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ON PADÉ AND BEST RATIONAL APPROXIMATION

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ABSTRACT. It is reasonable to expect that, under suitable conditions, Padé approximants should provide nearly optimal rational approximations to analytic functions in the unit disc. This is shown to be the case for e^z in the sense that main diagonal Padé approximants are shown to converge as expeditiously as best uniform approximants. Some more general but less precise related results are discussed.

The partial sums of the power series expansion of an entire function provide excellent uniform polynomial approximations to that function on the unit circle in the complex plane. In fact, as is elaborated later, best uniform approximations often behave asymptotically exactly like partial sum approximations. Analogously, one would expect Padé approximants, under suitable conditions, to provide nearly optimal uniform rational approximations. We examine this idea further. In particular, we show that the main diagonal Padé approximants to e^z are, up to a constant, as efficient as the best uniform rational approximations of corresponding degree. This complements related results of Saff ([6] and [7]).

Notation. Let Π_n denote the collection of polynomials of degree at most n. We say that r is an (n, m) rational function if $r = p/q$ where $p \in \Pi_n$ and $q \in \Pi_m$. If f is analytic in some neighbourhood U of zero then the (n, m) Padé approximant to f is an (n, m) rational function $s = p/q$ that satisfies

$$p(z) - q(z)f(z) = z^{n+m+1}g(z) \tag{1}$$

where $g(z)$ is analytic in U and where q is not identically zero. We will call s proper if $q(0) \neq 0$, that is, if s is analytic in some neighbourhood of zero. The proper (n, m) Padé approximant is unique.

Let $\|\cdot\|_K$ denote the supremum norm on a set K in the complex plane C. We say that an (n, m) rational function r is a best (n, m) rational approximation to f on K if r satisfies

$$\|f - r\|_K = \inf_{p \in \Pi_n,\, q \in \Pi_m} \|f - p/q\|_K.$$

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If f is continuous and K is compact then r exists although it need not be unique. (see [10 p. 350]).

Finally, let

$$D_\delta=\{z\in C: |z|<\delta\}$$
$$C_\delta=\{z\in C: |z|=\delta\}$$

and

$$A_\delta=\{f: f \text{ is analytic in } D_\delta\}.$$

A criterion for optimality. This first lemma provides a lower bound for the error in best rational approximation on C_δ, $\delta>0$.

LEMMA 1. *Suppose that f is analytic on D_ρ, $\rho>\delta>0$. Suppose that f has a proper (n, m) Padé approximant s. Let r be a best (n, m) rational approximant to f on C_δ. Then,*

$$\|f(z)-r(z)\|_{C_\delta}\geq \min_{z\in C_\delta}|f(z)-s(z)|.$$

Proof. Suppose

(2) $$\|f(z)-r(z)\|_{C_\delta}<\min_{z\in C_\delta}|f(z)-s(z)|$$

then, for $z\in C_\delta$,

$$|f(z)-r(z)|=|r(z)-s(z)-(f(z)-s(z))|<|f(z)-s(z)|.$$

Let $r=p/q$ and $s=u/v$ where p, q, u and v are polynomials. Then, for $z\in C_\delta$

$$|p(z)v(z)-q(z)u(z)-q(z)(v(z)f(z)-u(z))|<|q(z)(v(z)f(z)-u(z))|.$$

Since $v\cdot f-u$ has $n+m+1$ zeroes inside C_δ, Rouche's theorem implies that $p\cdot v-q\cdot u$ has $n+m+1$ zeroes inside C_δ and is thus identically zero. This contradicts (2). □

See [9] where variations of Lemma 1 are pursued in detail.

THEOREM 1. *If r_n is a best (n, n) rational approximation to e^z on C_1 and if s_n is the (n, n) Padé approximant to e^z then*

$$\|r_n(z)-e^z\|_{C_1}\geq \tfrac{1}{24}\|s_n(z)-e^z\|_{C_1}$$

and

$$\frac{(n!)^2}{4.5105(2n)!\,(2n+1)!}\leq\|r_n(z)-e^z\|_{C_1}\leq\frac{(5.3)(n!)^2}{(2n)!\,(2n+1)!}.$$

Proof. If p_n/q_n is the (n, n) Padé approximant to e^z then

(3) $$e^z-\frac{p_n(z)}{q_n(z)}=\frac{(-1)^n z^{2n+1}}{q_n(z)(2n)!}\int_0^1 e^{tz}t^n(1-t)^n\,dt$$

and

(4) $$q_n(z)=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{2n}{k}}\frac{(-z)^k}{k!}.$$

These well known identities may be found in [4. p. 431–6].

We first show that for $|z|=1$

(5) $$2-e^{1/2}\le|q_n(z)|\le e^{1/2}.$$

We rewrite (4) as

$$q_n(z)=\frac{n!}{(2n)!}\sum_{k=0}^{n}\frac{(2n-k)!}{k!\,(n-k)!}(-z)^k.$$

From which we deduce that

$$\begin{aligned}\|q_n\|_{C_1}&\le\frac{n!}{(2n)!}\left(\frac{(2n)!}{n!}+\frac{(2n-1)!}{(n-1)!}+\frac{(2n-2)!}{2!\,(n-2)!}+\cdots\right)\\&\le\frac{n!}{(2n)!}\left(\frac{(2n)!}{n!}+\frac{1}{2}\frac{(2n)!}{n!}+\frac{1}{4}\frac{(2n)!}{2!\,n!}+\cdots\right)\\&\le e^{1/2}\end{aligned}$$

and for $z\in C_1$,

$$\begin{aligned}|q_n(z)|&\ge\frac{n!}{(2n!)}\left(\frac{(2n)!}{n!}-\frac{(2n-1)!}{(n-1)!}-\frac{(2n-2)!}{2!\,(n-2)!}-\frac{(2n-3)!}{3!\,(n-3)!}\cdots\right)\\&\ge\left(1-\frac{1}{2}-\frac{1}{2^2\cdot 2!}-\frac{1}{2^3\cdot 3!}\cdots\right)\\&\ge 2-e^{1/2}.\end{aligned}$$

We now estimate $\int_0^1 e^{tz}(1-t)^n t^n$.

Clearly, for $z\in C_1$,

$$\left|\int_0^1 e^{tz}(1-t)^n t^n\,dt\right|\le\int_0^1 e^{t}(1-t)^n t^n\,dt\le\int_0^1\sum_{k=0}^{\infty}\frac{(1-t)^n t^{n+k}}{k!}\,dt$$

(6) $$\begin{aligned}&\le\sum_{k=0}^{\infty}\frac{n!\,(n+k)!}{(2n+k+1)!\,k!}\le\frac{n!\,n!}{(2n+1)!}\left(1+\frac{1}{2}+\frac{1}{2\cdot 2!}+\frac{1}{2\cdot 3!}+\cdots\right)\\&\le\frac{n!\,n!}{(2n+1)!}\left(\frac{1}{2}+\frac{1}{2}e\right).\end{aligned}$$

If we write $z = x + iy$ then,

$$\left|\int_0^1 e^{tz}(1-t)^n t^n\,dt\right| \geq \left|\int_0^1 e^{tx}[\cos(ty)](1-t)^n t^n\,dt\right|$$

$$(7) \qquad \geq \int_0^1 e^{-t}[\cos t](1-t)^n t^n\,dt$$

$$\geq \frac{(e^{-1/2}\cos(\frac{1}{2}) + e^{-1}\cos(1))}{2}\left(\frac{n!\,n!}{(2n+1)!}\right).$$

From (3), (5), (6), and (7) we deduce that for $z \in C_1$

$$(8) \qquad \frac{1}{4.5105}\frac{n!\,n!}{(2n)!\,(2n+1)!} \leq |e^z - p_n(z)/q_n(z)| \leq \frac{(5.3)n!\,n!}{(2n)!\,(2n+1)!}.$$

The result now follows from Lemma 1. □

There are only a few types of functions for which we can write down the Padé error in sufficiently nice form to allow explicit computations of the above variety. One class of functions which are amenable to similar techniques as those employed in Theorem 1 are functions of the form

$$(9) \qquad f(z) = \int_0^\gamma \frac{d\alpha(t)}{1+zt}$$

where $\alpha(t)$ is a real, non-decreasing function assuming infinitely many values on $[0, \gamma]$. Such functions are called Stieltjes series. If f is defined as in (9) and if $s_{n-1,n}$ is the $(n-1, n)$ Padé approximant to f then $s_{n-1,n}$ is proper and

$$(10) \qquad f(z) - s_{n-1,n}(z) = \frac{1}{p_n^2(-1/z)}\int_0^\gamma \frac{p_n^2(t)\,d\alpha(t)}{1+zt}$$

where $z \in C - (-\infty, -1/\gamma]$ and where p_n is a real polynomial of degree n with all its roots in the interval $[0, \gamma]$. (see [3]).

THEOREM 2. *Suppose $0 < \gamma < 1$ and suppose that*

$$f(z) = \int_0^\gamma \frac{d\alpha(t)}{1+zt}$$

where α is real, non-decreasing and assumes infinitely many values on $[0, \gamma]$. Let $s_{n-1,n}$ be the $(n-1, n)$ Padé approximant to f and let $r_{n-1,n}$ be a best uniform $(n-1, n)$ rational approximation to f on C_1. Then,

$$\|f - r_{n-1,n}\|_{C_1} \geq \left(\frac{1-\gamma}{1+\gamma}\right)^{2n+1} \|f - s_{n-1,n}\|_{C_1}.$$

Proof. Let p_n be defined as in (10) and let

$$M = \max_{|z|=1}\left|p_n^2\left(-\frac{1}{z}\right)\right| \quad \text{and} \quad m = \min_{|z|=1}\left|p_n^2\left(-\frac{1}{z}\right)\right|.$$

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Since $p_n^2(-1/z)$ has all its roots in $[-\infty, -1/\gamma]$ we deduce that

$$\frac{M}{m} \le \frac{\left(\frac{1}{\gamma}+1\right)^{2n}}{\left(\frac{1}{\gamma}-1\right)^{2n}} = \left(\frac{1+\gamma}{1-\gamma}\right)^{2n}.$$

From (10);

$$\|f(z)-s_{n-1}(z)\|_{C_1} \le \frac{1}{m(1-\gamma)} \int_0^\gamma p_n^2(t)\, d\alpha(t)$$

and

$$\min_{z\in C_1} |f(z)-s_{n-1}(z)| \ge \frac{1}{M(1+\gamma)} \int_0^\gamma p_n^2(t)\, d\alpha(t).$$

The result now follows from Lemma 1. □

The above theorem can be applied to $z^{-1}\log(1+\gamma z)$ on C_1. In this case best $(n-1, n)$ rational approximation can only improve on the $(n-1, n)$ Padé approximant by a factor of β^n.

The polynomial case. In the polynomial case one can, of course, say a good deal more (see, for example, [8]). Suppose $f = \sum_{k=0}^\infty a_k z^k$ is entire. Let p_n be the best uniform polynomial approximation of degree n to f on C_1 then

(11) $$\limsup_{n\to\infty} \frac{\|f-p_n\|_{C_1}}{\|f-\sum_{k=0}^n a_k z^k\|_{C_1}} = 1.$$

Statement (11) is an easy consequence of Lemma 1 and the observation that for infinitely many n

$$|a_n|\,(1-\varepsilon) \le \left|\sum_{k=n}^\infty a_k z^k\right| \le |a_n|(1+\varepsilon), \qquad z\in C_1.$$

Let

$$B_\rho = \left\{f\colon f = \sum_{k=0}^\infty a_k z^k \text{ and } |a_k|/|a_{k+1}| \ge \rho\right\}.$$

If $f \in B_\rho$, $\rho > 2$, and if p_n is the best uniform polynomial approximation to f on C_1 then

(12) $$\frac{\|f-p_n\|_{C_1}}{\|f-\sum_{k=0}^n a_k z^k\|_{C_1}} \ge \frac{\rho-2}{\rho}.$$

The above inequality follows from Lemma 1 and the observation that, for $z \in C_1$

$$|a_n| \left(\frac{\rho-2}{\rho-1}\right) \le \left|\sum_{k=n}^\infty a_k z^k\right| \le |a_n| \left(\frac{\rho}{\rho-1}\right).$$

The function $\rho/(z-\rho)\in B_\rho$. Rivlin [5] constructs explicit polynomial approximations to $\rho(z-\rho)$ and shows that (in the above notation)

$$\left\|\frac{\rho}{z-\rho}-p_n(z)\right\|_{C_1}=\frac{1}{\rho^{n-1}(\rho^2-1)}.$$

Also,

$$\frac{\rho}{z-\rho}=\sum_{k=0}^{\infty}\frac{z^k}{\rho^k}$$

and

$$\left\|\frac{\rho}{z-\rho}-\sum_{k=0}^{n}\frac{z^k}{\rho^k}\right\|_{C_1}=\frac{1}{\rho^n(\rho-1)}.$$

Thus, the constant $(\rho-2)/\rho$ in (12) cannot be replaced by any constant greater than $\rho/(\rho+1)$. In particular, (11) does not hold in general for non-entire functions.

The general case. If the (n, m) Padé approximant $s_{n,m}$ to $f\in A_\rho$ has no poles in A_ρ then it is reasonable to expect that $s_{n,m}$ should be a good uniform approximation on compact subsets. The final theorem, which is closely related to results of Gončar [2], shows that this is sometimes the case.

THEOREM. *Let $f\in A_{\rho^*}$ and suppose that p_n/q_m is the proper (n, m) Padé approximant to f. Let $\lambda<\rho<\rho^*$ and let p_n^*/q_m^* be a best uniform rational approximation to f on $\{z:|z|\le\rho\}$. If p_n/q_m has no poles in D_ρ then for $|z|<\lambda$,*

$$|f(z)-p_n(z)/q_m(z)|\le\frac{|z|^{m+n+1}}{\lambda^{m+n}}\left(1+\frac{2\lambda}{\rho-\lambda}\right)^{2m}\frac{\|f-p_n^*/q_m^*\|_{C_\lambda}}{\lambda-|z|}.$$

Proof. Let w_m be any polynomial of degree m, then for $z\in D_\lambda$,

$$f(z)-\frac{p_n(z)}{q_m(z)}=\frac{z^{m+n+1}}{q_m(z)w_m(z)}\cdot\frac{1}{2\pi i}\int_{C_\lambda}\frac{w_m(\zeta)q_m(\zeta)f(\zeta)\,d\zeta}{\zeta^{m+n+1}(\zeta-z)}.$$

This is an application of Cauchy's theorem (see [1, p. 336] for further discussion). If we take $w_m=q_m^*$ in the above equation we get

$$\begin{aligned}f(z)-\frac{p_n(z)}{q_m(z)}&=\frac{z^{m+n+1}}{q_m(z)q_m^*(z)}\cdot\frac{1}{2\pi i}\int_{C_\lambda}\frac{q_m^*(\zeta)q_m(\zeta)f(\zeta)\,d\zeta}{\zeta^{m+n+1}(\zeta-z)}\\&=\frac{z^{m+n+1}}{q_m(z)q_m^*(z)}\cdot\frac{1}{2\pi i}\int_{C_\lambda}\frac{q_m(\zeta)(q_m^*(\zeta)\,f(\zeta)-p_n^*(\zeta))\,d\zeta}{\zeta^{m+n+1}(\zeta-z)}\end{aligned}$$

where the second equality holds since $q_m\cdot p_n^*$ is of degree less than $m+n+1$. Thus,

$$\left|f(z)-\frac{p_n(z)}{q_m(z)}\right|\le\frac{|z|^{m+n+1}}{\lambda^{m+n}}\frac{\|q_m\cdot q_m^*\|_{C_\lambda}}{|q_m(z)q_m^*(z)|}\cdot\frac{\|f-p_n^*/q_m^*\|_{C_\lambda}}{\lambda-|z|}.$$

The result now follows from the observation that if $t_m \in \Pi_m$ has no zeroes in D_ρ then

$$\frac{\max_{z\in D_\lambda}|t_m(z)|}{\min_{z\in D_\lambda}|t_m(z)|} \le \left(1+\frac{2\lambda}{\rho-\lambda}\right)^m. \quad \square$$

If we apply the above theorem to convergence along the rows, that is for fixed m, then $(1+2\lambda/[\rho-\lambda])^m$ is constant. This suggests that rational approximations in this case may offer little improvement over Padé approximants, at least in the interior of the C_λ. This raises the following:

QUESTION 1. Suppose f is entire. Let $\rho>1$. Let s_n be the (n, n) Padé approximant to f and let r_n be a best uniform (n, n) approximant to f on $\overline{D_\rho}$. Suppose $\{s_n\}$ has a subsequence that converges to f uniformly on $\bar{D}_\rho$. Does there exist a constant $h_\rho>0$ depending only on ρ so that

$$\limsup_{n\to\infty} \frac{\|f-r_n\|_{C_1}}{\|f-s_n\|_{C_1}} > h_\rho?$$

Does $h_\rho \to 1$ as $\rho\to\infty$?

As posed, the above question avoids the difficult question of the existence of such a sequence. Related to the above is

QUESTION 2. Given any sequence of positive numbers δ_n tending to zero. Is it possible to find an entire function f so that for n sufficiently large

$$\|f-r_n\|_{C_1} \le \delta_n \, \|f-p_{2n}\|_{C_1}$$

where r_n is the best uniform (n, n) approximant to f and where p_{2n} is the $2n$th partial sum of the Taylor expansion of f?

If f is not entire then the above problem is trivially true for $1/(z-2)$.

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