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SET-THEORETIC BICONTEXTUALISM

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Abstract. Can we quantify over absolutely every set? Absolutists typically affirm, while relativists typically deny, the possibility of unrestricted quantification (in set theory). In the first part of this article, I develop a novel and intermediate philosophical position in the absolutism versus relativism debate in set theory. In a nutshell, the idea is that problematic sentences related to paradoxes cannot be interpreted with unrestricted quantifier domains, while prima facie absolutist sentences (e.g., "no set is contained in the empty set") are unproblematic in this respect and can be interpreted over a domain containing all sets. In the second part of the paper, I develop a semantic theory that can implement the intermediate position. The resulting framework allows us to distinguish between inherently absolutist and inherently relativist sentences of the language of set theory.

§1. Introduction.

Generality absolutism is the claim that it is possible to quantify over absolutely everything. Examples of absolutely general quantification are not hard to find and involve fundamental logical, mathematical, and metaphysical claims, such as 'Everything is self-identical' or 'Nothing is a member of the empty set'. Intuitively, these claims are absolutely unrestricted and their interpretation over a less than all-inclusive domain would be a fundamental misinterpretation. *Generality relativism* is the denial of generality absolutism, i.e., the claim that quantification is always restricted, and that no quantifier ever achieves absolute generality.[1]

Despite its initial appeal, many have questioned absolutism. The strongest argument against it, the *Argument from Paradox* as it is often called, comes from the semantic and set-theoretic paradoxes. In a nutshell, the argument begins by assuming absolutism and, with it, the existence of a domain containing absolutely everything. Now, according to standard model-theoretic semantics, domains of quantification are sets. Yet, Russell's paradox immediately implies that no set can be universal. The reasoning is well-known. Let M be a supposedly all-inclusive set, and consider M's Russell set $R_M := \{x \in M \mid x \notin x\}$. If R_M is in M, which it should be if M is universal, then it follows paradoxically that $R_M \in R_M \leftrightarrow R_M \notin R_M$. Thus, R_M is not in M, whence M is not universal after all. Since this argument can be generalized to apply to *any* kind of collection-like entity, the relativistic Argument from Paradox not only

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[1] Proponents of absolutism include Lewis [29], McGee [33, 34], Rayo & Uzquiano [40], and Williamson [49]. Proponents of relativism include Glanzberg [18], Lavine [28], and Studd [46].

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seemingly disqualifies universal set domains, but also any universal domain based on any other collection-like entity. It follows, relativists claim, that no domain can comprise absolutely everything.

Russell's paradox, and the set-theoretic paradoxes more generally, have led to far-reaching conceptual changes in the foundations of mathematics: the *naïve conception of set* implicitly at work in the early writings of Cantor and Frege, according to which every predicate has a set extension, has been replaced by the nowadays standard *iterative conception of set*. According to this, a set is anything obtainable from some previously given objects, the *urelements*, by iterated application of set construction mechanisms, such as the powerset and the union operation.[2] This elegant and sophisticated conception has been enormously successful, and has lead to the nowadays standard picture of the set universe as a *cumulative hierarchy of sets*.[3]

As Studd [46, chap. 2] among many others observes, though, that the cumulative-iterative account is inherently relativistic. Since each stage in the cumulative hierarchy has a successor stage that strictly extends all previous stages, no stage can be universal. Consequently, it would seem, the cumulative-iterative account is incompatible with absolutism. At the same time, the cumulative-iterative account provides an elegant and widely accepted solution to Russell's paradox and to the set-theoretic paradoxes more generally: every stage has a Russell set, which however enters the cumulative hierarchy only at a later stage. This blocks the paradox outright. But it also confronts us with a seemingly unavoidable decision: we must either embrace absolutism, and forgo the commonly accepted, elegant, and sophisticated solution to the paradoxes, or accept the standard cumulative-iterative account, and give up absolutism.

In a semantic setting, the situation is analogous. The semantic paradoxes—the paradoxes of semantic notions such as truth, satisfaction, and denotation—reveal a tension between naïve semantic principles, such as the interderivability of φ and "'φ' is true", and classical logic. Contextualist approaches to the semantic paradoxes propose to retain classical logic and interpret the paradoxes as evidence that semantic notions are non-naïve and hierarchical. Contextualism provides a solution to the semantic paradoxes that is philosophically well-motivated, technically elegant, and, importantly, fully in keeping with standard hierarchical treatments of the paradoxes of set theory. However, just like in the set-theoretic case, contextualism appears to be inherently relativistic.[4]

In a recent publication, Rossi [42] has shown that, in the truth-theoretical setting, this dilemma is by no means forced upon us, and that there is a way to combine a partial form of absolutism—the view that at least *some* sentences can be given an absolutely unrestricted interpretation—with contextualist, broadly hierarchical solutions to the paradoxes.[5] Following Cartwright [9], Rossi drops the assumption that domains have to be entities, and instead advocates an ontologically neutral reading of the notion

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2 One of the notable achievements of modern set theory is that urelements are dispensable for mathematical concerns. The cumulative hierarchy constructed without urelements is sometimes referred to as the hierarchy of *pure sets*. In what follows, I focus exclusively on sets that are pure in this sense.

3 See Boolos [5], Incurvati [21], Jech [22], Kanamori [23, 24], Kunen [27], Maddy [30, 31], and Potter [38].

4 For general background on contextualist approaches to the semantic paradoxes, see Glanzberg [16, 17], Murzi & Rossi [35], and Parsons [37].

5 This view is further expanded, and defended, in Murzi & Rossi [35].

of a 'domain'.[6] Moreover, he proposes to split up the sentences of the underlying language in those that lead to paradoxes, such as the Liar sentence λ, which says of itself that it isn't true, and those that are unproblematic in this respect, such as, e.g., '$5 + 7 = 12$' or 'Everything is self-identical'. He then proposes a *bipartite* semantics, where the unproblematic sentences are interpreted over an all-inclusive (non-objectual) 'domain', while problematic sentences are interpreted over restricted, set-sized domains. Thus, Rossi shows, we can combine absolutism for unproblematic sentences with contextualist approaches to the semantic paradoxes.

In this paper, I put forward a bipartite semantics for set theory that allows us to combine set-theoretic absolutism with the cumulative-iterative conception of sets. As I show below, just like one can develop a semantic theory that combines partial absolutism with standard hierarchical treatments of the semantic paradoxes, one can also develop a semantics for set theory that combines partial absolutism with the cumulative-iterative conception of sets. According to that position, the paradoxes show that *some* sentences must be relativistically interpreted, whence it is not reasonable to demand that *every* sentence be possibly given an absolutely unrestricted interpretation. At the same time, it *is* reasonable to demand that *some* sentences be given an absolutely unrestricted interpretation. This account, then, provides all the unrestricted quantification one can reasonably ask for, or so I argue. The upshot is a novel conception of quantification in set theory that allows us to retain the universal character of set theory—a theory whose subject matter is the entire cumulative hierarchy, which comprises absolutely all sets—and the open-ended, indefinitely extensible character of this very same hierarchy.

The primary aim is to develop a new semantics for set theory along the lines just sketched. This allows for an elegant, if standard, solution to the set-theoretic paradoxes, while preserving the possibility of quantifying over absolutely all sets. Crucially, I propose to use the bipartite and partially absolutistic semantics to identify a class of *inherently absolutistic sentences of the language of set theory*. These sentences do not have countermodels, neither in the standard sense of the cumulative hierarchy, nor in the all-inclusive model containing all sets. They express, then, the most fundamental facts about the set-concept and thereby shed new light on the concept of *set*.

The paper is structured as follows. §2 sets the stage by outlining the philosophical debate between absolutism and relativism in set theory, with particular emphasis on the cumulative-iterative conception and its role in responding to the paradoxes. §3 introduces the bicontextualist framework in two stages: first through a heuristic explanation (§3.1), and then through a formal development (§3.2–§3.6). §4 applies the framework to the set-theoretic paradoxes, showing that the bicontextualist treatment of paradoxes is in line with the standard treatment in nowadays set theory. §5 explores how the framework allows us to isolate core structural features of the set concept. §6 addresses a number of potential objections and replies. §7 concludes. An appendix provides background material on infinitary logic (A) and its use in formalizing set theory (B).

[6] Within this work, 'domains' (with quotation marks) indicates an ontologically neutral, non-standard reading of domains, according to which 'domains' are not entities, whereas domains (without quotation marks) simply refer to standard set-theoretic, entity-based domains.

§2. Relativism in set theory. The most influential critique of absolutism—again, understood as the claim that absolutely unrestricted quantification is possible—is based on the semantic and set-theoretic paradoxes. The modern version of the argument goes back to Dummett [12], who took the set-theoretic paradoxes to show that the notion of set is *indefinitely extensible*. But the core of the argument can already be traced back to Russell, who similarly took the paradoxes to show that we can never quantify over absolutely all sets.[7]

The argument runs as follows: The truth of absolutism requires a domain containing absolutely everything, where domains are standardly conceived as sets, or—more generally—collection-like entities. However, the notion of a set, and of a collection more generally, is indefinitely extensible, which is taken to imply that no set, or collection, can contain absolutely everything. Hence, the argument concludes, there cannot be a domain containing absolutely everything, and so absolutism must be false. This general version of the Argument from Paradox needs some unpacking, beginning with the somewhat mysterious notion of indefinite extensibility.

Following Incurvati, "a concept C is indefinitely extensible iff whenever we succeed in defining a set M of objects falling under C, there is an operation which, given M, produces an object falling under C but not belonging to M".[8] It is easy to show that the concept *set* is indefinitely extensible in this sense. Let M be a set and consider the subset $R_M \subseteq M$ which contains all and only those sets in M which are not members of themselves, i.e., $R_M := \{x \in M \mid x \notin x\}$. Now reason is that, if R_M is an element of M, we can ask whether R_M is contained in R_M (since R_M is defined to be a subset of M). If it is, it must satisfy the defining condition of R_M, that it is not self-membered, and so R_M cannot be an element of R_M. If R_M is not a member of itself, it satisfies the defining condition of R_M, and therefore must be a member of itself. This implies that $R_M \in R_M \leftrightarrow R_M \notin R_M$, which entails absurdity in any logic at least as strong as minimal logic. We can therefore reject that R_M is an element of M, which implies that there is a set (R_M) not contained in M. In Incurvati's terminology, we started from an object M falling under the concept 'set', and produced an object, R_M, also falling under the concept 'set', but not belonging to M. Since M was completely arbitrary, the reasoning applies to *any* set. Hence, the concept *set* is indefinitely extensible.

The formal derivation of Russell's paradox uses the *naïve comprehension schema*, according to which every definable property $\varphi(x)$ determines a set. Formally:

> **Naïve Comprehension:** For any formula $\varphi(x)$ of the language of set theory which lacks x free and contains no free occurrences of y, the following is an axiom:
>
> $$\exists y \forall x (x \in y \leftrightarrow \varphi(x)).$$

If we let $\varphi(x)$ be $x \notin x$, we get that $x \in y \leftrightarrow x \notin x$.

The indefinite extensibility of the set concept directly implies that there can be no universal set. To see this, assume that U is a universal set. Now consider its Russell set R_U: on pain of contradiction, R_U cannot be in U. So R_U is a set not contained in the supposedly universal set U, whence U cannot be universal after all. The argument,

[7] See Russell [43].
[8] See Incurvati [21, p. 27, notation adjusted].

which is just an instance of Russell's paradox, is fully general, i.e., it makes no specific assumptions about U. It therefore disqualifies any set from being universal.

In what follows, I will exclusively focus on sets. However, I should stress that the argument from indefinite extensibility is not limited to sets: any collection-like *entity* is equipped with a notion of membership, whence any supposedly all-inclusive collection gives rise to a subcollection containing all and only the non-self-membered collections. At this point, the only way to avoid a paradox would seem to require abandoning the assumption of a universal collection.

Relativists claim that the non-existence of a universal set directly proves absolutism wrong. Since, they reason, the all-inclusive domain postulated by absolutists contains absolutely every*thing*, and since sets are things, the all-inclusive domain of absolutism must contain absolutely every set—that is, the universal domain has to be a supercollection of the universal set. However, as the above reasoning shows, there can be no universal set, whence there can't be an all-inclusive domain after all either: no domain can be all-inclusive, which is to say that all quantifier domains, and quantification more generally, are restricted.

The set-theoretic paradoxes have led to far-reaching conceptual changes that affect, to this day, our understanding of the foundations of mathematics. The *iterative conception of set*, and its accompanying intended model of set theory, the *cumulative hierarchy*, have both been developed in reaction to the discovery of the paradoxes. Their immediate aim was to avoid the paradoxes; according to many, however, they genuinely capture essential properties of our conception of sets, or of sets themselves.[9] According to the iterative conception, "a set is something obtainable from the integers (or some other well-defined objects) by iterated application of the operation 'set of'".[10] Instead of postulating sets to be extensions of predicates (as in the naïve, i.e., *pre*-iterative conception), the iterative conception considers sets to be all and only those objects that we obtain from given objects (the *urelements*) by applying accepted *set-forming mechanisms*. Today's standard set theory ZFC is widely regarded as an axiomatization of the iterative conception.[11]

The cumulative hierarchy V organizes sets in an open-ended sequence of stages, beginning at stage V_0 with the empty set, and then iterating the powerset operation at successor stages $V_{\alpha+1}$ and taking unions at limit stages V_γ. The resulting hierarchy forms the ontological counterpart of the iterative conception, in the sense that it provides structurally *nice* models of ZFC.

The cumulative-iterative conception immediately blocks Russell's paradox. From an axiomatic point of view, ZFC replaces the naïve comprehension schema with the more restricted separation schema:

> **Separation:** For any formula $\varphi(x)$ of the language of set theory which lacks x free and contains no free occurrences of y and z, the following is an axiom:
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> $$\forall y \exists z \forall x (x \in z \leftrightarrow x \in y \land \varphi(x)).$$

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[9] See Boolos [5], Dummett [12], Gödel [19], Incurvati [21], Kanamori [23], Parsons [37], Shapiro [45], and Studd [46].

[10] Gödel [19, pp. 474–475].

[11] See Boolos [5], Kanamori [23], Maddy [30, 31], Potter [38], and Zermelo [50].

The difference between naïve comprehension and separation is that naïve comprehension postulates the existence of a set y containing all objects with the property $\varphi(x)$ *outright*, whereas separation only postulates the existence of a set z of all objects satisfying $\varphi(x)$ *which are members of a previously given set y*. This blocks the formal derivation of Russell's paradox, because when we substitute $x \notin x$ for $\varphi(x)$, we get $\forall y \exists z \forall x(x \in z \leftrightarrow x \in y \wedge x \notin x)$, which only gives us the set z of all non-self-membered sets *in y*.

From a conceptual or ontological point of view, the cumulative hierarchy avoids Russell's paradox in virtue of its open-ended character. For every stage V_α in V, the Russell set of V_α is not in V_α, but enters the hierarchy only at the later stage $V_{\alpha+1} = \mathcal{P}(V_\alpha)$—the powerset of V_α. By the open-endedness of the construction, each stage gets expanded in that way. As a result, we only get indefinitely many 'Russell sets', each of which can be harmlessly integrated in V.

As Studd [46, chap. 2] has observed, the cumulative-iterative account is inherently relativistic. Cantor's theorem implies that $\mathcal{P}(V_\alpha)$ is strictly larger than V_α and contains objects which are not in the previous stages. Hence, the sequence of increasing stages never reaches a final stage containing all sets, and no stage achieves absolute generality.

The cumulative-iterative account of set offers an elegant resolution to Russell's paradox by organizing sets in stages. However, as mentioned at the outset, this solution seemingly comes at the cost of having to abandon absolutism: it offers an understanding of sets in terms of an open-ended cumulative hierarchy—one that, by its very nature, would appear to preclude the existence of an all-encompassing absolute domain. In turn, this seemingly undermines the view of set theory as a comprehensive theory about *all* sets.

Relativists are of course aware of the problem and have sought to simulate absolutely general talk by means such as schemata (in the case of Russell) or modal operators (in Studd's case). But, it seems fair to say, these surrogates are all arguably inadequate for a number of reasons. As is well-known, schemata don't allow one to express unrestricted *existential* generalisations. And, as Studd himself acknowledges, primitive modal operators give rise to revenge paradoxes (see Studd [46, chap. 7]).

To summarize, the tension between absolutism and relativism arises from the fact that each view captures an essential aspect of our set-theoretic practice. Absolutism retains the intuitive idea that set theory is meant to be a theory of all sets. Relativism, in contrast, captures the insight that the set-concept is itself inherently hierarchical and open-ended. This latter feature is brought out with particular clarity in the cumulative-iterative conception of sets and its formal counterpart ZFC. However, one must not misunderstand ZFC as a technically convenient theory that is only designed to avoid paradoxes. Rather, it reflects a deeper philosophical understanding of set-concept as an inherently hierarchical concept, with sets being arranged stagewise in the cumulative hierarchy. As such, the cumulative-iterative conception—and with it, ZFC—provides the most compelling conceptual analysis of the set-concept currently available.

§3. Bicontextualism for set theory. At this point, it seems as if we've reached a dilemma: we must choose between unrestricted quantification, on the one hand, and the elegant but relativistic solutions to the paradoxes, on the other hand—we cannot have both. In the truth-theoretic case, Rossi [42] has recently shown that this trade-off can be overcome, and that it is possible to combine absolutism understood as the claim that absolutely unrestricted quantification is possible with the relativistic solutions to the paradoxes.

As Parsons [37] long pointed out, there are obvious similarities between the set-theoretic and the semantic paradoxes. More specifically, in the truth-theoretic case, contextualists postulate an open-ended hierarchy of contexts accompanied by a sequence of growing domains; in the set-theoretic case, the cumulative-iterative conception postulates an open-ended hierarchy of growing ranks. In keeping with this observation, I propose to generalise Rossi's bipartite approach to the semantic paradoxes to the set-theoretic paradoxes and to set theory more generally.

The first step is to notice that the relativistic Argument from Paradox and the prima facie examples for absolutism are, strictly speaking, not incompatible. Absolutists maintain that, e.g., $\forall x\, x = x$ is unproblematic, and requires an absolutely unrestricted interpretation of its quantifier. Relativists, in contrast, argue that we cannot interpret the quantifiers in the Liar sentence or in Russell's paradox over an unrestricted domain, on pain of contradiction, and conclude from this that *no* quantifier can have unrestricted range.

While the argument correctly rejects absolutism for sentences involved in paradoxical reasoning, it does not tell against an absolutely unrestricted interpretation of unproblematic sentences. The problem here is that the failure of absolutism for certain paradoxical sentences simply does not imply the failure of absolutism for *all* sentences.[12] While the paradoxes may indeed undermine a *strong version of absolutism*, understood as the view that it must be possible to provide an absolutely unrestricted interpretation of *all* sentences, they do not tell against absolutism's core tenet, that absolute generality is possible, and that it is possible to provide an absolutely unrestricted interpretation of sentences like $\forall x\; x = x$.

This position is a natural middle ground between strong absolutism and relativism—one that allows us to interpret unproblematic sentences like $\forall x \;\; x = x$ over an all-inclusive, non-objectual 'domain', and problematic sentences like λ over set-sized domain. In a truth-theoretic setting, Rossi [42] calls this *bicontextualism*. Due to the similarities between the semantic and truth-theoretic paradoxes, I apply bicontextualism's key ideas to set theory and the foundations of mathematics. As in the case of truth-theoretic bicontextualism, I allow the interpretation of unproblematic sentences from the language of set theory, such as $\neg\exists x \;\; x \in \emptyset$ over a 'domain' containing absolutely all sets, and problematic sentences, such as $\forall x(x \in y \leftrightarrow x \notin x)$ over restricted domains only.

Let me now develop the bicontextualist approach to set theory, beginning with a first informal presentation of the main ideas, before providing the technical details.

3.1. Heuristics. The key idea of *bicontextualism for sets* is to decide sentence by sentence between an absolutistic and a relativistic interpretation: *inherently relativistic sentences* are interpreted relativistically, in order to avoid paradoxes; in contrast, *inherently absolutistic sentences* are allowed to be interpreted over an all-inclusive 'domain' containing absolutely every set.

Bicontextualism's characteristic feauture is the abandonment of the principle of *unified interpretation*, according to which all sentences of the relevant object language

[12] This rejection of the Argument from Paradox is implicit in Rossi [42] and is further developed in Murzi & Rossi [35].

receive the same interpretation.[13] In contrast, (set-theoretic) bicontextualism allows for a bipartite interpretation of the language (of set theory), on which some sentences can be interpreted over an all-inclusive 'domain', while others can only be interpreted over restricted domains. Ontologically speaking, bicontexualism postulates a *non-objectual* 'domain' containing absolutely everything, and several *objectual* restricted and hierarchically organized domains living inside the big 'domain'. Unproblematic sentences can then be interpreted with the widest of all domains, while problematic and paradoxical sentences will always be restricted to one of the smaller domains inside the big 'domain'.

I should stress that I don't interpret talk of an all-inclusive 'domain' at face value; I rather understand it in a broadly Fregean sense (Williamson [49]). More precisely, although I take the paradoxes to show that all-inclusive domains cannot exist as entities, I also maintain, in line with Rossi's original presentation of bicontextualism, that we can nevertheless understand 'domains' through plurals: they are not reified objects, but rather *pluralities* of things.[14]

As for the restricted domains, I take them to be the strongly inaccessible rank-initial segments of the cumulative hierarchy, that is, domains of the form V_κ for inaccessible κ.[15] This choice is not arbitrary. As argued above, the cumulative-iterative conception captures the hierarchical and open-ended character of the set-concept, and ZFC is widely understood as its formal counterpart. By working with V_κ-domains, the framework retains the structural features that make the relativist approach to the paradoxes both philosophically attractive and technically robust. The restricted domains, then, are not just any set-sized models of set theory, but precisely those that arise from the iterative conception, as axiomatized by ZFC.[16]

It is important not to presuppose the distinction between inherently absolutistic and relativistic sentences, but to construct a semantic theory for the language of set theory in such a way that the distinction actually drops out of the theory. In order to do so, I work with a generalization of standard model theory that is compatible with generality absolutism. According to this approach, there is a *plural 'domain'* containing absolutely all sets.[17] And since the ranks of the cumulative hierarchy are sets, they are all contained in the universal 'domain'.[18]

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[13] See Rossi [42, sec. 4].

[14] For general background on plural logic, see Florio & Linnebo [15] and Oliver & Smiley [36].

[15] When I say that κ must be inaccessible, this also covers stronger notions of infinity. So if κ is Mahlo, measurable, or supercompact, then κ is also inaccessible, and so these ranks are taken into account.

[16] In addition to its conceptual and axiomatic virtues, ZFC is also the most widely accepted foundational theory among working set theorists. As such, focusing on ZFC aligns not only with philosophical considerations about the nature of sets, but also with the naturalistic observation that it serves as the de facto standard in contemporary set-theoretic research. Thanks to an anonymous referee for pressing me to clarify this point.

[17] More on the generalization of standard model theory and plural 'domains' in §3.2.

[18] Depending on the ambient set theory T, there are more or less V-ranks that are set-sized (i.e., T-provably exist). For instance, if $T = \mathsf{ZFC} - \mathsf{Inf}$ (the axiom of infinity), then only finite ranks exist. If, alternatively, $T = \mathsf{ZFC} + \exists! \kappa\ \mathsf{MC}(\kappa)$ (there exists exactly one measurable cardinal κ), than all V-ranks below κ exist. Either way, whatever provably exists according to the ambient theory T is contained as a set in the absolute 'domain'.

Formally, I use a language that has a name for every set—let $\mathcal{L}$ be such a language. When we interpret $\mathcal{L}$ over the 'domain' containing all sets, the resulting 'interpretation' is a total function. When we interpret $\mathcal{L}$ over a restricted domain, say some V_κ in V, the resulting interpretation is a partial function instead. The reason is that, since $\mathcal{L}$ has a name for every set, there are many more names in $\mathcal{L}$ than sets in V_κ (for any κ).[19] Names that do not correspond to sets in V_κ simply do not denote.[20]

I use a three-valued semantics with truth values $\{0, 1/2, 1\}$. If the interpretation function is a partial function, the semantics will be such that every sentence with denotationless terms has value $1/2$, and sentences which only contain denoting names have classical truth values. The main thought is that according to the restricted models, sentences referring objects outside the current domain are nonsensical or off-topic.[21]

In the next step, I interpret all $\mathcal{L}$-sentences over and over again, over all the restricted domains (V_κ's) *and* over the all-inclusive, maximal plural 'domain'. This yields classes of sentences validated by all the different models, which allow me to define the inherently absolutistic and relativistic sentences of $\mathcal{L}$. More specifically, notice that an inherently absolutistic sentence cannot just be taken to be a sentence that is true according to the universal 'domain'.[22] Think of the sentence $\exists\kappa \mathsf{IC}(\kappa)$, saying that there is an inaccessible cardinal. This sentence is false in all V-ranks below κ (assuming κ to be the first inaccessible). Hence, I will not count this sentence as inherently absolutistic, since it has at least one countermodel. I therefore take the inherently absolutistic sentences to be those that are true in the universal 'domain', and have no countermodel.

The inherently relativistic sentences are then defined via the inherently absolutistic sentences. To see this, consider, for some V_κ, the collection of all and only those sentences true in V_κ, and then subtract from it the sentences that are also true over the universal 'domain'. The resulting collection contains all and only those sentences true in V_κ *due to their relativistic nature*. My main claim, then, is that all the sentences of $\mathcal{L}$ that are classified as inherently absolutistic by the above semantics are to be interpreted over the all-inclusive 'domain', whereas all the sentences that are classified as inherently relativistic are to be interpreted over restricted domains. The upshot is a semantic theory that allows us to reconstruct the elegant relativistic approach to the set-theoretic paradoxes within a framework that also contains an all-inclusive 'domain', over which, in spite of the paradoxes, absolutistic sentences can be interpreted.

Using the three-valued semantics and the ability of $\mathcal{L}$ to name every set, we can also consider sentences such as the $\mathcal{L}$-formalization of "for every V_α, there is a V_β, such that

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[19] The use of large languages can be understood as a practical heuristic which does not require deep philosophical justification. If we extend the language of set theory $\mathcal{L}_\in$ to a larger language $\mathcal{L}'_\in$ by adding κ-many names, every $\mathcal{L}'_\in$-structure has a unique $\mathcal{L}_\in$-reduct. This means the semantic framework for such huge languages can always recover the $\mathcal{L}_\in$-fragment if required. However, the names serve a technical purpose, and so I include them in the semantics.

[20] Note the slight abuse of notation here: since $\mathcal{L}$ has a name for every set, it cannot be a set itself. Hence, an interpretation of $\mathcal{L}$ can also not be a set. The reader should bear in mind that all these notions are pluralities. For this reason, I use 'interpretation' (with quotation marks), and interpretation (without quotation marks), as in the case of 'domain' and domain. See fn. 6.

[21] See Beall [2].

[22] For simplicity, I often identify domains with models.

$\alpha < \beta$ and $V_\alpha \in V_\beta$". Such a sentence cannot be formalized in the standard language of set theory—at least not with its standard, absolutist meaning. The point is this: from within a given model V_κ, one can define an internal version of the cumulative hierarchy, with internal ordinals α, β, and internal ranks V_α, V_β, satisfying $V_\alpha \in V_\beta$.[23] The sentence is thus expressible and even provable in the internal language of V_κ. But this is a relativist interpretation: what the sentence expresses is a fact about how a given model internally simulates its own hierarchy. From an absolutist point of view, in contrast, the vocabulary of set theory refers not to what is definable inside some model, but to the actual cumulative hierarchy. In the bicontextualist semantics, the sentence is an example that requires the all-inclusive plural 'domain' containing all sets to express its intended meaning.

Both large cardinal axioms and Russell-style sentences are classified as inherently relativistic in the present framework. The reason is straightforward: their truth is not preserved across all domains. A sentence asserting the existence of a large cardinal κ is false in all V_α with $\alpha < \kappa$, and becomes true only in sufficiently large domains. Similarly, the sentence asserting the existence of the Russell set of V_α is false in V_α, but true in $V_{\alpha+1}$. In both cases, the sentence fails in some domain and is verified in a strictly larger one. According to the classification criterion developed here, this makes them inherently relativistic.

Note that what I call inherently absolutistic and inherently relativistic sentences is inspired by, though not identical with, the distinction between problematic and unproblematic sentences in Rossi [42]. While Rossi's approach primarily focuses on the distinction between paradoxical and non-paradoxical sentences, the set-theoretic case takes into account that any sentence that transcends some ranks of the cumulative hierarchy due to size-restrictions gives rise to wider, more-encompassing ranks. This is particularly important for the case of large cardinal axioms mentioned above. It is important that, even though paradoxical sentences and large cardinal axioms are classified as inherently relativistic by the semantics to be developed below, this does not mean that they are equally problematic in all respects. Nevertheless, from the relativist point of view, they exhibit a crucial structural similarity: both require an act of domain expansion. Working within a given V_α, the sentence asserting the existence of the Russell set of V_α, and the sentence asserting the existence of a set of size $\kappa > \alpha$, are both false in V_α and become true only in some V_β with $\beta > \alpha$. In this sense, both statements demand that we step outside the current set-theoretic universe in order to interpret them as true. The mechanism at work—failure at one level, truth at a higher one—is structurally the same.

Furthermore, what we classify as a paradox often depends on the background context. Russell's paradox is typically regarded as a paradox, but in standard axiomatic contexts (such as ZFC), it is treated as a theorem stating that a universal set cannot exist. The contradiction arises only when one assumes such a set exists. From this perspective, even paradigmatic paradoxes like Russell's share more with large cardinal principles than might initially be assumed: both are sentences whose evaluation highlights the limits of a given domain and the need for extension beyond it. In that respect, the semantic framework developed here justifies grouping them under

[23] As emphasized by an anonymous referee, this would just be formalized in the standard way as $\forall\alpha(Ord(\alpha) \rightarrow \exists\beta(Ord(\beta) \wedge \alpha < \beta \wedge V_\alpha \in V_\beta))$.

the same formal category of inherently relativistic sentences—while still allowing for important conceptual distinctions between them.

It is also important to clarify why the meta-theoretic assumption of large cardinals is unproblematic. Although sentences asserting the existence of large cardinals are classified as inherently relativistic within the object language, this only highlights that some models fail to accommodate them. It does not mean that such sentences are problematic in any stronger epistemic or metaphysical sense. From an absolutist standpoint, which bicontextualism partially retains, there is nothing puzzling about assuming the existence of inaccessible cardinals at the meta level, while simultaneously classifying them as non-absolutist truths about the set concept.[24]

3.2. The RU approach. Rayo and Uzquiano [40] develop a generalization of standard model theory based on higher-order logic that is compatible with generality absolutism.[25]. The main idea of their proposal (the RU approach, as I call it) is to rephrase key principles from standard model theory (*model, variable assignment, satisfaction*) for a first-order object language in a second-order metalanguage. For that purpose, a free monadic second-order variable X is taken to code the notion of a model for a first-order object theory.

Rayo and Uzquiano [40] combine their higher-order semantics with the plural interpretation for higher-order expressions also developed by Boolos.[26] On the plural interpretation, monadic second-order variables do not denote sets or set-like entities, as in standard semantics for second-order logic.[27] Rather, second-order variables introduce a new kind of denotation, so that they *plurally denote* first-order entities.[28] On the plural-interpreted RU approach, then, a 'model' for a first-order language is given by the values that the second-order variable X takes according to the second-order definition they give.

One key advantage of the plural-interpreted RU approach is that it allows us to dispense with what Cartwright [9] famously called the *all-in-one principle*. According to this principle, the domain of a model must always constitute a set or some other collection-like *entity*. In contrast, the RU approach avoids reifying 'domains' and 'models', as 'domains' are not treated as entities themselves but rather as the first-order entities plurally denoted by higher-order variables. As a result, the all-inclusive domain is not a 'thing' in the strict sense, making it immune to the relativistic argument from paradox, which only applies to reified domains, such as sets. This flexibility is pivotal: by rejecting the all-in-one principle and adopting a plural interpretation of higher-order variables, the RU approach circumvents set-theoretic size restrictions and aligns with generality absolutism.

One final note on the object theory: ontologically speaking, the relativistic solution to the paradoxes is based on the cumulative hierarchy. However, this is just a special class of models of set theory, since, after all, it doesn't include the nonstandard models of (first-order) ZFC whose existence follows from the compactness and Löwenheim–Skolem theorems. Following Zermelo, I want to rule out all these nonstandard models

[24] I am especially grateful to Deborah Kant and an anonymous referee whose constructive suggestions have significantly improved this section.

[25] Their paper is based on ideas developed by George Boolos in the 1980s (see Boolos [6, 7])

[26] Ibid.

[27] See Bacon [1], Bell [3], and Shapiro [44].

[28] See Florio & Linnebo [13–15] for general plural logic, and Rayo [39] for plural denotation.

and focus only on structurally nice models in the form of V. Consequently, I need to work with a theory strong enough to yield (quasi-)categorical axiomatizations. There are at least two options available: either work with a second-order version of ZFC, or with an infinitary one. Both give the desired results, and single out only the structurally nice models from the cumulative hierarchy. In what follows, I'll use the infinitary approach.[29] The choice made here is primarily technical, and ZFC_2 would work just as well. Nevertheless, the infinitary approach allows us to work with a form of first-order logic which requires just a modicum of modifications to the RU approach.[30] This means that I have to take an infinitary language as object languages and adjust the definition of satisfaction accordingly. This leads to the following definition.

DEFINITION 3.1. *Let $\mathcal{L}$ be a first-order language which satisfies the following requirements*:

1. *$\mathcal{L}$ has $\in$ as its only relation symbol, and has no function symbols*;
2. *$\mathcal{L}$ has a name for each set*;
3. *for every every ordinal α, $\mathcal{L}(\alpha)$ is the restriction of $\mathcal{L}$ to V_α*;
4. *for ordinals κ, λ, $\mathcal{L}_{\kappa\lambda}$ is the infinitary language resulting from $\mathcal{L}$ by allowing disjunctions of length κ and quantifier sequences of length λ.*

By requirement 1, the only relation constant is $\in$, and the language has no function constants. This simplifies the definitions and comes at no cost since functions can always be defined by relations that are univocal to the right. By requirement 2, the object language has a name for *every* set, which makes the language huge.[31] By requirement 3, the restricted languages $\mathcal{L}(\alpha)$ have names for every set in V_α, where α is the αth inaccessible rank and V_α is the αth inaccessible rank. By regularity of α, $|\mathcal{L}(\alpha)| = |V_\alpha| = |H(\alpha)| = \alpha$. By requirement 4, we have infinitary languages of the form $\mathcal{L}_{\kappa\lambda}$, which allow disjunctions of length $< \kappa$ and quantifier sequences of length $< \lambda$. Requirements 3 and 4 can be combined, so $\mathcal{L}_{\kappa\lambda}(\alpha)$ is the first-order language with a name for each set in V_α, allowing disjunctions of length $< \kappa$ and quantifier sequences of length $< \lambda$. In practice, however, I will only consider cases where $\lambda = \kappa$, so I will use $\mathcal{L}_\kappa$ as short for $\mathcal{L}_{\kappa\kappa}$ (and accordingly $\mathcal{L}_\kappa(\alpha)$) to simplify notation.

3.3. An absolutist-friend semantics for set theory. The definition of a higher-order model for $\mathcal{L}_\kappa(\kappa)$ is identical to the definition of a higher-order model for $\mathcal{L}$, so the difference between standard and infinitary first-order languages will only be important for the definition of the satisfaction predicate. The model predicate will be defined in such a way that it captures the case for $\mathcal{L}$ *and* for $\mathcal{L}(\kappa)$, the restriction of $\mathcal{L}$ to V_κ.

29 For the second-order version ZFC_2, the important theorem is the famous quasi-categoricity theorem from Zermelo [50] (see also Button & Walsh [8, chap. 8.A]), according to which models of ZFC_2 are exactly the strongly inaccessible rank-initials segments of the cumulative hierarchy. Regarding the infinitary version, a version of ZFC developed in an infinitary language is strong enough to characterize the *sets, hereditarily of cardinality less than* κ, $\mathcal{H}(\kappa)$ (see Karp [25]). Moreover, whenever κ is inaccessible, $V_\kappa = \mathcal{H}(\kappa)$, which yields the desired result. See Appendix B for a detailed presentation of the object theory together with the characterization theorem.

30 Using a second-order object theory instead, the definition of the RU model for the second-order language would have to be given in a third-order metalanguage. Then, I'd have to give an interpretation of the third-order variables used in the definitions. And, if I stick with the use of plurals in the metatheory, this would then require *superplurals* in the metametatheory. However, superplural resources are controversial (see Florio & Linnebo [15, chap. 9]).

31 Note that we cannot talk about the cardinality of $\mathcal{L}$ since $\mathcal{L}$ is not a set-theoretic object.

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This will correspond to the intuition given in §3.1 that the interpretation function, when used to interpret $\mathcal{L}$ over all sets, is a total function, and when restricted to a smaller model, is only a partial function. However, this makes it necessary to adapt the classical higher-order semantics in some way: the definitions given in the literature don't consider partial interpretation functions.[32] Formally, the restriction is implemented by requiring that only those constants in $\mathcal{L}(\kappa)$ have denotation, and those in $\mathcal{L} \setminus \mathcal{L}(\kappa)$ have no denotation. For the limit case, where $\mathcal{L}(\kappa) = \mathcal{L}$, $\mathcal{L} \setminus \mathcal{L}(\kappa) = \emptyset$.[33] The definition of a model for a first-order language is given in monadic second-order logic.

Definition 3.2 (First-Order RU-model). *For every unary second-order variable X, the second-order formula "X is an RU-Model for $\mathcal{L}(\kappa)$", in symbols $\mathbb{M}(X)$, is defined as*:

$$\begin{aligned}\mathbb{M}(X) :\leftrightarrow \exists x X(\langle \ulcorner\forall\urcorner, x\rangle) \wedge \forall x\Big[X(x) \rightarrow \big(\exists y(x = \langle \ulcorner\forall\urcorner, y\rangle)\vee \\ \exists u \exists v(x = \langle u, v\rangle) \vee \exists w \exists z(x = \langle \ulcorner\in\urcorner, \langle w, z\rangle\rangle)\big)\Big]\wedge \\ \forall x\Big[\mathsf{Con}_{\kappa}(x) \vee \mathsf{Var}_{\kappa}(x) \rightarrow \exists!\, y\big(X(\langle x, y\rangle) \wedge X(\langle \ulcorner\forall\urcorner, y\rangle)\big)\Big]\wedge \\ \forall x\Big[\mathsf{Con}_{\mathcal{L}\setminus\kappa}(x) \vee \mathsf{Var}_{\mathcal{L}\setminus\kappa}(x) \rightarrow \neg\exists y\big(X(\langle x, y\rangle)\big)\Big]\wedge \\ \forall w \forall z\Big[X(\langle \ulcorner\in\urcorner, \langle w, z\rangle\rangle) \rightarrow X(\langle \ulcorner\forall\urcorner, w\rangle) \wedge X(\langle \ulcorner\forall\urcorner, z\rangle)\Big].\end{aligned}$$

Remark 3.3.

- *An RU-model for $\mathcal{L}(\kappa)$ contains ordered pairs of three different kinds. There are pairs of the form $\langle \ulcorner\forall\urcorner, x\rangle$, encoding that* x *can be quantified on*; $\langle \ulcorner\in\urcorner, \langle x, y\rangle\rangle$, *coding that the pair $\langle x, y\rangle$ is in the extension of the $\in$-relation*; *and* $\langle x, y\rangle$, *where* x *is either an $\mathcal{L}(\kappa)$-constant or an $\mathcal{L}(\kappa)$-variable, which encodes that* y *interprets* x.
- *According to the above definition, X is a 'model' for the language $\mathcal{L}(\kappa)$ if and only if X is non-empty, and X contains interpretations for all $\mathcal{L}(\kappa)$-constants, -variables, and for all the* n*-ary $\mathcal{L}(\kappa)$ relation constants. Moreover, X does not interpret the constants (i.e., names) that are contained in the big language $\mathcal{L}$ but not in the smaller $\mathcal{L}(\kappa)$. This is due to the already mentioned intuition that when we talk about objects outside V_κ with $\mathcal{L}(\kappa)$, this is off-topic.*
- *Recall that I use the plural interpretation for the higher-order variables, so when I speak about X as* a model, *this is actually misleading. Rather, the objects that collectively satisfy the above definition, when combined, interpret the object language $\mathcal{L}(\kappa)$. I will nevertheless often fall back to standard* model-talk *for simplicity. However, the reader should bear in mind that this can always be translated back into more complicated* plural-talk. *Consequently, expressions such as* the model *should not falsely be read as reifications of pluralities into collection-like entities.*[34]

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32 See, e.g., Rayo & Uzquiano [40], Rayo & Williamson [41], Button & Walsh [8, sec. 12.A], Trueman [47, sec. 7.A], and Rossi [42].

33 This is again a slight abuse of notation, since $\mathcal{L}$ is not a set-theoretic object, and so set-theoretic operations are not defined for $\mathcal{L}$. However, the intuition should be clear enough.

34 I will, again, use 'model' and model (with and without quotation marks), as in the case of 'domain' and domain. See fn. 6.

- *Note that unlike the standard approach (in model theory but also the approach presented in Rayo & Uzquiano [40]), in the above definition, the 'model' already contains the variable assignment. This follows the approach developed by Rossi [42] and is only to simplify definitions. It is important for the definition of variations because I have to define them to be variations of 'models' and not just of assignments, which I do next.*

3.3.1. RU-domains and RU-variations. Next I will define the domain component and the variations of variable assignments in the higher-order framework. These definitions will be crucial for the definition of the satisfaction predicate. I start with the domain component.

Definition 3.4 (RU-domain). *For all second-order variables Y and X, the second-order formula "Y is the RU-domain of the RU-model X", in symbols $\mathbb{D}(X, Y)$, is defined as:*

$$\mathbb{D}(X, Y) :\leftrightarrow \mathbb{M}(X) \wedge \forall x\Big(Y(x) \leftrightarrow \exists y(X(y) \wedge y = \langle \ulcorner \forall \urcorner, x\rangle)\Big)$$

The domain component Y of an RU-model X contains all and only those objects (sets), which the RU-model can quantify on.

Next I will formalize the notion of a variation of a variable assignment. As already anticipated in Remark 3.3, variable assignments are part of the definition of an RU-model, and thus a variation of an assignment is a variation of an RU-model. Since the focus is on infinitary object languages, a variation is always defined for a sequence of variables. Let the object language be $\mathcal{L}_\kappa$. Here is the definition of a variation for a sequence of variables $\vec{x_\alpha}$ of length α for $\alpha < \kappa$.

Definition 3.5 (RU-Variant). *For any sequence of first-order variable $\vec{x_\alpha}$, and any second-order variables X, Y, the second-order formula "the RU-model Y is an $\vec{x_\alpha}$-variation of X", where $\vec{x_\alpha}$ is a variable sequence of length $\alpha < \kappa$, in symbols $\mathbb{V}(X, Y, \vec{x_\alpha})$, is defined as:*

$$\begin{aligned}\mathbb{V}(X, W, \vec{x_\alpha}) :\leftrightarrow\ & \mathbb{M}(X) \wedge \mathbb{M}(Y) \wedge \exists Z(\mathbb{D}(X, Z) \wedge \mathbb{D}(Y, Z))\wedge \\ & \forall y(y \neq x_1 \wedge \ldots \wedge y \neq x_\alpha \rightarrow \forall z(X(\langle y, z\rangle) \leftrightarrow Y(\langle y, z\rangle))).\end{aligned}$$

According to Definition 3.5, Y is an RU-$\vec{x_\alpha}$-variant of X if and only if both X and Y are RU-models with the same domain Z, and if they both agree on the interpretation of all variables, expect possibly those in $\vec{x_\alpha}$.

3.4. The satisfaction predicate. Recall that I want to have a semantics that allows me to assign every sentence that has denotationless terms the intermediate truth value 1/2. All other sentences, i.e., all those in which each term denotes, shall receive a classical truth value. For that purpose, I use a *weak Kleene satisfaction predicate*. In the literature on weak Kleene, the third truth value is sometimes called *infectious*, because as soon as a sentence φ has this truth value, any complex sentence that you build with φ also has this truth value. There might be different intuitions on that point. For instance, one might argue that even though φ is meaningless in this sense, a sentence, such as $\varphi \vee 0 = 0$, should still come out true. However, I stick with the off-topic interpretation. The resulting picture is that of a sequence of relativistic models $V_1, \ldots, V_\alpha, \ldots$ (where V_1 is the first inaccessible, V_α is the αth inaccessible and so on) and additionally a plural 'domain', to be denoted by V, which contains all sets. Moreover, I use the languages $\mathcal{L}_1, \ldots \mathcal{L}_\alpha, \ldots$ as well as the big language $\mathcal{L}$ containing names for every set. However, the

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language I'm ultimately interested in is the big language $\mathcal{L}$. When we interpret $\mathcal{L}$ over the sequence of V_κ's, i.e., *as we move up in the cumulative hierarchy*, more and more $\mathcal{L}$-sentences receive a classical evaluation. The limit case is when we look at $\mathcal{L}$ and the all-inclusive 'domain' V, where all terms denote, and where satisfaction is fully classical. Here is the higher-order definition of the weak Kleene satisfaction predicate Sat_κ for the object language $\mathcal{L}_\kappa$ that allows for κ-sequences of quantifiers and disjunctions.

DEFINITION 3.6 (Weak Kleene Satisfaction). *Let X and Y be second-order variables. The second-order formula "the RU-model X with RU-domain Y satisfies the $\mathcal{L}_\kappa$-sentence φ", in symbols $\mathsf{Sat}_\kappa(\varphi, X, Y)$, is defined as follows*: $\mathsf{Sat}_\kappa(\varphi, X, Y)$:*iff*

1. $\mathbb{M}(X)$ *and* $\mathbb{D}(X, Y)$ *and*
2. $\varphi \equiv v_i = v_j$, *there are* y_i, y_j *s.t.* $X(\langle v_i, y_i\rangle)$, $X(\langle v_j, y_j\rangle)$ *and* $y_i = y_j$, *or*
3. $\varphi \equiv v_i \neq v_j$, *there are* y_i, y_j *s.t.* $X(\langle v_i, y_i\rangle)$, $X(\langle v_j, y_j\rangle)$ *and* $y_i \neq y_j$, *or*
4. $\varphi \equiv v_i \in v_j$, *there are* y_i, y_j *s.t.* $X(\langle v_i, y_i\rangle)$, $X(\langle v_j, y_j\rangle)$ *and* $X(\langle \ulcorner\in\urcorner, \langle v_i, v_j\rangle\rangle)$, *or*
5. $\varphi \equiv v_i \notin v_j$, *there are* y_i, y_j *s.t.* $X(\langle v_i, y_i\rangle)$, $X(\langle v_j, y_j\rangle)$ *and* $\neg X(\langle \ulcorner\in\urcorner, \langle v_i, v_j\rangle\rangle)$, *or*
6. $\varphi \equiv \neg\neg\psi$ *and* $\mathsf{Sat}_\kappa(\psi, X, Y)$, *or*
7. $\varphi \equiv \bigvee \Phi$, $\Phi = \{\varphi_\alpha : \alpha < \kappa\} \subseteq \mathsf{For}_\kappa$, *there is some* $\varphi_\alpha \in \Phi$ *s.t.* $\mathsf{Sat}_\kappa(\varphi_\alpha, X, Y)$, *and for all* $\varphi_\beta \in \Phi$, $\mathsf{Sat}_\kappa(\varphi_\beta, X, Y)$ *or* $\mathsf{Sat}_\kappa(\neg\varphi_\beta, X, Y)$, *or*
8. $\varphi \equiv \neg \bigvee \Phi$, $\Phi = \{\varphi_\alpha : \alpha < \kappa\} \subseteq \mathsf{For}_\kappa$, *and for all* $\varphi_\alpha \in \Phi$, $\mathsf{Sat}_\kappa(\neg\psi_\alpha, X, Y)$, *or*
9. $\varphi \equiv \forall \vec{x}_\kappa \psi(Q)$, *and for every W s.t.* $\mathbb{V}(X, W, \vec{x}_\kappa)$, $\mathsf{Sat}_\kappa(\psi(\vec{x}_\kappa), W, Y)$, *or*
10. $\varphi \equiv \neg\forall \vec{x}_\kappa \psi(\vec{x}_\kappa)$, *and for some W s.t.* $\mathbb{V}(X, W, \vec{x}_\kappa)$, $\mathsf{Sat}_\kappa(\neg\psi(\vec{x}_\kappa), W, Y)$,

where For_κ *is a set of* $\mathcal{L}_\kappa$*-formulae.*[35]

The second-order formula $\mathsf{Sat}_\kappa(\varphi, X, Y)$ generates the set of $\mathcal{L}_\kappa$-sentences that are true in the RU-model X with RU-domain Y. Let us adopt the following convention.

REMARK 3.7 (Convention on the enumeration of inaccessible ranks). *The enumeration of V-ranks is from now on restricted to inaccessible ranks, i.e., by* V_1, *I mean* the first inaccessible rank, *by* V_2, *I mean* the second inaccessible rank, ..., *by* V_α, *I mean* the αth inaccessible rank *of the cumulative hierarchy, and so on*.

With Remark 3.7 in place, fix a standard model V_α, i.e., the αth inaccessible rank of the cumulative hierarchy. This generates the set of sentences that are true in V_α.

DEFINITION 3.8. *Let* V_α *be the* α*th inaccessible rank of the cumulative hierarchy. The set of* $\mathcal{L}_\kappa(\alpha)$*-truths of* V_α, *in symbols* $\mathsf{Sat}^\alpha_\kappa$, *is the extension of* Sat_κ *over* V_α, *defined for the restricted language* $\mathcal{L}_\kappa(\alpha)$:

$$\mathsf{Sat}^\alpha_\kappa := \{\varphi \in \mathcal{L}_\kappa(\alpha) \mid \mathsf{Sat}_\kappa(\varphi, X, V_\alpha)\}.$$

[35] To adapt Definition 3.6 to the strong Kleene satisfaction scheme, replace clause 7 by

7^*. $\varphi \equiv \bigvee \Phi$, $\Phi = \{\varphi_\alpha : \alpha < \kappa\} \subseteq \mathsf{For}_\kappa$ and for some $\varphi_\alpha \in \Phi$, $\mathsf{Sat}_\kappa(\psi_\alpha, X, Y)$.

The requirement of 7^* for a disjunction to be satisfied by a model is that one disjunct is satisfied regardless of whether the other disjuncts have a classical truth value. Clause 7 requires instead that all disjuncts have a classical truth value.

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The set of all $\mathcal{L}_\kappa$-truths over the plurality of all sets is just the extension of Sat relative to the RU-model which has V as its domain, i.e., the extension of $\mathsf{Sat}^{\mathsf{V}}_\kappa(\varphi, X, \mathsf{V})$. I also use the overline notation for complements, i.e.,

$$\overline{\mathsf{Sat}^{\mathsf{V}}_\kappa} = \{\neg\varphi \mid \varphi \in \mathsf{Sat}^{\mathsf{V}}_\kappa\},$$

and $\overline{\mathsf{Sat}^{\alpha}_\kappa}$ is the complement of $\mathsf{Sat}^{\alpha}_\kappa$, respectively.

Fixing $\mathcal{L}_\kappa$ and assuming an unbounded sequence of inaccessible models, the above definitions produce the following sequence:

$$\mathsf{Sat}^{\mathsf{V}}_\kappa, \mathsf{Sat}^{1}_\kappa, \mathsf{Sat}^{2}_\kappa, \dots, \mathsf{Sat}^{\alpha}_\kappa, \dots$$

where $\mathsf{Sat}^{\mathsf{V}}_\kappa$ corresponds to the model with domain V, and $\mathsf{Sat}^{\alpha}_\kappa$ corresponds to the model with domain V_α.

3.5. Combining absolutism and realism in the RU approach. Armed with the higher-order semantics and the weak Kleene satisfaction predicate, let me now define the notions of *inherently absolutistic* and *inherently relativistic sentences* of the language $\mathcal{L}_\kappa$, i.e., the language of set theory that has a name for every set and that allows for disjunctions and quantifier sequences of length κ.

I start with the definition of the inherently absolutistic sentences. I first give the definition and unpack it afterwards.

DEFINITION 3.9. *The* absolutely general truths *of $\mathcal{L}_\kappa$, in symbols* Abs_κ, *are defined as follows*:

$$\mathsf{Abs}_\kappa := \mathsf{Sat}^{\mathsf{V}}_\kappa \setminus (\mathsf{Sat}^{\mathsf{V}}_\kappa \cap \bigcup_{\alpha\in\Omega} \overline{\mathsf{Sat}^{\alpha}_\kappa}).$$

The absolutely general $\mathcal{L}_\kappa$-truths are certainly evaluated by the 'model' with 'domain' V, as it can be seen in the first part of the definition. However, we cannot just take these $\mathcal{L}_\kappa$-truths, because the plurality of all sets, under certain large cardinal assumptions, validates sentences, such as 'There are 36 inaccessible cardinals', which are not validated by all ranks smaller than the 37th inaccessible rank of V. We need to subtract all the $\mathcal{L}_\kappa$-sentences on which the 'model' with 'domain' V and all the smaller models disagree. This is done in the second part of the definition: $\bigcup_{\alpha\in\Omega} \overline{\mathsf{Sat}^{\alpha}_\kappa}$ is the union of all complements of collections of $\mathcal{L}_\kappa$-sentences that are true in some standard model. In effect, I subtract all sentences that are true in V, but false in one of the V_α's.

One might ask why not just take as the absolutely general truths just those sentences on which all models agree, i.e., why not define them as follows:

$$\mathsf{Abs}^*_\kappa := \mathsf{Sat}^{\mathsf{V}}_\kappa \cap \bigcap_{\alpha\in\Omega} \mathsf{Sat}^{\alpha}_\kappa.$$

The answer is that this leaves open the possibility that there are sentences which are true according to V, but which are left undecided by all the V_α's. One may have different intuitions about whether or not to count such cases as absolutely general truths. However, I take Abs_κ and not Abs^*_κ as the official definition.

According to the bicontextualist semantics, sentences like the examples 'Nothing is a member of the empty set' or 'Everything is self-identical' are contained in the collection Abs_κ. These sentences are true according to a model that has V as its domain. Moreover, no standard model falsifies any of these sentences.

Let me now define the inherently relativistic truths. In this case, we cannot simply identify the inherently relativistic sentences with the relativistic truths of some standard model, since this would only give us a subcollection of the inherently relativistic sentences. Thus, I start with the relativistic truths of some V_α, i.e., the sentences that are true *in a standard model* by virtue of their inherently relativistic character, and give the generalization to the whole collection of inherently relativistic truths afterwards.

DEFINITION 3.10. *The* relativistic $\mathcal{L}_\kappa(\alpha)$-truths of V_α, *in symbols* $\mathsf{Rel}^\alpha_\kappa$, *are defined as follows*:

$$\mathsf{Rel}^\alpha_\kappa := \mathsf{Sat}^\alpha_\kappa \setminus \mathsf{Abs}_\kappa.$$

$\mathsf{Rel}^\alpha_\kappa$ contains sentences of $\mathcal{L}_\kappa$ which are true in the standard model V_α (the αth inaccessible rank of the cumulative hierarchy), but which are not inherently absolutistic. Consider, for example, the sentence 'There are 36 inaccessible cardinals'. This sentence is true according to the 'model' that has V as its 'domain' (under appropriate large cardinal assumptions), and hence it is in $\mathsf{Sat}^\mathsf{V}_\kappa$. However, it is not contained in any Sat^n_κ for $1 \leq n \leq 36$, and so it is in, say, $\overline{\mathsf{Sat}^{15}_\kappa}$. Consequently, by the definition of inherently absolutistic truths, it is *not* in Abs_κ. But from V_{37} on, the sentence becomes true, and so it is in $\mathsf{Sat}^\alpha_\kappa$ and also in $\mathsf{Rel}^\alpha_\kappa$ for $\alpha > 36$. So we have, for every V_α, the inherently relativistic truths of V_α. To get the inherently relativistic truths *simpliciter*, we take the union of all the relativizations to standard models.

DEFINITION 3.11. *The* inherently relativistic truths *of* $\mathcal{L}_\kappa$, *in symbols* Rel_κ, *are the union of all relativizations of inherently relativistic truths to standard models of the form* V_α:

$$\mathsf{Rel}_\kappa := \bigcup_{\alpha \in \Omega} \mathsf{Rel}^\alpha_\kappa.$$

For example, Rel_κ contains all large cardinal sentences of the form 'There are α inaccessible cardinals' as discussed above, since they all have countermodels and enter $\mathsf{Rel}^\beta_\kappa$ for $\beta > \alpha$. On this approach, then, large cardinal sentences of this form are considered to be relativistic sentences.

REMARK 3.12 (Bicontextualist classification of inherently absolutistic and inherently relativistic truths). *The bicontextualist semantics for the language of set theory classifies sentences as inherently absolutistic and inherently relativistic as follows*:

- *the* inherently absolutistic truths *are just the absolutely general truths of* $\mathcal{L}_\kappa$, *given by* Abs_κ,
- *the* inherently relativistic truths *of* $\mathcal{L}_\kappa$ *are defined as the union of all relativizations to standard models* V_α, *given by* Rel_κ.

The above classification distinguishes two kinds of truths of the language of set theory: we have the inherently absolutistic truths, which express facts about sets that have no countermodels, neither in the cumulative hierarchy nor in the form of the plurality of all sets, and we have the inherently relativistic truths, which express facts about sets that can be true relative to some models and false relative to others.

My proposal is that we think of this distinction as giving each $\mathcal{L}_\kappa$-sentence its intended interpretation. So, for example, the sentence $\forall x(x = x)$ contained in Abs_κ, which is taken to be inherently absolutistic, has as its intended interpretation the plurality of all sets. On the other hand, all sentences contained in one of the collections

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$\mathsf{Rel}_\kappa^\alpha$ always have only restricted quantifier ranges and are therefore to be inherently relativistic. We can approach inherently relativistic sentences in a coarse-grained way, and say that all sentences in Rel_κ are inherently relativistic, or we can approach them in a finer-grained way and consider the relativizations of relativistic truths to standard models as given by the $\mathsf{Rel}_\kappa^\alpha$'s. On this approach, the natural relativisation of a relativistic truth is given by the first model that makes it true. So the natural relativisation of 'there are 36 inaccessible cardinals' is given by the 37th standard model V_{37}. In any case, the intended interpretation for inherently relativistic sentences is not given by a model whose domain is the plurality of all sets.

3.6. A bicontextualist notion of logical consequence. Having defined the sequence of $\mathsf{Rel}_\kappa^\alpha$ and Abs_κ, we can define a bicontextualist notion of consequence that respects intended interpretations. Note that here, we focus on standard models and consider the sequence of $\mathsf{Rel}_\kappa^\alpha$'s rather than Rel_κ.

DEFINITION 3.13. *Let V_α and $\mathcal{L}_\kappa(\alpha)$ be as above (for $\alpha \leq \kappa$), and let $\{\Gamma, \varphi\}$ be a set of $\mathcal{L}_\kappa(\kappa)$-sentences. The fact that the RU-model with RU-domain V_α is a model of φ, in symbols $V_\kappa \models_\kappa^\alpha \varphi$, is defined as follows*:

$$V_\alpha \models_\kappa^\alpha \varphi \textit{ if and only if } \varphi \in \mathsf{Rel}_\kappa^\alpha \cup \mathsf{Abs}_\kappa.$$

The fact that the argument from Γ to φ is bicontextually valid, in symbols $\Gamma \models_\kappa^\alpha \varphi$, is defined as follows:

$$\Gamma \models_\kappa^\alpha \varphi \textit{ if and only if, if all sentences in } \Gamma \textit{ are in } \mathsf{Rel}_\kappa^\alpha \cup \mathsf{Abs}_\kappa, \textit{ so is } \varphi.$$

It is important that the $\models_\kappa^\alpha$-relation is defined for $\mathcal{L}_\kappa(\alpha)$, i.e., the restriction of $\mathcal{L}_\kappa$ to V_α. The reason for this is that otherwise, sentences from larger languages could get into $\mathsf{Rel}_\kappa^\alpha \cup \mathsf{Abs}_\kappa$ via Abs_κ. For example, if φ is a sentence in $\mathcal{L}_\kappa$ but not in $\mathcal{L}_\kappa(\alpha)$ that is in Abs_κ—i.e., an absolutely unrestricted truth that is validated by some V_β for $\beta > \alpha$, remains true from there on, but was undecided by all ranks below V_β—we would end up in the absurd situation where V_α validates sentences about objects outside V_α. To get a bicontextualist notion of logical consequence that respects intended interpretations and is not restricted only to some standard model but considers the plurality of all sets, just replace $\mathsf{Rel}_\kappa^\alpha$ by Rel_κ in Definition 3.13.

§4. Set-theoretic bicontextualist treatment of the paradoxes. Let me now explain how the bicontextualist semantics treats set-theoretic paradoxes. I will focus on Russell's paradox because other paradoxes, such as the Burali-Forti paradox, are treated in a similar way. Consider some rank V_α of the cumulative hierarchy and the sentence

$$\exists x \forall y (y \in x \leftrightarrow y \notin y).$$

Since there is no such set x in V_α, this sentence is false in the sense of both $\models$ and of $\models_\kappa^\alpha$. This is true for any model in the cumulative hierarchy, and also for the absolutist model. However, such a sentence corresponds to an application of the naive comprehension scheme, and not, as it is now standard, to an application of the separation scheme. Nevertheless, it shows that there is no universal Russell set, i.e., there is no set containing all and only the non-self-membered sets. This is also true according to set-theoretic bicontextualism, i.e., according to $\models_\kappa^\alpha$. So we get the following lemma.

LEMMA 4.1. *There is no universal Russell set, i.e., no set R such that*

$$\forall x(x \in R \leftrightarrow x \notin x).$$

Proof. If there were an R such that $\forall x(x \in R \leftrightarrow x \notin x)$, then $R \in R \leftrightarrow R \notin R$. Contradiction. □

Now consider the case of separation. Instead of considering a sentence postulating the existence of a universal Russell set as above, let's consider the case where the universal quantifier is syntactically restricted to the model V_α:[36]

$$\rho_{V_\alpha} := \exists x \forall y(y \in x \leftrightarrow y \in V_\alpha \wedge y \notin y).$$

There is no set of all non-self-membered sets of V_α inside V_α, so in the classical sense $V_\alpha \not\models \rho_{V_\alpha}$. However, since R_α is a subset of V_α, and thus an element of the powerset of V_α, R_α is in $V_{\alpha+1}$ and so $V_{\alpha+1} \models \rho_{V_\alpha}$. Again, if we ask for a Russell set of $V_{\alpha+1}$, we get similar results: $V_{\alpha+1} \not\models \rho_{V_{\alpha+1}}$ but $V_{\alpha+2} \models \rho_{V_{\alpha+1}}$, and so on, all the way up the hierarchy.

Now consider the bicontextualist notion of logical consequence $\models^\alpha_\kappa$. Since $\mathcal{L}_\kappa(\alpha)$ has no term for V_α, the sentence ρ_{V_α} remains undecided over V_α. In the bicontextualist semantics, this is implemented by the fact that the truth value of ρ_{V_α} in V_α is $1/2$. At the next stage, $V_{\alpha+1}$, the sentence becomes true, and so $V_{\alpha+1} \models^\alpha_\kappa \rho_{V_\alpha}$. But again, $\rho_{V_{\alpha+1}}$ has truth value $1/2$ and becomes true only from $V_{\alpha+2}$ on, in the sense described in §2.

Note, however, that there is a slight difference between the relativistic treatment of the paradoxes in the standard setting and the one presented here: while according to $\models$, the existence of a Russell set relative to V_α is strictly false in V_α, according to $\models^\alpha_\kappa$, the corresponding sentence is undecided in V_α. All Russell sentences of this form remain undecided until a point is reached where they are declared true, and then they remain true from that point on. As a consequence of that, the fact that every set has a Russell set is true in the absolute sense. This allows me to prove the following claim.

LEMMA 4.2. *It is an absolutely general truth that every set has a Russell set, i.e., for every set A, there is a Russell sentence*

$$\rho_A := \exists x \forall y(y \in x \leftrightarrow y \in A \wedge y \notin y)$$

such that $\rho_A \in \mathsf{Abs}_\kappa$.

Proof. Let A be a set, α be the smallest ordinal such that $A \in V_\alpha$, and $\kappa > \alpha$ be inaccessible. Then, $V_\kappa \models \mathsf{ZFC}$, and so by the axiom of separation, there is a set $R_A = \{x \in R_A \mid x \notin x\}$. Since $R_A \subseteq A$, and since V_α is transitive, $R_A \in V_\alpha$, so $V_\kappa \models \rho_A$. Since only languages of the form $\mathcal{L}(\kappa)$ have names for A, the truth value of ρ_A is $1/2$ in all V_λ for λ inaccessible $< \kappa$, and from V_κ on, the truth value of ρ_A is 1. Moreover, R_A is a set, and hence it is among the sets, i.e., $R_A \preceq \mathsf{V}$. Hence, $\rho_A \in \mathsf{Sat}^{\mathsf{V}}_\kappa$. This implies that $\rho_A \in \mathsf{Abs}_\kappa$. □

Note that the lemma could also be formulated without making use of the constants A and quantifying over all sets. The sentence would then be

$$\forall z \exists x \forall y(y \in x \leftrightarrow y \in z \wedge y \notin y).$$

[36] Note that this is not the standard notion of a syntactic restriction of the universal quantifier that is in place here, as this would be of the form $\forall y(y \in V_\alpha \rightarrow \ldots)$. However, the right-hand side of the biconditional ensures that $y \in V_\alpha$, which is enough.

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This sentence also comes out true in all strongly inaccessible ranks, as well as in the universal 'domain'. The reason to work with constants is that, from the absolutist point of view, this is a stronger claim. It says that for every set A, the sentence "The Russell set of A exists" is an inherently absolutistic truth of bicontextualism.[37]

What the treatment of the paradoxes in the bicontextualist semantics shows is that sentences of the form ρ_{V_α} must always carry the restriction to a previously given set. But this does not rule out absolutism in general. As the semantics of the previous section suggests, unproblematic or inherently absolutistic sentences can be interpreted over a domain containing all sets without any problems whatsoever. Moreover, the relativistic treatment of the paradox is compatible with absolutism.

§5. Extracting core properties. Bicontextualism for sets addresses a fundamental shortcoming of the standard semantics for set theory, namely, it allows for a truly absolutistic interpretation of the inherently absolutistic sentences of the language of set theory. In my view, an adequate set theory must be about all sets—not just some. My framework captures this requirement through the absolutistic aspect of the semantics, which ensures that, under my interpretation, set theory is truly concerned with its subject matter—the entire universe of sets. As we've seen, this is achieved by allowing the quantifiers of inherently absolutistic sentences to range over absolutely all sets.

The distinction between absolutistic and relativistic sentences is crucial. The bicontextualist semantics allows for an absolutely unrestricted interpretation of those sentences that are intuitively about *all* sets. In particular, sentences like the axiom of extensionality, which provides an identity criterion for sets, are inherently absolutistic. Such sentences cannot have a countermodel, neither in the sense of an initial segment of the cumulative hierarchy, nor in the all-inclusive 'model'. Consequently, I suggest, inherently absolutistic claims express fundamental core properties about the set concept. In contrast, the paradoxes can be seen as placing limitations on the possible interpretation of some of the sentences in the language of set theory. More precisely, the quantifiers of the sentences occurring in the paradoxes cannot have an unrestricted quantifier range, on pain of inconsistency. Set-theoretic bicontextualism respects these limitations by restricting the quantifiers of problematic sentences.

The class Abs_κ contains absolutely general truths of the language of set theory. These sentences are characterized by two features. First, every sentence $\varphi \in \mathsf{Abs}_\kappa$ is true according to the RU model that has all sets in its 'domain'. Second, no sentence $\varphi \in \mathsf{Abs}_\kappa$ has a counter-model in the cumulative hierarchy. Examples are the axiom of extensionality, self-identity claims, and the like. A sentence φ which has these two features expresses fundamental properties of the set concept. Not only is it true in the all-inclusive RU model—i.e., $\varphi \in \mathsf{Sat}_\mathsf{V}$—but also any level in the cumulative hierarchy large enough to decide φ makes it true—i.e., $\varphi \notin \overline{\mathsf{Sat}_\alpha}$ for each α.

The question which sentences go into Abs_κ can be answered, at least partly, via absoluteness results. More precisely, Σ_1 upwards absoluteness and Π_1 downwards absoluteness generalize to the absolutist case: whenever a Σ_1-sentence is validated by some V_α, it remains true not only throughout the hierarchy of V_α's, but also in the

[37] Thanks to an anonymous referee for highlighting this point.

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limit case where the model has as its 'domain' the universe V. Similarly, whenever a Π_1-sentence is validated by the absolutist model V, it remains true in every rank of the cumulative hierarchy. Note, however, that in the case of Π_1 downwards absoluteness, we cannot use the language $\mathcal{L}$, which has a name for every set. The reason is that if φ is a Π_1-sentence of $\mathcal{L}$ of the form $\forall x\psi(x)$, where ψ contains a term that some smaller language $\mathcal{L}(\alpha)$ does not contain, then φ has truth value $1/2$ in V_α even though it has truth value 1 according to V.

The situation is somewhat different for Δ_0-sentences, since their absoluteness depends crucially on the transitivity of the sets, and the universe of sets, taken to be a plurality, is not transitive because pluralities are flat. To see this, consider $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\{\emptyset, \{\emptyset\}\}\}$, which is transitive. However, if we only look at $\{\{\emptyset\}\}$, it is not transitive because it has an element, $\{\emptyset\}$, which is not a subset of $\{\{\emptyset\}\}$, because $\emptyset \notin \{\{\emptyset\}\}$. But of course $\{\{\emptyset\}\}$ is a set, and so it is in the universe of sets. This means that we cannot argue downwards from V by Δ_0 absoluteness. What we can do, however, is to argue upwards, because whenever a Δ_0-sentence is true in some V_α, the set to which the quantifiers are restricted is also in V, and so the sentence remains true. The other direction does not hold. All of the following theorems are generalization of the Δ_0-, Σ_1-, and Π_1-absoluteness theorems.[38]

THEOREM 5.1. *Let φ be a Δ_0-formula of $\mathcal{L}$. $\varphi \in \mathsf{Sat}_\kappa^\alpha$ for some α if and only if $\varphi \in \mathsf{Sat}_{\mathsf{V}}$.*

Proof. If φ contains no quantifiers (i.e., is atomic, or of φ is of the form $\psi \wedge \chi$, $\psi \vee \chi$, $\neg\psi$, or $\psi \rightarrow \chi$ where both ψ and χ contain no quantifiers), then $\varphi \in \mathsf{Sat}_\kappa^\alpha$ if and only if $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$.

Let φ be $(\exists x \in y)\psi(x)$, and $y \in V_\alpha$. Assume for induction that $\psi(x) \in \mathsf{Sat}_\kappa^\alpha$ if and only if $\psi(x) \in \mathsf{Sat}_\kappa^{\mathsf{V}}$.

$\Rightarrow$: If $\varphi \in \mathsf{Sat}_\kappa^\alpha$, then $(\exists x \in y)\psi(x) \in \mathsf{Sat}_\kappa^\alpha$ for some $x \in V_\alpha$ such that $x \in y$. But then, since $V_\alpha \preceq \mathsf{V}$, x and y are in V. Hence, by induction hypothesis, $(\exists x \in y)\psi(x) \in \mathsf{Sat}_\kappa^{\mathsf{V}}$.

$\Leftarrow$: Assume $(\exists x \in y)\psi(x) \in \mathsf{Sat}_\kappa^{\mathsf{V}}$ for some y in V_α. Then, for some $x \in y$, $\psi(x) \in \mathsf{Sat}_\kappa^{\mathsf{V}}$. Since V_α is transitive, $x \in V_\alpha$, and so by induction hypothesis, we get $\psi(x) \in \mathsf{Sat}_\kappa^\alpha$ for $x \in V_\alpha$, and hence $(\exists x \in y)\psi(x) \in \mathsf{Sat}_\kappa^\alpha$. □

COROLLARY 5.2. *Let φ be a Δ_0-formula of $\mathcal{L}$. If $\varphi \in \mathsf{Sat}_\kappa^\alpha$ for some α, then $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$.*

Proof. Immediate from Theorem 5.1. □

THEOREM 5.3. *Let φ be of the form $\exists x\psi(x)$, where $\psi(x)$ is a Δ_0-formula of $\mathcal{L}$ with all free variables displayed. Then $\varphi \in \mathsf{Sat}_\kappa^\alpha$ implies $\varphi \in \mathsf{Sat}_\kappa^\beta$ for all $\beta > \alpha$, and $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$.*

Proof. Let φ be $\exists x\psi(x)$, where $\psi(x)$ is a Δ_0-formula with all free variables displayed and assume that $\varphi \in \mathsf{Sat}_\kappa^\alpha$, i.e., $\mathsf{Sat}_\kappa(\exists x\psi(x), X, V_\alpha)$. Then, there is some $y \in V_\alpha$ such that $\mathsf{Sat}_\kappa(\psi(y), X, V_\alpha)$. Then, since $\psi(y)$ is Δ_0, by Theorem 5.1, $\mathsf{Sat}_\kappa(\psi(y), X, \mathsf{V})$, and since y is a set and therefore in V, $\mathsf{Sat}_\kappa(\exists x\psi(x), X, \mathsf{V})$, i.e., $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$. □

THEOREM 5.4. *Let φ be of the form $\forall x\psi(x)$, where $\psi(x)$ is a Δ_0-formula of $\mathcal{L}(\alpha)$ with all free variables displayed. Then $\varphi \in \mathsf{Sat}_\kappa^\alpha$ implies $\varphi \in \mathsf{Sat}_\kappa^\beta$ for all $\beta < \alpha$, and $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$ implies $\varphi \in \mathsf{Sat}_\kappa^\alpha$ for all α.*

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[38] See Devlin [10, sec. I.8].

Proof. Let φ be $\forall x\psi(x)$, where $\psi(x)$ is a Δ_0-formula with all free variables displayed and assume that $\varphi \in \mathsf{Sat}_\kappa^{\mathsf{V}}$, i.e., $\mathsf{Sat}_\kappa(\forall x\psi(x), X, \mathsf{V})$. Then, for each $y \preceq \mathsf{V}$, $\mathsf{Sat}_\kappa(\psi(y), X, \mathsf{V})$. Fix any V_α. Since $V_\alpha \preceq \mathsf{V}$ and since $\psi(y)$ is Δ_0, by Theorem 5.1, $\mathsf{Sat}_\kappa(\psi(y), X, V_\alpha)$ for all $y \in V_\alpha$. But then, $\mathsf{Sat}_\kappa(\forall x\psi(x), X, V_\alpha)$, i.e., $\varphi \in \mathsf{Sat}_\kappa^\alpha$. □

The above results can be used to get a clearer picture of which sentences are in Abs_κ: since we have Σ_1 upwards absoluteness, all Σ_1-sentences that are true in the first inaccessible segment will remain true throughout the whole bicontextualist framework. Consequently, all Σ_1-truths of V_1 are in Abs_κ. Moreover, since all Π_1-sentences that are true in the universe remain true no matter how far down the hierarchy we go, they also have no counter-models. Consequently, all Π_1-truths of V are in Abs_κ. Finally, all Δ_0-sentences that are true in any V_α remain true in all other V_β's *and* in V, so they're also contained in Abs_κ. So, if

$$A \subseteq_{\Sigma_1} \mathsf{Sat}_\kappa^\alpha, \qquad B \subseteq_{\Pi_1} \mathsf{Sat}_\kappa^{\mathsf{V}}, \qquad C \subseteq_{\Delta_0} \bigcup_{\alpha\in\Omega} \mathsf{Sat}_\kappa^\alpha$$

are the sets of Σ_1-sentences in $\mathsf{Sat}_\kappa^\alpha$, of Π_1-sentences in $\mathsf{Sat}_\kappa^{\mathsf{V}}$, and of Δ_0-sentences from all the $\mathsf{Sat}_\kappa^\alpha$'s, then

$$A \cup B \cup C \subseteq \mathsf{Abs}_\kappa.$$

Moreover, the ZFC-axioms are absolutely general truths of the language of set theory. This is a corollary of the following theorem.

THEOREM 5.5. *The universal domain satisfies the* ZFC *axioms, i.e., for all* $\varphi \in \mathsf{ZFC}$, $\mathsf{Sat}_\kappa(\varphi, X, \mathsf{V})$.

The proof consists of verifying that all ZFC-axioms hold in V. I write $\mathsf{V} \models \varphi$ for $\mathsf{Sat}_\kappa(\varphi, X, \mathsf{V})$. Most cases can be adopted from the proof of that inaccessible ranks are models of ZFC. However, it is crucial to know that V, the universe of sets, is closed under set-of operations as encoded by ZFC *by assumption*. This follows from the definition of the cumulative hierarchy V.

There is another important aspect: many steps in the proof that inaccessible ranks are models of ZFC start by showing that if a particular set or sets are contained in some V_κ, then other sets are contained in V_κ as well. For instance, if $x \in V_\kappa$, then so is the powerset of x, its unionset, etc. Then, the proof proceeds by using Δ_0-absoluteness to show that not only is the particular set contained in V_κ, but V_κ also *thinks* that it contains the particular set, and therefore satisfies the axiom in question. This last step is crucial. It relies on the background assumption that the particular set we're looking for is contained in the background universe which, by assumption, has all the sets, is a cumulative hierarchy, and therefore is always right about what sets there are and how they are structured. Then, since the definition of these sets is Δ_0, it follows that the truth of the particular axioms in the background universe carries over to the V_κ. Theorem 5.5 operates directly on the background universe. This just makes explicit what is usually taken to be implicit in the proof. However, this has the direct implication that the second step—using Δ_0-absoluteness to show that the V_κ satisfies the respective axiom—is not needed anymore. Therefore, most steps of the proof aim to show that if some set is given, then its powerset, unionset, etc. is also a set. This implies that they are contained in the background universe, and so the background

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universe, which is always right about what sets there are and how they are structured, satisfies the respective axioms. Here is the proof.

Proof.

Extensionality Let $x, y \preceq \mathsf{V}$ be distinct. Then, there is a set $z \in x \wedge z \notin y$ (or *vice versa*), and since z is a set, $z \preceq \mathsf{V}$. Then,

$$\mathsf{V} \models \forall x \forall y (x \neq y \rightarrow \neg \forall u (u \in x \leftrightarrow u \in y)),$$

which is equivalent to the axiom of extensionality.

Separation Let $\varphi(y)$ be a first-order formula such that $F = \{y \mid \varphi(y)\}$. Let x be a set. Then the intersection $F \cap x = \{y \in x \mid \varphi(y)\}$ exists by first-order Separation. Hence, $F \cap x$ is a set and therefore $F \cap x \preceq \mathsf{V}$.

Pairing Let x, y be sets. Since V is closed under set-of operations, $a = \{x, y\}$ is a set and therefore in V.

Union Let x be a set. Then, all $y \in x$ are sets as well, and $z = \{y \mid y \in x\} = \cup x$ is also a set and therefore contained in V.

Powerset If x is a set, then each subset $y \subseteq x$ is a set, and so is their collection $\mathcal{P}(x)$.

Infinity $V_\omega \preceq \mathsf{V}$.

Foundation The set-theoretic universe is well-founded by assumption.

Replacement Let $\psi(u, v)$ be a first-order formula such that $G = \{\langle u, v\rangle \mid \psi(u, v)\}$ and suppose that ψ defines a functional relation on x, i.e., for every $u \in x$, there exists a unique v such that $\psi(u, v)$. Then, by the first-order Replacement schema, the image set $y = \{v \mid \exists u \in x \;\; \psi(u, v)\}$ exists and is therefore a set. Hence, $G[x] \preceq \mathsf{V}$.

Choice The universe of sets is well-orderable by assumption. □

COROLLARY 5.6. *The* ZFC *axioms are absolutely general truths of the language of set theory.*

Proof. Each inaccessible rank of V is a ZFC-model. Moreover, by Theorem 5.5, the ZFC-axioms are in $\mathsf{Sat}^{\mathsf{V}}_{\kappa}$. Consequently, by Definition 3.9, the ZFC-axioms are in Abs_{κ}. □

§6. Objections and replies.

Let me now consider two potential objections against the semantics for set theory developed above. The first one is based on the observation that forms of absolutism can be recovered in ZFC via reflection principles or via inner models. The second is that set-theoretic bicontextualism is too revisionary and might therefore be anti-naturalistic.

6.1. Reflection principles and inner models. A first objection to bicontextualism for sets is that absolutism in set theory can be achieved via reflection principles or via inner models. Therefore, an extra absolutist interpretation is redundant, since truths about all sets can be extracted from reflection or inner models.

The main idea of reflection principles is the following: the cumulative hierarchy is so complex that it cannot be characterized by any formula φ of the language of set theory. The reason is that any such formula that is true in the universe of all sets is already true in some initial segment V_α. And so any attempt to characterize the cumulative

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hierarchy as the collection of all things that satisfy φ fails, because there is always a V_α such that φ characterizes V_α.

For example, V is closed under replacement and powerset, but there is some V_κ, for κ an inaccessible cardinal, which is already closed under replacement and powerset, and hence, closure under replacement and powerset is not a unique characterization of V. Gödel therefore concludes that "[t]he universe of all sets is structurally indefinable".[39] The truth of any such description is already *reflected* by some rank.

Since reflection principles are usually presented as biconditionals, they work both ways:[40] not only are truths about all sets reflected by initial segments, but also facts about the initial segments hold in the whole hierarchy. In this way, we can extract facts about the cumulative hierarchy, i.e., facts where the quantifiers can be taken to be unrestricted, from facts about initial segments.

Similarly, one could argue that absolutism can be mimicked by an inner model. Starting from the language of set theory ZFC, and a standard model V_κ of ZFC, an *inner model* is a definable transitive class $\mathcal{N}$ in V_κ, such that $\in^{\mathcal{N}} = \in^{V_\kappa} \restriction dom(\mathcal{N})^2$, $\mathcal{N} \models$ ZFC, and where $\mathcal{N}$ contains all the ordinals of V_κ. This last fact is crucial: Since inner models contain all ordinals, it is—depending on which inner model exactly we consider—at least close to absolutism. But if this is the case, then inner models at least allow us to mimic absolutism.

However, reflection principles and inner models do not quite provide what the bicontextualist wants, i.e., a maximally general interpretation of the set-theoretic quantifiers in the case of unproblematic sentences. Moreover, neither reflection principles nor inner models show that paradoxes do not force us into relativism. So even if absolutism can be achieved by such techniques, this does not mean that bicontextualism can be achieved by reflection or inner models.[41]

6.2. Anti-naturalism. Another line of criticism might arise from a naturalistically minded philosopher, and might look roughly as follows: Mathematicians work with ZFC (or some extensions thereof with large cardinal or forcing axioms), but they do not use bicontextualist semantics, and so neither should the philosopher.

I believe that, in the end, such criticism is misguided. There is no doubt that ZFC is the widely accepted canonical axiomatization of set theory, and I neither intend nor have the ability to change this. However, it is important to clarify what my proposal is, and what it is not.

Bicontextualism, at its core, classifies sentences in the language of set theory as either inherently absolutistic or inherently relativistic. The debate between absolutists and relativists is primarily a philosophical one, and my main goal is to contribute to this discussion. My use of languages with names for every set and the RU approach should be seen as methodological tools in pursuit of this philosophical aim.

Since the standard first-order language of set theory $\mathcal{L}_\in$, which includes only $\in$ as a non-logical constant, is a sublanguage of my all-inclusive language $\mathcal{L}$, which contains a name to every set, we can easily recover the inherently absolutistic $\mathcal{L}_\in$-sentences from the inherently absolutistic $\mathcal{L}$-sentences. In essence, I am using a distinct methodological

39 Wang [48, p. 280].

40 See, e.g., Devlin [10, pp. 25–26].

41 Thanks to Leon Horsten and Giorgio Venturi for emphasizing this point.

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tool to achieve a philosophical goal, rather than challenging the mathematical canon. There is nothing anti-naturalistic about this approach.

What I explicitly do not intend is to settle inherently mathematical and set-theoretical questions such as 'What sets are there?'. This question can be reformulated as 'What is the ultimate large cardinal axiom to adopt?'. Answering this question is a task for set theorists. If, however, set theorists settle on a particular axiomatization, say $\mathsf{ZFC} + \mathsf{V} = \mathsf{UltL}$, then my approach can still be applied to identify the inherently absolutistic and the inherently relativistic sentences.

6.3. Isn't that just generality absolutism? The main claim of absolutism is that for a given language $\mathcal{L}$, there is an interpretation according to which the $\mathcal{L}$-quantifiers range over absolutely everything. Absolutists do not claim, however, that the $\mathcal{L}$-quantifiers *always* range over absolutely everything, but only that they *sometimes* range over absolutely everything. Since bicontextualism also sometimes allows unrestricted quantification, a natural objection to bicontextualism is that it is, after all, just a version of generality absolutism. This, however, is not correct.

Even though one might paraphrase absolutism and bicontextualism as the claim that quantifiers *sometimes* have an all-inclusive quantifier domain, the interpretation of *sometimes* is a different one: according to absolutism, *sometimes* means, that there is one $\mathcal{L}$-interpretation according to which all $\mathcal{L}$-quantifiers range over an unrestricted domain. According to bicontextualism, *sometimes* means that only a special class of $\mathcal{L}$-sentences is adequate for an absolutistic interpretation.

This explains why bicontextualism really is an intermediate position: according to relativism, no sentence can ever achieve absolute generality, and according to absolutism, all sentences can sometimes achieve absolute generality. According to bicontextualism, however, some sentences can never achieve absolute generality (relativistic aspect), while other sentences can sometimes achieve absolute generality (absolutistic aspect). In this sense, bicontextualism is neither fully absolutistic, nor fully relativistic.

§7. Conclusion. As noted above, relativism is a strategy for avoiding set-theoretic paradoxes. Dummett has argued that paradoxes prove absolutism wrong. But the main lesson of this paper is that paradoxes do not force us into strict relativism, since absolutism can be combined with a relativistic treatment of paradoxes. A more cautious assessment shows that paradoxes only force us into relativism only with respect to sentences that are crucial to the paradoxes. Unproblematic sentences can be interpreted absolutistically even in the light of the paradoxes.

The development of a semantic framework that implements such an intermediate position between absolutism and relativism provides us with a semantic criterion for distinguishing between inherently absolutistic and inherently relativistic sentences. This shows that some sentences of the language of set theory are most naturally interpreted as expressing facts about only some sets, while other sentences are most naturally interpreted as expressing facts about all sets.

Moreover, a partly absolutist, partly relativist semantics such as bicontextualism for sets can be used to extract core properties of the set concept. The semantics can be used to isolate a class of sentences of the language of set theory with the special status of being true by the model containing all sets in its domain, and that are not satisfied by any standard model in the cumulative hierarchy.

§A. Infinitary logic. This brief summary of infinitary logic is not intended to be a complete presentation.[42] I start with a first-order language $\mathcal{L}$ and show how to extend its syntax and semantics to get an infinitary language $\mathcal{L}_{\kappa\lambda}$, which allows disjunctions of length $< \kappa$ and quantifier sequences of length $< \lambda$.

The first step is the syntax: we extend $\mathcal{L}$ by a set of variables, $\mathsf{Var}_{\kappa\lambda}$ of cardinality κ, and a logical operator $\bigvee$ for infinite disjunction. The atomic formulae of $\mathcal{L}_{\kappa\lambda}$ are just the atomic formulae of $\mathcal{L}$.

As for complex formulas, we add an extra clause to the recursive definition, saying that whenever Φ is a set of $\mathcal{L}_{\kappa\lambda}$-formulas such that $|\Phi| < \kappa$, then $\bigvee \Phi$ is a $\mathcal{L}_{\kappa\lambda}$-formula. $\bigvee \Phi$ has the form $\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_\alpha \vee \dots$ for all $\varphi_\alpha \in \Phi$, and we can also write $\bigvee_{\alpha\in\beta} \varphi_\alpha$ (assuming that $|\Phi| = \beta$.)

We add a similar clause for the quantifier case: for X a set of $\mathcal{L}_{\kappa\lambda}$-variables such that $|X| < \lambda$, $\exists X\varphi$ is a formula. We can also write $(\exists x_\alpha)_{\alpha<\beta}\varphi$, provided that $|X| = \beta$. Infinite conjunctions and infinite universal quantification are defined according to the usual conventions:

- $\bigwedge \Phi := \neg \bigvee\{\neg\varphi : \varphi \in \Phi\}$;
- $\forall X\varphi := \neg\exists X\neg\varphi$.

In this characterization, standard first-order logic is just the language that allows only finite disjunctions and finite sequences of quantifiers. In the above terminology, we can write $\mathcal{L}$ as $\mathcal{L}_{\omega\omega}$.

Now for the semantics. An $\mathcal{L}_{\kappa\lambda}$-structure is just an $\mathcal{L}$ structure, and we just need to add the following clauses corresponding to infinite disjunction and infinite existential quantification to the standard first-order definition of satisfaction, where $\mathcal{M}$ is an $\mathcal{L}$-structure and σ is a variable assignment relative to $\mathcal{M}$:

- $\mathcal{M}, \sigma \models \bigvee \Phi$ iff $|\Phi| < \kappa$ and for some $\varphi_\alpha \in \Phi$, $\mathcal{M}, \sigma \models \varphi_\alpha$,
- $\mathcal{M}, \sigma \models \exists X\varphi(X)$ iff there is an X-variant σ_0 such that $\mathcal{M}, \sigma_0 \models \varphi(X)$.

Validity and *logical truths* are defined as usual.

However, there are some restrictions on the cardinals κ and λ:

- $\lambda \leq \kappa$: if $\lambda > \kappa$, there would be $\mathcal{L}_{\kappa\lambda}$-formulas with at most κ-many free variables, but with λ-many quantifiers, most of which would have no free variables to bind.
- κ must be a regular cardinal. If κ is singular, the languages $\mathcal{L}_{\kappa\lambda}$ and $\mathcal{L}_{\kappa^+\lambda}$ have the same expressive power in the sense that an $\mathcal{L}_{\kappa^+\lambda}$-sentence φ can always be converted into an $\mathcal{L}_{\kappa\lambda}$-sentence φ^* with the same meaning. However, this is not a requirement for λ because there is a difference in meaning between 'there are at least λ-many', and 'there are at least λ^+-many', even if λ is singular.[43]

To give just a very brief idea of the power of infinitary languages, consider how such languages can uniquely characterize important mathematical notions such as *finiteness*. This can be done by a single $\mathcal{L}_{\omega_1,\omega}$-sentence, the language allowing countably infinite disjunctions and only finite sequences of quantifiers:

$$\bigvee_{n<\omega} \exists x_1, \dots, \exists x_n \forall y(y = x_1 \vee \dots \vee y = x_n).$$

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42 For a more detailed presentation, the reader is referred to Bell [4], Karp [25], and Marker [32].

43 See Dickmann [11, p. 85 and p. 139] for details.

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We can use them to uniquely characterize important mathematical structures such as the natural numbers. The first-order models isomorphic to $\mathbb{N}$ are just the class of models of the $\mathcal{L}_{\omega_1\omega}$-sentence

$$\mathsf{PA}^- \wedge \forall x \bigvee_{n<\omega} x = succ^n(0),$$

where PA^- is PA without the induction scheme, and $succ^n(0)$ is the result of applying the successor operations n-times to 0.[44]

§B. Characterization of $\mathcal{H}(\kappa)$ in $\mathcal{L}_{\kappa^+\kappa^+}$. We can uniquely characterize the structure $\mathcal{H}(\kappa)$, the class of sets hereditarily of power $\leq \kappa$, in $\mathcal{L}_{\kappa^+\kappa^+}$. To do this, I will first define the $\mathcal{H}(\kappa)$'s. Then, I will show how to properly characterise the notion of well-foundedness in $\mathcal{L}_{\omega_1\omega_1}$. Finally, I show that any structure $\mathcal{M}$ on which $\in$ is well-founded (i.e., which satisfies the $\mathcal{L}_{\omega_1\omega_1}$-characterization of well-foundedness) and which satisfies the $\mathcal{L}_{\kappa^+\kappa^+}$-sentences, which intuitively says that all sets are at most of power κ and that the domain consists of transitive sets only, is isomorphic to $\mathcal{H}(\kappa)$.

Let's start with the definition of the $\mathcal{H}(\kappa)$'s: recall that a set X is *transitive* if $x \in X$ implies $x \subseteq X$. The *transitive closure* of a set X, $TC(X)$, is defined as the intersection of the set of all transitive supersets of X, i.e.,

$$TC(X) := \bigcap \{Y : X \subseteq Y \text{ and } Y \text{ is transitive}\}.$$

Definition B.1 (Sets Hereditarily of Power $\leq \kappa$). *For any cardinal κ, the set of all sets hereditarily of cardinality less than κ, $H(\kappa)$, is defined as follows*:

$$H(\kappa) := \{X : |TC(X)| < \kappa\}.$$

If GCH holds, then for all ordinals α, $H(\alpha) = V_\alpha$. However, if GCH does not hold, for $\kappa > \omega$, the cardinality of $H(\kappa)$ is $2^{<\kappa}$, while the cardinality of $V_\kappa = \beth_\kappa$. However, $H(\kappa) = V_\kappa$ whenever $\beth_\kappa = \kappa$, i.e., for all fixed points of the $\beth$-function.[45] But the $\beth$-function has arbitrarily large fixed points: if κ is strongly inaccessible, $\kappa = \beth_\kappa$.[46] So, after all, $H(\kappa) = V_\kappa$ whenever κ is strongly inaccessible, independent of GCH.

The above definition gives the class of sets hereditarily of power $\leq \kappa$. We can now define the corresponding structure.

Definition B.2. *Let $H(\kappa)$ be the set of all sets hereditarily of cardinality less than κ. $\mathcal{H}(\kappa)$ is defined as the structure having $H(\kappa)$ as its domain and restricting $\in$ to $H(\kappa)$, formally*

$$\mathcal{H}(\kappa) := \langle H(\kappa), \in \cap (H(\kappa) \times H(\kappa))\rangle.$$

Next we define the notion of well-foundedness in $\mathcal{L}_{\omega_1\omega_1}$. Recall that a relation E on W is well-founded iff every non-empty subset $W_0 \subseteq W$ has an E-least element. More precisely, we say that E is *well-founded* on E, i.e., for every $\emptyset \neq W_0 \subseteq W$, there exists an $x \in W_0$ such that for all $y \in W_0 : \neg yEx$. Alternatively, well-foundedness can be characterized by the absence of infinite descending chains.

44 For more on both, see Keisler [26] or Marker [32].
45 See Kunen [27, p. 78, lemma I.13.29].
46 Ibid. Lemma I.13.31.

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LEMMA B.3. *Let W be a set and E be a relation of W. Then, E is well-founded on W if and only if there is no infinite descending E-chain of elements of W.*

Proof. $\Rightarrow$: Let E be well-founded on W and suppose, for *reductio*, that $\langle w_n \rangle_{n\in\omega}$ is an infinite descending E-chain in W, i.e., that all tuples of the form $w_n E w_{n+1}$ are in the extension of E. Let $S = \{w_0, w_1, w_2, ...\}$ and assume that w_i is the E-least element in S, i.e., there are no $w_j \in S$ such that $w_j E w_i$. This element exists because E is well-founded. But then, there is w_{i+1} such that $w_{i+1} E w_i$. Contradiction.

$\Leftarrow$: We prove the contrapositive. Suppose E is not well-founded in W. Then, there exists $\emptyset \neq S \subseteq W$ with no E-minimal element. By a weak form of the axiom of choice, we can construct a sequence $\langle w_n \rangle_{n\in\omega}$ such that $\forall n \in \mathbb{N} : w_{n+1} E w_n$. This is an infinitely descending E-chain. □

In ZFC, the axiom of foundation expresses as much of the well-foundedness of $\in$ as is possible in first-order logic. However, due to the compactness of first-order logic, there are ZFC-models in which $\in$ is not well-founded. This is mainly due to the inability of first-order logic to exclude infinitely descending $\in$-chains. All first-order logic can do is to rule out *first-order definable* infinite descending $\in$-chains such as $x \in x$ (which gives rise to $x \in x \in x \in x ...$) and $x_n \in x_{n-1} ... \in x_0 \in x_n$.[47]

By the above lemma, it suffices to exclude infinitely descending $\in$-chains in order to make $\in$ well-founded. This can be done in $\mathcal{L}_{\omega_1\omega_1}$ by the following sentence:

$$\varphi_1 := \neg(\exists x_n)_{n<\omega}(\bigwedge_{n<\omega} x_{n+1} \in x_n).$$

LEMMA B.4. $\mathcal{M} \models \varphi_1$ *iff* $\in$ *is well-founded on* M.

Proof. If $\mathcal{M} \models \varphi_1$, then it does not have infinitely descending $\in$-chains. Thus, by Lemma B.3, $\in$ is well-founded on M. If $\in$ is well-founded on M, then every $\emptyset \neq R \subseteq M$ has a $\in$-least element. Hence, by Lemma B.3, there are no infinitely descending $\in$-chains on M, and so $\mathcal{M} \models \varphi_1$. □

Next we add $\mathcal{L}_{\kappa^+\kappa^+}$-sentences which express that the elements of the domain are subsets of the domain and give an upper bound on the cardinality of the subsets of the domain, as well as the axiom of extensionality.

Let's start with extensionality, which is a first-order sentence:

$$\mathsf{Ext} := \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

Next are the $\mathcal{L}_{\kappa^+\kappa^+}$-sentences, first the sentence expressing that the elements of the domain are subsets of the domain:

$$\varphi_2 := (\forall x_\alpha)_{\alpha<\kappa} \exists y \forall z (z \in y \leftrightarrow \bigvee_{\alpha<\kappa} (z = x_\alpha)),$$

and the upper bound on cardinality:

$$\varphi_3 := \forall y [\exists z (z \in y) \rightarrow (\exists x_\alpha)_{\alpha<\kappa} \forall z (z \in y \leftrightarrow \bigvee_{\alpha<\kappa} (z = x_\alpha))].$$

Now I can give the axiomatization of set theory in infinitary logic.

[47] See Shapiro [44, p. 113 ff.] for more details.

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Definition B.5 (ZFC_κ). *The theory* ZFC_κ *consists of the axioms of extensionality, pairing, union, infinity, power set, and choice, the axiom schemata of separation and replacement,* φ_1 *(the infinitary version of foundation), as well as the axioms* φ_2 *and* φ_3 *characterizing the size of* ZFC_κ*-models.*

Finally, here is the characterization of $H(\kappa)$ in $\mathcal{L}_{\kappa^+\kappa^+}$.

Theorem B.6 Hanf & Scott [20]. *Let* $\mathsf{Ext} \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ *be a* $\mathcal{L}_{\kappa^+\kappa^+}$*-sentence, and* $\mathcal{M} = \langle M, E\rangle$ *be an* $\mathcal{L}_{\kappa^+\kappa^+}$*-structure, where* E *interprets* $\in$ *in* $\mathsf{Ext} \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$*. Then*

$$\mathcal{M} \models \mathsf{Ext} \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \text{ if and only if } \mathcal{M} \cong \mathcal{H}(\kappa).$$

Proof. We follow Karp [25].

$\Leftarrow$: It suffices to show that $\mathcal{H}(\kappa) \models \mathsf{Ext} \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$. $H(\kappa)$ is transitive, so Ext and φ_2 hold in $\mathcal{H}(\kappa)$; it is well-founded, so φ_1 holds in $\mathcal{H}(\kappa)$; and all elements are of power $< \kappa$, so φ_3 holds in $\mathcal{H}(\kappa)$.

$\Rightarrow$: The proof is done in two steps.

1. If $\mathcal{M} \models \mathsf{Ext} \wedge \varphi_1$, then $\mathcal{M}$ is isomorphic to a structure $\mathcal{N} = \langle N, \in\rangle$, where N is a transitive set.[48]
2. If $\mathcal{N}$ is a transitive model in which φ_2 and φ_3 hold, then $N \subseteq H(\kappa)$ and $H(\kappa) \subseteq N$.

1.: Since $\mathcal{M} \models \varphi_1$, E is well-founded on M, and so there exists an E-minimal element $m_0 \in M$. Since $\mathcal{M} \models \mathsf{Ext}$, m_0 is unique. Call it 0^E. We need the following lemma.

Lemma B.7. *Induction holds for* E *in* $\mathcal{M} = \langle M, E\rangle$*, i.e., for any property* ψ*, if*

1. $\psi(0^E)$ *and,*
2. *if* $x \in M$ *and* $\psi(y)$ *for all* yEx*, then* $\psi(x)$*,*

then $\forall x \in M$*,* $\psi(x)$. □

Proof of Lemma B.7. Let $S := \{x \in M : \neg\psi(x)\} \subseteq M$, and suppose $S \neq \emptyset$. By φ_1, S has an E-minimal element s_0. By (1), $x_0 \neq 0^E$. Then, x_0 is the E-least element in M such that $\neg\psi(x_0)$. Then, $\psi(y)$ for all yEx_0, and hence, by (2), $\psi(x_0)$, which contradicts $\neg\psi(x_0)$. So $S = \emptyset$. This concludes the proof of Lemma B.7. □

Now we can use E-induction to define the transitive collapse ψ of E. For $x \in M$, let

$$\pi(x) := \{\pi(z) : zEx\}. \tag{1}$$

Then, $\pi : M \to \pi(M)$. Obviously, $\pi(M)$ is transitive and π is surjective. Now we show that π is injective.

Define a rank function relative to E and M:

$$\mathsf{rank}^E(x) = \begin{cases} 0, & \text{if } \forall y(\neg yEx), \\ \mathsf{sup}(\{\mathsf{rank}^E(y) + 1 : yEx\}), & \text{otherwise.} \end{cases}$$

Injectivity follows by induction on $\mathsf{rank}^E(x)$: for the base case, let $x, y \in M$ be such that $\mathsf{rank}^E(x) = \mathsf{rank}^E(y) = 0$. Then, $\pi(x) = \{x' \in M : x'Ex\} = \emptyset = \{y' \in M : y'Ey\} = \pi(y)$. Now assume that if $\mathsf{max}(\mathsf{rank}^E(x), \mathsf{rank}^E(y)) < \alpha$, then $\pi(x) = \pi(y)$.

48 This is essentially an application of Mostowski's collapsing theorem (see Jech [22, theorem 6.15, p. 69].

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Let $a, b \in M$ be such that $\mathsf{rank}^E(a) = \mathsf{rank}^E(b) = \alpha$ and assume $\pi(a) = \pi(b)$. If dEa, then $\pi(d) \in \pi(a) = \pi(b)$, so for some $e \in M$, eEb and $\pi(d) = \pi(e)$, and by IH, $d = e$, so dEb. The same holds for the other direction. So $\forall x \in M(xEa \leftrightarrow xEb)$. Since E is extensional on M, $a = b$.

Finally, π preserves structure, i.e.,

$$aEb \leftrightarrow \pi(a) \in \pi(b).$$

For if bEa, then, by 1, $\pi(a) \in \pi(b)$. On the other hand, if $\pi(a) \in \pi(b)$, again by 1, $\pi(c) = \pi(b)$ for some $c \in M$ such that cEa. But then $c = b$ since π is injective, and so bEa.

This shows that if $\mathcal{M} \models \mathsf{Ext} \wedge \varphi_1$, then $\mathcal{M}$ is isomorphic to the structure

$$\langle \pi(M), \{\langle \pi(a), \pi(b)\rangle \in \pi(M) \times \pi(M) : aEb\}\rangle,$$

the transitive collapse (or Mostowski collapse) of E and M. Call this structure $\mathcal{N} = \langle N, \in \rangle$.

2.: Now suppose that $\mathcal{N} \models \varphi_2 \wedge \varphi_3$. We show that $N = H(\kappa)$: since $\mathcal{N} \models \varphi_2 \wedge \varphi_3$, the elements of N are transitive subsets of N with power at most $< \kappa$, so $N \subseteq H(\kappa)$. We now show $H(\alpha) \subseteq N$ by induction on $\alpha < \kappa$. For $\alpha = 0$, $H(\alpha) = \emptyset \subseteq S$. Assume that for all $0 < \beta < \kappa$, $H(\beta) \subseteq S$. Let $x \in H(\beta^+)$. Then $|TC(x)| < \beta^+ \leq \kappa$. Then, since $H(\beta^+)$ is transitive, so is x. So $x = TC(x)$, and so $|x| < \beta^+$. Since $\mathcal{N} \models \varphi_2$, there is some $y \in N$ such that for all $x_\gamma \in x$ (for all $\gamma \leq \beta^+$), $x_\gamma \in y$, and nothing else is in y. So by Ext, $x = y \in N$. Since x was arbitrary, $H(\beta^+) \subseteq N$. This concludes the proof of Theorem B.6.

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