ON MEROMORPHIC OPERATORS, I

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1. Introduction. If X is a complex Banach space and $B(X)$ denotes the space of bounded linear operators on X, then the class $\mathfrak{M}$ of *meromorphic* operators consists of those T in $B(X)$ such that the non-zero points of $\sigma(T)$ are poles of the resolvent $R_\lambda(T)$. If we also require that each non-zero eigenvalue of T have finite multiplicity, members of the class $\mathfrak{R} \subseteq \mathfrak{M}$ so defined have been called operators of Riesz type. $\mathfrak{M}$ and $\mathfrak{R}$ have been studied in **(2, 6, 7)** and **(1, 4)** respectively.

In this paper, an asymptotic characterization for $\mathfrak{M}$, somewhat similar to that obtained by Ruston **(4)** for $\mathfrak{R}$, is devised and the application of the usual operational calculus to $\mathfrak{M}$ is studied.

2. We shall use $\mathfrak{F}$ to denote the subclass of $\mathfrak{M}$ consisting of those operators T whose spectrum consists of a finite number of poles of $R_\lambda(T)$.

THEOREM 1. *Let T_1 and T_2 belong to $\mathfrak{F}$ and commute. Then $T_1 + T_2$ and $T_1 T_2$ belong to $\mathfrak{F}$.*

Proof. Suppose that $\sigma(T_i) = \{\lambda_{ij}: j = 1, 2, \ldots, n_i\}$, $i = 1, 2$, such that λ_{ij} is a pole of $R_\lambda(T_i)$ of order m_{ij}. Now define $f_i(\lambda) = \prod_j(\lambda - \lambda_{ij})^{m_{ij}}$. By **(5**, p. 307**)**, we know that $f_i(T_i) = 0$. Now consider the function

$$f(\lambda) = \prod_{k,j}(\lambda - \lambda_{1k} - \lambda_{2j})^t$$

where $t = 2\max_{i,j} m_{ij}$. We shall show that $f(T_1 + T_2) = 0$. In fact, $f(T_1 + T_2)$ can be expanded by the binomial theorem into a finite linear combination of terms of the form

$$l = \prod_k(T_1 - \lambda_{1k})^{\Sigma_j s_{kj}} \cdot \prod_j(T_2 - \lambda_{2j})^{n_1 t - \Sigma_k s_{kj}}, \tag{2.1}$$

where s_{kj} are integers, $0 \leqslant s_{kj} \leqslant t$. Suppose that $\sum_j s_{kj} < m_{1k}$ for some k, say $k = k_0$; then $s_{k_0 j} < m_{1k_0}$ for all j. Hence

$$n_1 t - \sum_k s_{kj} \geqslant n_1 t - [(n_1 - 1)t + m_{1k_0}] \geqslant t - m_{1k_0} \geqslant m_{2j}$$

by the definition of t. Thus (2.1) contains a factor $f_i(T_i)$ for $i = 1$ or 2. Hence $f(T_1 + T_2) = 0$. Now it is well known from the Gelfand theory that, since T_1 and T_2 commute, $\sigma(T_1 + T_2)$ is a subset of the vector sum of the $\sigma(T_i)$, so that $\sigma(T_1 + T_2)$ is a finite set. Suppose that $\lambda_0 \in \sigma(T_1 + T_2)$ is an essential

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singularity of $R_\lambda(T_1 + T_2)$. Then by a known theorem **(5**, p. 307**)**, if $f(T_1 + T_2) = 0$, then $f(\lambda)$ is identically zero in some neighbourhood of λ_0. Since this is clearly not the case, we conclude that $T_1 + T_2 \in \mathfrak{F}$.

By a similar argument, we show that $T_1 T_2 \in \mathfrak{F}$. In this case, define

$$f(\lambda) = \prod_{k,j} (\lambda - \lambda_{1k}\lambda_{2j})^t.$$

Then by using a binomial expansion and making some rearrangements, we find that $f(T_1 T_2)$ is a finite linear combination of terms of the form

$$T_1^{n_1 n_2 t - \Sigma_{k,j} s_{kj}} \cdot \prod_j \lambda_{2j}^{\Sigma_k s_{kj}} \cdot l$$

so that the previous argument shows that $f(T_1 T_2) = 0$. Since

$$\sigma(T_1 T_2) \subseteq \sigma(T_1) \cdot \sigma(T_2),$$

the result follows.

Remark. The commutativity condition in this theorem is essential, for the non-commuting operators defined below are elements of $\mathfrak{F}$ but neither their sum nor their product lies in $\mathfrak{F}$.

Let $X = l^1$ and write $\bar{x}$ for the vector with components $x_1, x_2, \ldots$; define A and B by the relations

$$A\bar{x} = \bar{x} + (0, x_1, 0, x_3, 0, x_5, 0, \ldots),$$
$$B\bar{x} = -\bar{x} + (0, 0, x_2, 0, x_4, 0, x_6, \ldots).$$

Then it is not difficult to show that $\sigma(A) = \{1\}$ and

$$R_\lambda(A) = I(\lambda - 1)^{-1} + (A - I)(\lambda - 1)^2$$

so that $A \in \mathfrak{F}$.

Similarly $\sigma(B) = \{-1\}$ and $B \in \mathfrak{F}$. Moreover $AB \neq BA$ since by direct calculation $AB\bar{x}$ and $BA\bar{x}$ have third components equal to $x_2 - x_3$ and $x_1 + x_2 - x_3$ respectively.

The operator $A + B$ is studied in **(5**, p. 266**)** where it is shown that $\sigma(A + B)$ is the unit disk. Finally, it is possible to calculate the matrix which represents $R_\lambda(AB)$. If this matrix has elements $r_{ij}(\lambda)$, then

$$r_{ij}(\lambda) = \begin{cases} 0, & \text{if } j > i, \\ (1 + \lambda)^{-1} & \text{if } j = i, \\ (-1)^{i-j+1}(1 + \lambda)^{-j}\lambda^{c_{ij}} & \text{if } j < i, \end{cases}$$

where

$$\begin{aligned} c_{ij} &= 0 && \text{if } j = 1, 2, \\ c_{ij} &= c_{i,j-1} = \tfrac{1}{2}(j - 2) && \text{if } i, j \text{ are even}, \\ &\phantom{= c_{i,j-1}} = \tfrac{1}{2}(j - 3) && \text{if } i, j \text{ are odd}. \end{aligned}$$

By a well-known formula $(r_{ij}(\lambda))$ represents a bounded linear operator in l^1 if and only if $\sup_i \sum_j |r_{ij}(\lambda)|$ is finite. This is equivalent to requiring the absolute convergence of the series

$$\frac{1}{1+\lambda} + \frac{1}{(1+\lambda)^2} + \frac{\lambda}{(1+\lambda)^3} + \frac{\lambda}{(1+\lambda)^4} + \frac{\lambda^2}{(1+\lambda)^5} + \ldots.$$

But this series is absolutely convergent if and only if $|\lambda| < |1 + \lambda|^2$. Hence $\sigma(A)$ cannot be a finite set.

THEOREM 2. *If $T \in \mathfrak{F}$ and T^{-1} exists in $B(X)$, then $T^{-1} \in \mathfrak{F}$.*

Proof. By the spectral mapping theorem, $\sigma(T^{-1}) = \{\lambda: \lambda^{-1} \in \sigma(T)\}$ so that $\sigma(T^{-1})$ is a finite set. If $\lambda_0^{-1} \in \sigma(T^{-1})$, we can write a Laurent expansion for $R_\lambda(T^{-1})$ in the neighbourhood of λ_0^{-1} and a similar expression for $R_\lambda(T)$ in the neighbourhood of λ_0. Let A_n and B_n be the coefficients of $(\lambda - \lambda_0^{-1})^{-n}$ and $(\lambda - \lambda_0)^{-n}$ in the respective expansions. If we write $N(\lambda_0; T)$ for a disk of centre λ_0 such that $\sigma(T) \cap N(\lambda_0; T) = \{\lambda_0\}$ and define $f_n(\lambda)$ as equal to $(\lambda - \lambda_0)^{n-1}$ for $\lambda \in N(\lambda_0; T)$ and equal to zero elsewhere, $g_n(\lambda)$ equal to $(\lambda - \lambda_0^{-1})^{n-1}$ for $\lambda \in N(\lambda_0^{-1}, T^{-1})$ and equal to zero elsewhere, then it is well known (**5**, p. 305) that $A_n = g_n(T^{-1})$ and $B_n = f_n(T)$. If $h(\lambda) = 1/\lambda$, then $A_n = g_n[h(T)]$. By (**5**, p. 303), we can therefore write $A_n = (g_n \circ h)(T)$.

Now

$$\begin{aligned}(g_n \circ h)(\lambda) &= (-1)^{n-1}(\lambda - \lambda_0)^{n-1}(\lambda\lambda_0)^{-(n-1)} && \text{for } \lambda \in N(\lambda_0; T),\\ &= 0 && \text{elsewhere.}\end{aligned}$$

Defining $G_n(\lambda) = (-\lambda\lambda_0)^{-(n-1)}$, we get that $g_n \circ h = G_n f_n$. Thus

$$A_n = (g_n \circ h)(T) = (G_n f_n)(T) = G_n(T) f_n(T) = (-\lambda_0 T)^{-(n-1)} B_n.$$

Hence $A_n = 0$ for n sufficiently large. In fact, the order of λ_0 as a pole of $R_\lambda(T)$ is equal to the order of λ_0^{-1} as a pole of $R_\lambda(T^{-1})$.

3. Characterization of $\mathfrak{M}$. If $A, B \in B(X)$ and $AB = BA = 0$, we shall write $A \perp B$. Define

$$\lambda(A) = \inf\{||A - V||: V \in \mathfrak{F}_0\}$$

where $\mathfrak{F}_0 = \{V \in \mathfrak{F}: A - V \perp V\}$. Clearly $\lambda(A)$ is well defined since $0 \in \mathfrak{F}_0$.

THEOREM 3. $\mathfrak{M} = \{T \in B(X): [\lambda(T^n)]^{1/n} \to 0\}$.

Proof. Let $T \in \mathfrak{M}$ and take $\epsilon > 0$. Define $\sigma = \{\lambda: |\lambda| > \epsilon; \lambda \in \sigma(T)\}$. Then by the definition of $\mathfrak{M}$, σ is a spectral set. Let the associated spectral projection E_σ have range R_σ and null space N_σ. Define $T_\epsilon = TE_\sigma$ and $S_\epsilon = T(I - E_\sigma)$. Then we show that (i) $T_\epsilon \in \mathfrak{F}$ and (ii) $\sigma(S_\epsilon) \subseteq \{\lambda: |\lambda| \leqslant \epsilon\}$.

(i) Since E_σ is continuous, R_σ is closed and hence may be considered as a Banach space. Let T_1 be defined in $B(R_\sigma)$ by $T_1 x = Tx$ for $x \in R_\sigma$. Since R_σ and N_σ completely reduce T, we can write for $x \in X$, $x = x_1 + x_2$, $x_1 \in R_\sigma$, $x_2 \in N_\sigma$. Consider

$$(\lambda - T_\epsilon)^k x = (\lambda - TE_\sigma)^k (x_1 + x_2) = (\lambda - T_1)^k x_1 + \lambda^k x_2.$$

Then, for $\lambda \neq 0$,

$$\text{(3.1)}\quad \begin{aligned} R[(\lambda - T_\epsilon)^k] &= R[(\lambda - T_1)^k] \oplus N_\sigma = [R[(\lambda - T)^k] \cap R_\sigma] \oplus N_\sigma,\\ N[(\lambda - T_\epsilon)^k] &= N[(\lambda - T_1)^k] = N[(\lambda - T)^k] \cap R_\sigma\end{aligned}$$

where for any operator S, $R(S)$ and $N(S)$ denote the range and null space respectively. It is well known that $\sigma(T_1) = \sigma$. Suppose that $\lambda \neq 0$ and $\lambda \notin \sigma$. Then $R(\lambda - T_\epsilon) = R_\sigma \oplus N_\sigma = X$ and $N(\lambda - T_\epsilon) = \{0\}$. Thus $\lambda \in \rho(T_\epsilon)$, which means that $\sigma(T_\epsilon) \subseteq \sigma \cup \{0\}$. Thus $\sigma(T_\epsilon)$ is a finite set.

We next show that $T_\epsilon \in \mathfrak{M}$. By **(5**, pp. 273, 310**)**, it suffices to show that if $\lambda \neq 0$, $\alpha(\lambda - T_\epsilon) = \delta(\lambda - T_\epsilon) < \infty$ and, if p_λ is their common value, that the range of $(\lambda - T_\epsilon)^{p_\lambda}$ is closed. But these facts follow from (3.1), **(5**, p. 306**)**, and the assumption that $T \in \mathfrak{M}$. (For definitions of α, δ, σ, and ρ, see **(5)**.)

Finally we must show that if $\lambda = 0$ belongs to $\sigma(T_\epsilon)$, then it is a pole of $R_\lambda(T_\epsilon)$. Now

$$\begin{aligned} T_\epsilon^k x &= (TE_\sigma)^k(x_1 + x_2) \\ &= T^k x_1 \qquad \text{if } k > 0 \\ &= T_1^k x_1. \end{aligned}$$

Now $\lambda = 0$ lies in $\rho(T_1)$ so that $N(T_1^k) = \{0\}$ and $R(T_1^k) = R_\sigma$. Hence $N(T_\epsilon^k) = N_\sigma$ and $R(T_\epsilon^k) = R_\sigma$ for each $k > 0$ so that $\alpha(T_\epsilon) = \delta(T_\epsilon) = 1$. Also $R(T_\epsilon) = R_\sigma$, which is closed, and it is known **(5**, pp. 273, 310**)** that, since $\lambda = 0$ is isolated in $\sigma(T_\epsilon)$, we can conclude that $T_\epsilon \in \mathfrak{F}$.

(ii) Let $\sigma' = \sigma(T) - \sigma$ and define $E_{\sigma'}$, $R_{\sigma'}$, $N_{\sigma'}$ as in (i) above, replacing σ by σ' in each definition. Then $S_\epsilon = TE_{\sigma'}$ and, exactly as in (i),

$$\sigma(S_\epsilon) \subseteq \sigma' \cup \{0\}.$$

We now proceed to a proof of the theorem. We know that the spectral radius of S_ϵ is no greater than ϵ so that $\lim_n ||S_\epsilon^n||^{1/n} \leqslant \epsilon$. But it is clear that $T_\epsilon \perp S_\epsilon$ so that $T^n = (S_\epsilon + T_\epsilon)^n = S_\epsilon^n + T_\epsilon^n$. Hence $\lim_n ||T^n - T_\epsilon^n||^{1/n} \leqslant \epsilon$. By Theorem 1, since $T_\epsilon \in \mathfrak{F}$, $T_\epsilon^n \in \mathfrak{F}$. Moreover $T^n - T_\epsilon^n \perp T_\epsilon^n$ so that

$$\lambda(T_\epsilon^n) \leqslant ||T^n - T_\epsilon^n||$$

and hence $\lim_n [\lambda(T^n)]^{1/n} \leqslant \epsilon$.

Conversely, let $[\lambda(T^n)]^{1/n} \to 0$ and take $\epsilon > 0$. Then for some $N(\epsilon)$, $\lambda(T^n) < \epsilon^n$ whenever $n > N(\epsilon)$. Fix $q > N(\epsilon)$. Then there exists $V \in \mathfrak{F}$ such that $T^q - V \perp V$ and $||T^q - V|| < \epsilon^q$. Write $U = T^q - V$. Then

$$\sigma(U) \subseteq \{\lambda: |\lambda| \leqslant \epsilon^q\}.$$

Now $U \perp V$ and it is a simple matter to verify from the identity

$$(\lambda - U)(\lambda - V) = \lambda[\lambda - (U + V)] \tag{3.2}$$

that

$$\sigma(U) \cup \sigma(V) = \sigma(T^q) \cup \{0\}. \tag{3.3}$$

Since $\sigma(V)$ is finite, $\sigma(T^q)$ has at most finitely many points outside $\{\lambda: |\lambda| = \epsilon^q\}$. Each such point is a pole of $R_\lambda(T^q)$, for since from (3.3) $\rho(T^q) - \{0\} = \rho(U) \cap \rho(V)$, then if $\lambda \in \rho(T^q)$, we can obtain from (3.2) that

$$R_\lambda(U)R_\lambda(V) = \lambda^{-1}R_\lambda(T^q) \qquad \text{if } \lambda \neq 0.$$

Now outside $\{\lambda: |\lambda| = \epsilon^q\}$, $R_\lambda(U)$ is holomorphic and $R_\lambda(V)$ is meromorphic so that $R_\lambda(T^q)$ is meromorphic outside this circle. Moreover, since

$$\lambda^q - T^q = (\lambda - T)(\lambda^{q-1}T + \ldots + T^{q-1}),$$

we obtain $R_\lambda(T) = R_{\lambda^q}(T^q)(\lambda^{q-1} + \lambda^{q-2}T + \ldots + T^{q-1})$ so that $R_\lambda(T)$ is meromorphic outside the circle $\{\lambda: |\lambda| = \epsilon\}$. Since ϵ is arbitrary, it follows that $T \in \mathfrak{M}$.

4. Perturbation theory in $\mathfrak{M}$. The nature of the spectrum of a meromorphic operator restricts the possibilities for additive perturbation. For even the addition of ϵI produces an operator with a non-zero point of accumulation in its spectrum. The subclass $\mathfrak{R}$ of Riesz operators has much more satisfactory properties in this respect; indeed $\mathfrak{R}$ acts as a "stable kernel" for $\mathfrak{M}$.

Results obtained in **(1)** include the following:

(i) if $T_1, T_2 \in \mathfrak{R}$ and $T_1 T_2 = T_2 T_1$, then $T_1 + T_2$, $T_1 T_2 \in \mathfrak{R}$.

(ii) if $T \in \mathfrak{R}$ and $S \in B(X)$, then $TS \in \mathfrak{R}$ if $TS = ST$.

(iii) if $\{T_n\}$ is a sequence in $\mathfrak{R}$ with uniform limit S, $T_n S = ST_n$ for n sufficiently large implies that $S \in \mathfrak{R}$.

It has been seen that $\mathfrak{F}$ displays the first of these properties. The second clearly fails, however; for $I \in \mathfrak{F}$ and commutes with any $T \in B(X)$. If in l^1 we define a sequence of operators T_n with matrix representations

$$(t_{ij}^{(n)}) = \text{diag}(1, \tfrac{1}{2}, \ldots, 1/n, 0, 0, \ldots)$$

which converge to and commute with operator T with matrix representation diag $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, then we see that (iii) is untrue for $\mathfrak{F}$, since $T \notin \mathfrak{F}$. However, we can obtain the following perturbation theorem.

THEOREM 4. *Suppose $T \in \mathfrak{M}$ and $V_0 \in \mathfrak{F}$. Let V_0 commute with T and also have the property: if $V \in \mathfrak{F}$ and V commutes with T^n for all n, then V commutes with V_0^n. Then TV_0 is meromorphic.*

Proof. Let

$$\begin{aligned}
\mathfrak{S}_1 &= \{V \in \mathfrak{F}; (TV_0)^n - V \perp V\},\\
\mathfrak{S}_2 &= \{V \in \mathfrak{S}_1; V = UV_0^n \text{ for some } U \in \mathfrak{F}\},\\
\mathfrak{S}_3 &= \{U \in \mathfrak{F}; UV_0^n \in \mathfrak{S}_1\},\\
\mathfrak{S}_4 &= \{U \in \mathfrak{F}; T^n - U \perp U\}.
\end{aligned}$$

Clearly $\mathfrak{S}_1 \supseteq \mathfrak{S}_2$. We shall prove that $\mathfrak{S}_3 \supseteq \mathfrak{S}_4$. Let $U \in \mathfrak{S}_4$. Then $U \in \mathfrak{F}$ and $T^n U = UT^n = U^2$. Hence, by assumption, $V_0^n U = UV_0^n$. Moreover, $T^n U \cdot V_0^{2n} = UT^n V_0^{2n} = U^2 V_0^{2n}$ can be written

$$T^n V_0^n U V_0^n = U V_0^n T^n V_0^n = (UV_0^n)^2$$

so that $T^n V_0^n - UV_0^n \perp UV_0^n$. Also since U and V_0^n commute, $UV_0^n \in \mathfrak{F}$ by Theorem 1. Hence $U \in \mathfrak{S}_3$.

Now

$$\begin{aligned}
\inf_{V \in \mathfrak{S}_1} \|(TV_0)^n - V\|^{1/n} &\leqslant \inf_{V \in \mathfrak{S}_2} \|(TV_0)^n - V\|^{1/n}\\
&\leqslant \inf_{U \in \mathfrak{S}_3} \|(TV_0)^n - UV_0^n\|^{1/n} \leqslant \|V_0\| \inf_{U \in \mathfrak{S}_3} \|T^n - U\|^{1/n}\\
&\leqslant \|V_0\| \inf_{U \in \mathfrak{S}_4} \|T^n - U\|^{1/n}.
\end{aligned}$$

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By Theorem 3, the last quantity converges to zero. Hence so does the first and the same theorem gives the required result.

5. Functions of a meromorphic operator.

THEOREM 5. *Let T be meromorphic with the non-zero points of its spectrum denoted by $\{\lambda_n\}$. Let $f(\lambda)$ be analytic on some open set D which contains $\sigma(T)$ and let $f(0) = 0$. Then $f(T)$, defined by the usual operational calculus, is meromorphic.*

Moreover, let E_n denote the spectral projection associated with T and the single point λ_n. For any non-zero point μ_0 in $\sigma[f(T)]$, define $S(\mu_0) = \{t: f(\lambda_t) = \mu_0\}$. Then the spectral projection associated with $f(T)$ and μ_0 is given by

$$\sum_{s\in S(\mu_0)} E_s.$$

Proof. First, we show that μ_0 is isolated in $\sigma[f(T)]$. Suppose it is not; then using the spectral mapping theorem, we can conclude that $\{\lambda_n\}$ contains a subsequence $\{\lambda_{n_K}\}$ such that $f(\lambda_{n_K}) \to \mu_0$. But $\{\lambda_n\}$ is a null sequence so that, by the continuity of f, $f(\lambda_{n_K}) \to f(0) = 0$. Hence $\mu_0 = 0$, contrary to assumption.

We now show that μ_0 is a pole of $R_\mu[f(T)]$. Suppose μ is fixed in $\rho(f(T))$. There exists an open set U such that $\sigma(f(T)) \subseteq U \subseteq f(D)$ and such that μ lies in the complement of U. Write $V = f^{-1}(U)$ so that $\sigma(T) \subseteq V \subseteq D$, and for $\lambda \in V, f(\lambda) \neq \mu$. It is known **(5)** that we can always find a Cauchy domain S inside D such that $\sigma(T) \subseteq S \subseteq \bar{S} \subseteq V$. Write C for the positively oriented boundary of S. Then we can write

$$R_\mu[f(T)] = \frac{1}{2\pi i} \oint_C [\mu - f(\lambda)]^{-1} R_\lambda(T) d\lambda.$$

We now use the Mittag–Leffler type expansion of $R_\lambda(T)$ as given in **(7**, pp. 428–9**)**. In fact

$$R_\lambda(T) = \sum_{n=1}^{\infty} [S_n(\lambda) - P_n^{(p_n)}(\lambda)] + \sum_{n=0}^{\infty} \lambda^{-n} Q_n$$

for each $\lambda \in \rho(T)$, where

$$S_n(\lambda) = \sum_{k=1}^{q_n} (\lambda - \lambda_n)^{-k} (T - \lambda_n)^{k-1} E_n,$$

$$P_n^{(p)}(\lambda) = \sum_{k=1}^{p} \lambda^{-k} T^{k-1} E_n,$$

and q_n is the order of λ_n as a pole of $R_\lambda(T)$.

The starting point of the theory in the last-mentioned paper is a proof of the fact that positive integers p_n and operators Q_n in $B(X)$ can be chosen such that the representation of $R_\lambda(T)$ is uniformly convergent on compact subsets of $\rho(T)$. Thus we can use the representation to write

$$R_\mu[f(T)] = \sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\oint_C[\mu - f(\lambda)]^{-1}[S_n(\lambda) - P_n^{(p_n)}(\lambda)]d\lambda\right] + \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_C[\mu - f(\lambda)]^{-1}\lambda^{-n}Q_n d\lambda\right].$$

Thus $R_\mu[f(T)]$ is the sum of operators with scalar coefficients of the form

$$I_{n,k} = \frac{1}{2\pi i}\oint_C[\mu - f(\lambda)]^{-1}(\lambda - \lambda_n)^{-k}d\lambda,$$

$$I_k = \frac{1}{2\pi i}\oint_C[\mu - f(\lambda)]^{-1}\lambda^{-k}d\lambda.$$

In fact

$$R_\mu[f(T)] = \sum_{n=1}^{\infty}\left[\sum_{k=1}^{q_n} I_{n,k}(T - \lambda_n)^{k-1}E_n - \sum_{k=1}^{p_n} I_k T^{k-1}E_n\right] + \sum_{n=0}^{\infty} I_n Q_n.$$

By construction, $[\mu - f(\lambda)]^{-1}$ is analytic inside and on C. Hence we can write

$$I_{n,k} = \frac{1}{(k-1)!}\left\{\frac{d^{k-1}}{d\lambda^{k-1}}[\mu - f(\lambda)]^{-1}\right\}_{\lambda=\lambda_n},$$

$$I_k = \frac{1}{(k-1)!}\left\{\frac{d^{k-1}}{d\lambda^{k-1}}[\mu - f(\lambda)]^{-1}\right\}_{\lambda=0}.$$

To evaluate these expressions, we shall adopt the following notation: $\Phi = \mu - f(\lambda)$, $\Theta = \Phi^{-1}$, $D \equiv d/d\lambda$. Then since $\Phi\Theta = 1$, we can use Leibniz's rule to get

$$\sum_{s=0}^{n-1}\binom{n}{s}D^s\Phi D^{n-s}\Theta = -\Theta D^n\Phi, \qquad n = 1, 2, \ldots, k.$$

We may consider the above as a system of n linear equations in the unknowns $D\Theta, D^2\Theta, \ldots, D^n\Theta$. Using Crámer's rule, we get $D^k\Theta$ equal to:

$$\Phi^{-k}\begin{vmatrix} -\Theta D\Phi & 0 & 0 & 0 & \cdots\ 0\ 0 & \Phi \\ -\Theta D^2\Phi & 0 & 0 & 0 & \cdots\ 0\ \Phi & \binom{2}{1}D\Phi \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ -\Theta D^{k-1}\Phi & \Phi & \binom{k-1}{1}D\Phi & \binom{k-1}{2}D^2\Phi & \cdots\ \cdot\ \cdot & \binom{k-1}{k-2}D^{k-2}\Phi \\ -\Theta D^k\Phi & \binom{k}{1}D\Phi & \binom{k}{2}D^2\Phi & \binom{k}{3}D^3\Phi & \cdots\ \cdot\ \cdot & \binom{k}{k-1}D^{k-1}\Phi \end{vmatrix}.$$

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If we use this relation to evaluate $I_{n,k}$ (to evaluate I_k) we find that it is analytic except for a pole of order not greater than k at $\mu = f(\lambda_n)$ (at $\mu = 0$). Using this information together with the expansion of $R_\mu[f(T)]$, we see that the latter has a pole at each non-zero $f(\lambda_n)$. By the spectral mapping theorem, this gives the result.

We now turn our attention to the statement about the spectral projections. First we must show that $S(\mu_0)$ is a finite set. If $S(\mu_0)$ were infinite, then $\{\lambda_s: s \in S(\mu_0)\}$ would be an infinite set and hence have $\lambda = 0$ as its only point of accumulation. By the continuity of f, this would mean that $\{f(\lambda_s): s \in S(\mu_0)\}$ would have the same property. But $\{f(\lambda_s): s \in S(\mu_0)\} = \{\mu_0\}$.

Now suppose that $g_\mu(\lambda)$ is defined as equal to 1 when $\lambda \in N(\mu; f(T))$ and zero elsewhere. (Recall the definition of $N(\mu; f(T))$ from the proof of Theorem 2.) Then $g_{\mu_0}(f(T))$ defines E_0, the spectral projection associated with μ_0 and $f(T)$. By **(5**, p. 303**)**, $E_0 = (g_{\mu_0} \circ f)(T)$.

Now

$$\begin{aligned}(g_{\mu_0} \circ f)(\lambda) &= 1 \qquad \text{for } \lambda \in \textstyle\bigcup_{t \in S(\mu_0)} N(\lambda_t; T), \\ &= 0 \qquad \text{elsewhere,}\end{aligned}$$

i.e.

$$(g_{\mu_0} \circ f)(\lambda) = \textstyle\sum_{t \in S(\mu_0)} f_{\lambda_t}(\lambda)$$

where $f_{\lambda_t}(\lambda)$ is defined as equal to 1 when $\lambda \in N(\lambda_t; T)$ and zero elsewhere. Hence

$$E_0 = \textstyle\sum_{t \in S(\mu_0)} f_{\lambda_t}(T) = \sum_{t \in S(\mu_0)} E_t.$$

Remarks. 1. It is obvious that the omission of the condition $f(0) = 0$ makes the theorem untrue. For consider $f(\lambda) = 1 + \lambda$. Then $f(T) = I + T$ and if $\sigma(T)$ has a point of accumulation at $\lambda = 0$, $\sigma(f(T))$ will have a point of accumulation at $\lambda = 1$. However, the condition was used only to establish that $\sigma(f(T))$ had no non-zero points of accumulation. For a given T, a weaker condition on f may suffice.

2. An examination of the proof shows that if q_0 is the order of μ_0 as a pole of $R_\mu[f(T)]$, then $q_0 \leqslant \max\{q_t: t \in S(\mu_0)\}$.

3. Let $\mathfrak{A}$ be any collection of operators in $B(X)$. We shall say that $\mathfrak{A}$ is *f-invariant* if, given $T \in \mathfrak{A}$ and f analytic on some open set containing $\sigma(T)$ with $f(0) = 0$, then $f(T) \in \mathfrak{A}$.

Corollary 1. *$\mathfrak{M}$, $\mathfrak{R}$, $\mathfrak{F}$, and the class $\mathfrak{C}$ of compact operators in $B(X)$ are f-invariant.*

Proof. The assertion regarding $\mathfrak{M}$ is part of Theorem 6; the proof of that concerning $\mathfrak{R}$ is given in **(1)**. Suppose that $T \in \mathfrak{F}$; then $f(T)$ lies in $\mathfrak{M}$ and has finite spectrum. We need only show that if $0 \in \sigma[f(T)]$, then $\lambda = 0$ is a pole of $R_\lambda[f(T)]$. Since $T \in \mathfrak{F}$, we can write

$$R_\lambda(T) = \sum_{n=1}^{t} S_n(\lambda) + \phi(\lambda)$$

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where $\sigma(T)$ consists of t poles of $R_\lambda(T)$ and $\phi(\lambda)$ is an entire function. If we now examine the proof of the theorem, we can conclude that all the points of $\sigma[f(T)]$ are poles of $R_\lambda[f(T)]$.

Finally, suppose that $T \in \mathfrak{C}$. Now for $f(0) = 0$, we can find $s \geqslant 1$ such that $f(\lambda) = \lambda^s g(\lambda)$ with $g(0) \neq 0$ and $g(\lambda)$ analytic wherever $f(\lambda)$ is analytic. Hence $f(T) = T^s g(T)$ and since $T \in \mathfrak{C}$ and $\mathfrak{C}$ is an ideal, $f(T) \in \mathfrak{C}$. This same argument would also be valid to prove the f-invariance of $\mathfrak{R}$.

THEOREM 6. *Let T be meromorphic and $\mathfrak{A}_0(T)$ be the collection of functions $f(\lambda)$ which are locally analytic in some open set containing $\sigma(T)$ and have a zero at $\lambda = 0$. Then, if we write A_0 for the Banach algebra generated by $\{f(T): f \in \mathfrak{A}_0(T)\}$, $A_0 \subseteq \mathfrak{M}$.*

Proof. Let $\phi: A_0 \to C(X_0)$ be the Gelfand representation of A_0 where X_0 is the space of maximal ideals of A_0 with the usual weak topology. Since $I \in A_0$, X_0 is compact. For $P \in A_0$, write $\hat{P}$ for $\phi(P)$. Then we can identify X_0 with $\sigma(T)$, for the map $\psi: X_0 \to \sigma(T)$ defined by $\psi(x) = \hat{T}(x)$ is a continuous surjection. Moreover, if $\hat{T}(x_1) = \hat{T}(x_2)$, then $f[\hat{T}(x_1)] = f[\hat{T}(x_2)]$ for all $f \in \mathfrak{A}_0(T)$. But it is well known that $f \circ \hat{T} = \widehat{f(T)}$ so that $\widehat{f(T)}(x_1) = \widehat{f(T)}(x_2)$ for all $f \in \mathfrak{A}_0(T)$. Since the set $\{f(T): f \in \mathfrak{A}_0(T)\}$ is dense in A_0, $\hat{S}(x_1) = \hat{S}(x_2)$ for each $S \in A_0$. But it is known that $\{\hat{S}: S \in A_0\}$ separates the points of X_0. Hence $x_1 = x_2$. This permits us to conclude that ψ is a homeomorphism and to identify X_0 with $\sigma(T)$.

Suppose that $S \in A_0$ and that $\sigma(S)$ has a point of accumulation μ_0. Then there exists a sequence $\{\mu_n\}$, μ_n distinct, $\mu_n \in \sigma(S)$, such that $\mu_n \to \mu_0$. Since $\sigma(S)$ is the range of $\hat{S}$ and we are identifying X_0 with $\sigma(T)$, there must be distinct λ_n in $\sigma(T)$ such that $\hat{S}(\lambda_n) = \mu_n$. But since $T \in \mathfrak{M}$, $\lambda_n \to 0$, so that $\hat{S}(\lambda_n) \to 0$ and hence $\mu_0 = 0$. Thus $\sigma(S)$ has no non-zero points of accumulation. Moreover,

$$\begin{aligned}\sigma(S) &= \{\mu\colon \mu = \hat{S}(x) \text{ for some } x \in X_0\} \\ &= \{\mu\colon \mu = \lim_{n\to\infty} \widehat{f_n(T)}(x) \text{ for some } x \in X\} \\ &= \{\mu\colon \mu = \lim_{n\to\infty} f_n[\hat{T}(x)] \text{ for some } x \in X\} \\ &= \{\mu\colon \mu = \lim_{n\to\infty} f_n(\lambda) \text{ for some } \lambda \in \sigma(T)\}.\end{aligned}$$

(A discussion of the Gelfand theory used above can be found in **(3)**.) We now wish to show that if $\lambda_k \in \sigma(T)$, $f_n \in \mathfrak{A}_0(T)$, $f_n(T) \to S$, and

$$\mu_k = \lim_{n\to\infty} f_n(\lambda_k)$$

such that $\mu_k \neq 0$, then μ_k is a pole of $R_\lambda(S)$. We already know that μ_k is isolated in $\sigma(S)$. Let C be the boundary of a small circle such that C lies in $\rho(S)$, μ_k lies inside C, and the remaining points of $\sigma(S)$ lie outside C. Moreover, let us arrange that $\lambda = 0$ does not lie on C. For each n, no more than a finite number of elements of $\sigma[f_n(T)]$ lie on C, for if an infinite number of elements of $\sigma[f_n(T)]$ were on C, they would have limit point on C, since C is compact. But $f_n(T)$ is meromorphic.

Let $M = \sup_{\lambda \in C} \|R_\lambda(S)\|$ and suppose C_n is a contour formed by indenting C to avoid $\sigma(S) \cup \sigma[f_n(T)]$. It is obviously always possible to do this in such a way that, for every preassigned $\delta > 0$, C_n is the boundary of a Cauchy domain and such that if $M_n = \sup_{\lambda \in C_n} \|R_\lambda(S)\|$, then $|M_n - M| < \delta$, for $R_\lambda(S)$ is continuous on C.

Now we can write

$$\frac{1}{2\pi i} \oint_{C_n} [R_\lambda(f_n(T)) - R_\lambda(S)] d\lambda = E(\sigma_n; f_n(T)) - E(\mu_k, S)$$

where σ_n is the spectral set obtained for $f_n(T)$ by taking those elements of $\sigma[f_n(T)]$ which lie within C_n, and $E(\sigma_n; f_n(T))$, $E(\mu_k; S)$ are the spectral projections associated with $\sigma_n, f_n(T)$ and $\{\mu_k\}, S$, respectively. There exists $N(\delta) > 0$ such that $\|f_n(T) - S\| < 1/(M + \delta)$ whenever $n > N(\delta)$. Thus for $n > N(\delta)$, $\|f_n(T) - S\| < 1/M_n$ so that

$$\|f_n(T) - S\| \, \|R_\lambda(S)\| < 1 \qquad \text{for } n > N(\delta) \text{ and } \lambda \in C_n.$$

Thus, for $n > N(\delta)$ and $\lambda \in C_n$, the series

$$\sum_{k=0}^{\infty} [f_n(T) - S]^k [R_\lambda(S)]^{k+1}$$

is convergent, with sum $K(\lambda)$, which we compute by multiplying the above series by $I - [f_n(T) - S]R_\lambda(S)$. It is a simple matter to verify that the product is $R_\lambda(S)$ and that $I - [f_n(T) - S]R_\lambda(S) = R_\lambda(S)[\lambda - f_n(T)]$. Hence $K(\lambda)R_\lambda(S)[\lambda - f_n(T)] = R_\lambda(S)$ and since $\lambda \in \rho[f_n(T)] \cap \rho(S)$, we can deduce that $K(\lambda) = R_\lambda[f_n(T)]$. Thus we can write

$$R_\lambda[f_n(T)] - R_\lambda(S) = \sum_{k=1}^{\infty} [f_n(T) - S]^k [R_\lambda(S)]^{k+1}.$$

Moreover, since $\|f_n(T) - S\| \, \|R_\lambda(S)\| < 1$, the series is uniformly convergent on C_n, and termwise integration around C_n is valid. We observe, however, that for any integer $t > 1$,

$$[R_\lambda(S)]^t = \frac{1}{1-t} \frac{d}{d\lambda} \{[R_\lambda(S)]^{t-1}\} \qquad \text{(see } \mathbf{(5}, \text{ p. } 257\mathbf{)})$$

so that for $t > 1$,

$$\oint_{C_n} [R_\lambda(S)]^t \, d\lambda = 0.$$

Thus

$$\oint_{C_n} \{R_\lambda[f_n(T)] - R_\lambda(S)\} d\lambda = 0$$

whenever $n > N(\delta)$. But this implies that $E(\mu_k; S) = E(\sigma_n; f_n(T))$ for $n > N(\delta)$. Now since $f_n(T) \in \mathfrak{M}$, σ_n consists of a finite number of points, say $f_n(\lambda_1^{(n)}), f_n(\lambda_2^{(n)}), \ldots, f_n(\lambda_{t_n}^{(n)})$. Hence

$$E(\sigma_n; f_n(T)) = \sum_{k=1}^{t_n} E(f_n(\lambda_k^{(n)}); f_n(T))$$
$$= \sum_{k=1}^{t_n} \Big[\sum_{s \in N_k^{(n)}} E_s \Big]$$

where $N_k^{(n)} = \{s: f_n(\lambda_s) = f_n(\lambda_k^{(n)})\}$, and making use of Theorem 5

$$= \sum_{s \in N_n} E_s$$

where $N_n = \{S: f_n(\lambda_s) \text{ lies inside } C_n\}$. Hence, for $n > N(\delta)$, N_n must be a fixed set of integers. Denote this fixed set by N. Define k_n to be the greatest order of the poles which $R_\lambda(T)$ has at the points $\{\lambda_s: s \in N_n\}$. Since N_n is a finite set, $k_n < \infty$. Moreover, for $n > N(\delta)$, k_n is a finite constant, say K.

Now for $s \in N_n$,

$$[f_n(\lambda_s) - f_n(T)]^{k_n+1} E_s = 0 \tag{5.1}$$

Consider s fixed in N_n. Then $\{f_n(\lambda_s)\}$, $n = N(\delta), N(\delta) + 1, \ldots$, is a sequence within C. Now we have seen earlier that all such sequences converge to elements of $\sigma(S)$. In this case, obviously, $f_n(\lambda_s) \to \mu_k$ as $n \to \infty$. Thus, from (5.1), taking the limit as $n \to \infty$, we get

$$[\mu_k - S]^{k+1} E_s = 0 \qquad \text{for each } s \in N.$$

Therefore

$$[\mu_k - S]^{k+1} E(\mu_k; S) = [\mu_k - S]^{k+1} \textstyle\sum_{s \in N} E_s = 0.$$

Hence $R_\lambda(S)$ has a pole at μ_k so that we can conclude that $S \in \mathfrak{M}$.

6. Meromorphic indices. In the proof of Theorem 5, mention was made of a sequence $\{p_n\}$ of positive integers. We now suppose that it is possible to choose $p_n \equiv p$ for all n. Following Derr and Taylor **(2)**, we say that T has *absolute index* p if

$$\sum_{n=m}^{\infty} ||S_n(\lambda) - P_n^{(p)}(\lambda)||$$

converges uniformly outside any circle $\{|\lambda| = \delta: |\lambda_k| < \delta \text{ for } k \geqslant m\}$. If p is the least integer for which this is true, then p is the *minimal absolute index*. The same condition on

$$\sum_{n=m}^{\infty} [S_n(\lambda) - P_n^{(p)}(\lambda)]$$

define *uniform index* and *minimal uniform index* relative to the enumeration $\{\lambda_n\}$ of the non-zero elements of $\sigma(T)$.

THEOREM 7. *Let T be meromorphic and $f \in \mathfrak{A}_0(T)$. Let $f(\lambda)$ have a zero of order s at $\lambda = 0$. Then if T has minimal absolute index p, $f(T)$ has minimal absolute index not exceeding p/s.*

Proof. Let E_n be defined as in Theorems 5 and 6. Now it is shown in **(2)** that T has minimal absolute index p if and only if

$$\sum_{n=1}^{\infty} ||T^q E_n||$$

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converges when $q = p$ but diverges when $q = p - 1$. Define

$$g(\lambda) = f(\lambda)/\lambda^s, \qquad \lambda \neq 0,$$
$$g(0) = \lim_{\lambda\to 0} f(\lambda)/\lambda^s.$$

Then $f(\lambda) = \lambda^s g(\lambda)$ for all λ in the domain of definition of f and $g(\lambda)$ is analytic wherever $f(\lambda)$ is analytic. If $\{\mu_n\}$ is an enumeration of the non-zero elements of $\sigma[f(T)]$, then

$$\sum_{n=1}^{\infty} ||[f(T)]^j E(\mu_n; f(T))|| = \sum_{n=1}^{\infty} \left\| [g(T)]^j \, T^{js} \sum_{s \in S(\mu_n)} E_s \right\|$$
$$\leqslant ||[g(T)]^j|| \sum_{k=1}^{\infty} ||T^{js} E_k||$$

where $E(\mu_n; f(T))$ is defined in Theorem 6 and $S(\mu_n)$ in Theorem 5, and the last step is justified since rearrangements are permissible in an absolutely convergent series. The assertion of the theorem follows.

THEOREM 8. *Let T be meromorphic, let $f \in \mathfrak{A}_0(T)$, and let f have a zero of order s at $\lambda = 0$. Let the non-zero elements of $\sigma(T)$ be given an enumeration $\{\lambda_k\}$ in such a way that $f(\lambda_k) = \mu_s$ or zero for $n_s \leqslant k < n_{s+1}$ where $\{n_s\}$ is some strictly increasing sequence of positive integers with $n_1 = 1$ and $\{\mu_s\}$ is some enumeration of the non-zero elements of $\sigma[f(T)]$. Suppose T has minimal uniform index p relative to $\{\lambda_k\}$ and that q is the least integer greater than or equal to p/s. Then $f(T)$ has minimal uniform index $m \leqslant q$ relative to $\{\mu_s\}$.*

If, in addition, the convergence of

$$\sum_{i=1}^{\infty} T^j \left(\sum_{k \in S(\mu_i)} E_k \right)$$

implies that of

$$\sum_{k=1}^{\infty} T^j E_k,$$

i.e. that the removal of parentheses does not affect convergence, then $m = q$.

Proof. We observe first that the non-zero elements of $\sigma(T)$ can always be enumerated in such a manner as the theorem assumes. For only a finite number of elements of $\sigma(T)$ can be zeros of f; otherwise f would be identically zero in some neighbourhood of the origin, contrary to assumption. Moreover, if an infinite number of elements of $\sigma(T)$ are mapped by f onto a single element of $\sigma[f(T)]$, then since $T \in \mathfrak{M}$, the continuity of f would imply that such an element must be zero.

As shown in **(2)**,

$$\sum_{k=1}^{\infty} T^j E_k$$

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converges if and only if $j \geqslant p$. Thus $\sum_k' T^p E_k$ converges where $\sum_k'$ indicates summation over only those k such that $f(\lambda_k) \neq 0$. By the construction of the enumeration,

$$\sum_k' T^p E_k = \sum_{s=1}^{\infty} T^p E(\mu_s; f(T)).$$

Define $g(\lambda)$ as in Theorem 7. Then

$$\sum_{s=1}^{\infty} [g(T)]^q T^{p+r} E(\mu_s; f(T))$$

is convergent for any non-negative integer r. Choose $r = qs - p$. Since $p/s \leqslant q$, r is non-negative. Thus

$$\sum_{s=1}^{\infty} [g(T)]^q \, T^{qs} E(\mu_s; f(T))$$

converges, i.e.

$$\sum_{s=1}^{\infty} [f(T)]^q \, E(\mu_s; f(T))$$

is convergent so that $m \leqslant q$.

Finally, define $S = \{\lambda\colon \lambda \in \sigma(T);\ g(\lambda) \neq 0\}$; then S is a spectral set, for $\sigma(T) - S \subseteq \{\lambda\colon \lambda \in \sigma(T);\ f(\lambda) = 0\}$ so that $\sigma(T) - S$ consists of a finite number of non-zero points of $\sigma(T)$. Let E be the spectral projection associated with S and T. Then the range of E, being closed, can be considered as a Banach space, which we shall denote by Y. Define T_1 in $B(Y)$ by $T_1 x = Tx$ for $x \in Y$. Then $f(T_1)$ and $g(T_1)$ are well defined. We prove that (a) $g(T_1)$ has a bounded inverse in $B(Y)$ and (b) for any function $h(\lambda)$ which is analytic on an open set containing $\sigma(T)$, then $h(T)E_n = h(T_1)E_n$ whenever $\lambda_n \in S$. The first of these assertions can be deduced from (**5**, p. 290), since $\sigma(T_1) = S$ and $g(\lambda)$ is non-zero on S. To prove (b), we show as a preliminary step that $R_\lambda(T_1)E_n = R_\lambda(T)E_n$ for $\lambda \in \rho(T)$ and $\lambda_n \in S$. Suppose that

$$\sigma(T) - S = \{\lambda_s\colon s \in \kappa\}$$

where κ is a finite set. In particular, if $\lambda_n \in S$, $n \notin \kappa$. Hence $E = I - \sum_{s\in\kappa} E_s$ so that if $x \in N(E)$, then $\sum_{s\in\kappa} E_s x = x$. Hence $E_n(\sum_{s\in\kappa} E_s x) = E_n x$ and thus $E_n x = 0$. Thus $N(E) \subseteq N(E_n)$. Since

$$X = R(E) \oplus N(E) = R(E_n) \oplus N(E_n),$$

it is easy to deduce that $R(E) \supseteq R(E_n)$, so that $(\lambda - T_1)E_n = (\lambda - T)E_n$. For $\lambda \in \rho(T)$, since $\rho(T) \subseteq \rho(T_1)$, $R_\lambda(T_1)E_n = R_\lambda(T)E_n$. If C is the boundary of a suitable Cauchy domain which contains $\sigma(T)$, we can write

$$\begin{aligned} h(T)E_n &= \frac{1}{2\pi i} \oint_C h(\lambda) R_\lambda(T) E_n \, d\lambda \\ &= \frac{1}{2\pi i} \oint_C h(\lambda) R_\lambda(T_1) E_n \, d\lambda = h(T_1)E_n. \end{aligned}$$

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Suppose now that

$$\sum_{n=1}^{\infty} [f(T)]^j E(\mu_n; f(T))$$

is convergent. This series can be written as

$$\sum_{n=1}^{\infty} [g(T_1)]^j T_1^{js}\Big(\sum_{k\in S(\mu_n)} E_k\Big)$$

since $\{\lambda_k: k \in S(\mu_n)\} \subseteq S$ for each n.

Because $g(T_1)$ has a bounded inverse, we can deduce the convergence of

$$\sum_{n=1}^{\infty} T_1^{js}\Big(\sum_{k\in S(\mu_n)} E_k\Big), \qquad \text{i.e. of } \sum_{n=1}^{\infty} T^{js}\Big(\sum_{k\in S(\mu_n)} E_k\Big).$$

By assumption, this implies the convergence of

$$\sum_{n=1}^{\infty} T^{js} E_k.$$

Hence $js \geqslant p$ so that $m \geqslant q$.

This concludes the proof.

Remark. The above theorem generalizes (**2**, Theorem 12).

References

1. S. R. Caradus, *Operators of Riesz type*, Pacific J. Math., *18* (1966), 61–71.

2. J. Derr and A. E. Taylor, *Operators of meromorphic type with multiple poles of the resolvent*, Pacific J. Math., *12* (1962), 85–111.

3. L. H. Loomis, *An introduction to abstract harmonic analysis* (Princeton, 1953).

4. A. F. Ruston, *Operators with Fredholm theory*, J. London Math. Soc., *29* (1954), 318–326.

5. A. E. Taylor, *Introduction to functional analysis* (New York, 1958).

6. ——— *Mittag Leffler expansions and spectral theory*, Pacific J. Math., *10*, 3 (1960), 1049–1066.

7. ——— *Spectral theory and Mittag Leffler type expansions of the resolvent*, Proc. Int. Symp. Linear Spaces, Jerusalem, 1960, pp. 426–440.

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