suppose that $x = g(x)$. Then $\varphi(x) = (\{x\}, \{x\})$. Now $\varphi(y) = (\emptyset, \langle y\rangle)$ is not possible, because we have assumed that $\varphi(x) \leq \varphi(y)$. Therefore, it must be that $\varphi(y) = (R(y)^{\blacktriangledown}, R(y)^{\blacktriangle})$. This gives $x \in R(y)^{\blacktriangledown} = \{y, g(y)\}$ by (4.3), so that $x = y$ or $x = g(y)$. Also the second equality gives $x = g(x) = g(g(y)) = y$. Analogously, $g(y) = y$ means that $\varphi(y) = (\{y\}, \{y\})$. If $\varphi(x) = (\emptyset, \langle x\rangle)$, then $x \in \langle x\rangle \subseteq \{y\}$ implies that $x = y$

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and $g(x) = g(y) = y = x$. If $\varphi(x) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$, then $R(x)^{\blacktriangledown} = \{x, g(x)\} \subseteq \{y\}$ gives $x = g(x) = y$.

Therefore, we may assume that $x \neq g(x)$ and $y \neq g(y)$. We divide the rest of the proof into four cases:

(i) $x < g(x)$ and $y < g(y)$;
(ii) $x < g(x)$ and $y > g(y)$;
(iii) $x > g(x)$ and $y < g(y)$;
(iv) $x > g(x)$ and $y > g(y)$.

(i) Let $x < g(x)$ and $y < g(y)$. Then $x, y \in \mathcal{A}$ which yields $\varphi(x) = (\emptyset, R(x)) = (\emptyset, \langle x \rangle)$ and $\varphi(y) = (\emptyset, R(y)) = (\emptyset, \langle y \rangle)$. By $\varphi(x) \leq \varphi(y)$ we get $x \in \langle x \rangle \subseteq \langle y \rangle$, which is possible only if $x = y$ by Lemma 4.6.

(ii) Suppose that $x < g(x)$ and $y > g(y)$. Hence x and $g(y)$ are atoms. We have that $\varphi(x) = (\emptyset, R(x)) = (\emptyset, \langle x \rangle)$ and $\varphi(y) = \{R(y)^{\blacktriangledown}, R(y)^{\blacktriangle}\}$. Now $R(y)^{\blacktriangle} = R(g(y))^{\blacktriangle} = \bigcup\{\langle z \rangle \mid z \simeq g(y)\}$. Because $x \in \langle x \rangle \subseteq \bigcup\{\langle z \rangle \mid z \simeq g(y)\}$, we get $x \in \langle z \rangle$ for some $z \simeq g(y)$. Then $z \leq g(g(y)) = y$. Since x and z are atoms, we obtain $x = z$, and therefore $x \leq y$.

(iii) If $x > g(x)$ and $y < g(y)$, then $y \in \mathcal{A}$ and $x \in \mathcal{J} \setminus \mathcal{A}$. Therefore $\varphi(x) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$ and $\varphi(y) = (\emptyset, R(y))$. Now $R(x)^{\blacktriangledown} = \{x, g(x)\} \neq \emptyset$, which contradicts that $\varphi(x) \leq \varphi(y)$. Hence this case is not possible.

(iv) Assume that $x > g(x)$ and $y > g(y)$. Then $g(x)$ and $g(y)$ are atoms and $x, y \in \mathcal{J} \setminus \mathcal{A}$. We know that $\varphi(x) = (\{x, g(x)\}, R(x)^{\blacktriangle})$ and $\varphi(x) = (\{y, g(y)\}, R(y)^{\blacktriangledown}\}$. By $\varphi(x) \leq \varphi(y)$, $\{x, g(x)\} \subseteq \{y, g(y)\}$. If $x = y$, then the proof is complete, and if $x = g(y)$, then $x < y$, because $g(y) < y$.

Finally, we show that the map φ is onto $\mathcal{J}(RS)$. Because R is induced by the irredundant covering $\{\langle x \rangle \mid x \in \mathcal{A}\}$, there are two kinds of element in $\mathcal{J}(RS)$: for each $x \in \mathcal{A}$, there is the rough set $(\langle x \rangle^{\blacktriangledown}, \langle x \rangle^{\blacktriangle})$, and for each $x \in \mathcal{A}$ such that $|R(x)| = |\langle x \rangle| \geq 2$, there exists $(\emptyset, \langle x \rangle)$ in $\mathcal{J}(RS)$. So if $j = (\emptyset, \langle x \rangle)$, then $\varphi(x) = j$. Suppose that $j = (\langle x \rangle^{\blacktriangledown}, \langle x \rangle^{\blacktriangle})$ for some $x \in \mathcal{A}$. Because $x \in \mathcal{A}$, $x = g(x)$ or $x < g(x)$. If $x = g(x)$, then $\langle x \rangle = \{x\}$, $j = (\{x\}, \{x\})$ and $\varphi(x) = j$. If $x < g(x)$, then $\varphi(g(x)) = (R(g(x))^{\blacktriangledown}, R(g(x))^{\blacktriangle}) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) = j$. □

Lemma 5.2. *For all $x \in \mathcal{J}$, $\varphi(g(x)) = g(\varphi(x))$.*

Proof. Because $\mathbb{L}$ is a completely distributive Kleene algebra, there are, by (J1)–(J3), three kinds of element x in $\mathcal{J}$ with respect to the map g: (i) $x < g(x)$; (ii) $x = g(x)$; and (iii) $x > g(x)$. Based on this, we divide the proof into three cases.

(i) If $x < g(x)$, then $\varphi(x) = (\emptyset, R(x))$ and $|R(x)| = |R(g(x))| \geq 2$. Therefore

$$\begin{aligned}\varphi(g(x)) &= (R(g(x))^{\blacktriangledown}, R(g(x))^{\blacktriangle}) = (R(x)^{\blacktriangledown}, R(x)^{\blacktriangle}) \\ &= g((\emptyset, R(x))) = g(\varphi(x)).\end{aligned}$$

(ii) If $x = g(x)$, then $R(x) = \{x\}$ and

$$\varphi(g(x)) = \varphi(x) = (\{x\}, \{x\}) = g(\varphi(x)).$$

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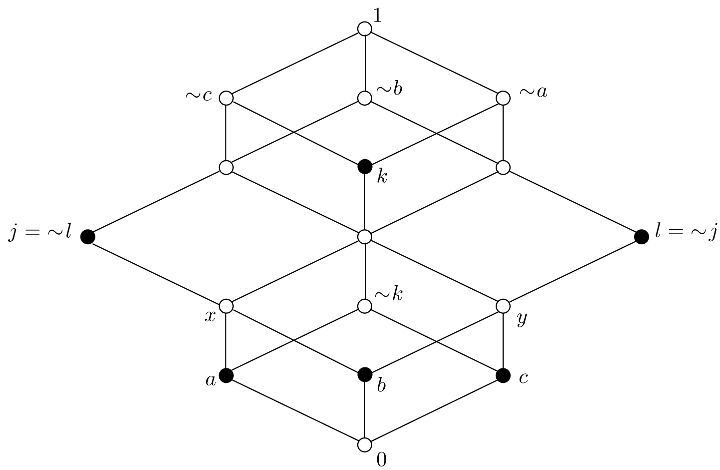

FIGURE 1. A finite regular pseudocomplemented Kleene algebra.

(iii) If $x > g(x)$, then $g(x) < g(g(x))$ and, as in (i), $|R(x)| = |R(g(x))| \geq 2$. Thus

$$\varphi(g(x)) = (\emptyset, R(g(x))) = (\emptyset, R(x)) = g((R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})) = g(\varphi(x)). \qquad \square$$

We proved in [JR11, Corollary 2.4] that if $\mathbb{L} = (L, \vee, \wedge, \sim, 0, 1)$ and $\mathbb{K} = (K, \vee, \wedge, \sim, 0, 1)$ are two De Morgan algebras defined on algebraic lattices and $\varphi \colon \mathcal{J}(L) \to \mathcal{J}(K)$ is an order-isomorphism such that $\varphi(g(j)) = g(\varphi(j))$ for all $j \in \mathcal{J}(L)$, then the algebras $\mathbb{L}$ and $\mathbb{K}$ are isomorphic. By Lemmas 5.1 and 5.2, we can establish the following representation result.

THEOREM 5.3. *Let $\mathbb{L} = (L, \vee, \wedge, \sim, ^*, 0, 1)$ be a regular pseudocomplemented Kleene algebra defined on an algebraic lattice. There exists a set U and a tolerance R on U such that $\mathbb{L} \cong \mathbb{RS}$.*

EXAMPLE 5.4. Let us consider the Kleene algebra $\mathbb{L}$ depicted in Figure 1.

The (completely) join-irreducible elements are marked with filled circles. Now $\mathcal{A} = \{a, b, c\}$ and $\mathcal{J} \setminus \mathcal{A} = \{j, k, l\}$. Note that the elements $\sim x$ are denoted for each $x \in \mathcal{J}$. Because the lattice L is finite, it is algebraic. It is easy to observe that $\mathcal{J}$ has at most two levels. Therefore the pseudocomplemented Kleene algebra $\mathbb{L}$ is regular. By Theorem 5.3, there exists a set U and a tolerance R on U such that the Kleene algebra $\mathbb{RS}$ determined by R is isomorphic to $\mathbb{L}$. Next, we will illustrate this construction.

Now $g(a) = j$, $g(b) = k$ and $g(c) = l$. This means that the tolerance $\simeq$ on $\mathcal{A}$ is such that $a \simeq b$ and $b \simeq c$. For simplicity, we denote $a \vee b$ by x and $b \vee c$ by y. The sets

$$\langle a \rangle = \{a, j, x\}, \quad \langle b \rangle = \{b, k, x, y\}, \quad \langle c \rangle = \{c, l, y\}$$

form an irredundant covering of $U = \{a, b, c, j, k, l, x, y\}$ inducing R.

$$R(a) = R(j) = \langle a \rangle, \quad R(b) = R(k) = \langle b \rangle, \quad R(c) = R(l) = \langle c \rangle,$$
$$R(x) = \langle a \rangle \cup \langle b \rangle, \quad R(y) = \langle b \rangle \cup \langle c \rangle.$$

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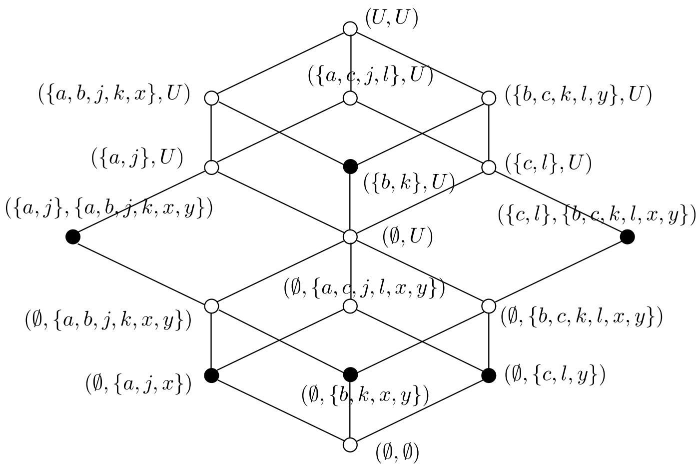

FIGURE 2. A rough set Kleene algebra isomorphic to the Kleene algebra of Figure 1.

The rough set system *RS* induced by the tolerance *R* is depicted in Figure 2. The original Kleene algebra $\mathbb{L}$ is isomorphic to the Kleene algebra $\mathbb{RS}$.

6. Some concluding remarks

Rough set lattices determined by quasiorders and by tolerances induced by irredundant coverings form Kleene algebras such that their underlying lattices are algebraic. Their set of completely join-irreducible elements $\mathcal{J}$ are such that each $x \in \mathcal{J}$ is comparable with $g(x) \in \mathcal{J}$. In this work, we have shown that rough set algebras determined by tolerances induced by irredundant coverings are such that $\mathcal{J}$ has at most two levels, and we proved in case of pseudocomplemented De Morgan algebras defined on algebraic lattices that these are exactly the (congruence-)regular ones. In case of an equivalence E, the set $\mathcal{J}$ of *RS* is such that each $x \in \mathcal{J}$ is comparable *only* with $g(x)$. This means that *RS* is isomorphic to $\mathbf{2}^I \times \mathbf{3}^K$, where I is the set of singleton E-classes and K is the set of E-classes having at least two elements. The regular distributive double pseudocomplemented lattice *RS* defined by an equivalence is, in fact, a regular double Stone algebra, and each regular double Stone algebra isomorphic to a direct product of chains of **2** and **3** defines an equivalence E such that the rough set algebra $\mathbb{RS}$ is isomorphic to the original regular double Stone algebra.

Obviously, we may divide the class of Kleene algebras defined on algebraic lattices into two classes: the ones in which $\mathcal{J}$ has at most two levels, and those whose $\mathcal{J}$ has at least three levels. As we have shown in this work, if $\mathcal{J}$ has at most two levels, then these algebras can be represented by tolerances which are induced by an irredundant covering. On the other hand, consider a Kleene algebra L defined on an algebraic lattice such that there are at least three levels in $\mathcal{J}$. Now we may apply the results of [JR11], where we proved that the ones corresponding to rough sets

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determined by quasiorders are exactly those which satisfy the *interpolation property*: if $x, y \leq g(x), g(y)$ for some $x, y \in \mathcal{J}$, then there exists $z \in \mathcal{J}$ such that $x, y \leq z \leq g(x), g(y)$. Note that rough set systems defined by equivalences satisfy, trivially, this interpolation property, because each $x \in \mathcal{J}$ is comparable only with $g(x)$. Therefore the condition $x, y \leq g(x), g(y)$ is never true for $x \neq y$. Note also that we showed in [JR11, Example 4.4] that the height of $\mathcal{J}$ can be arbitrarily high.

In the future, it would be interesting to study what other kinds of rough set structure can be characterised as the class of Kleene algebras defined on algebraic lattices, by defining conditions on the set $\mathcal{J}$ of completely join-irreducible elements.

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