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POINCARE DUALITY AND THE RING OF COINVARIANTS

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ABSTRACT. It is shown that, in characteristic zero, a finite subgroup of a general linear group is generated by pseudo-reflections if and only if its ring of coinvariants satisfies Poincaré duality.

1. **Introduction.** Let $G \subset \mathrm{GL}(V)$ be a finite subgroup where V is a finite dimensional vector space over a field $\mathbb{F}$ of characteristic 0. Let $S = S(V)$ be the symmetric algebra of V. The action of G on V extends to a multiplicative action on S. The *ring of invariants* is given by

$$R = S^G = \{x \in S \mid g \cdot x = x \text{ for all } g \in G\}.$$

If one lets

$$I = \text{ the graded ideal of } S \text{ generated by } R_+ = \sum_{i \geq 1} R^i$$

then one can also form the *ring of coinvariants* S/I. It is well known that the assertion that $G \subset \mathrm{GL}(V)$ is a pseudo-reflection group (*i.e.* generated by its pseudo-reflections) is equivalent to either of the following conditions

(1.1) R is a polynomial algebra

(1.2) S is a free R module.

As a convenient reference for invariant theory and pseudo-reflection groups see Stanley's discussion in [1]. The main result of this note is to give another characterization of pseudo-reflection groups, this time in terms of the ring of coinvariants. This characterization, as we will see, is a corollary of the work of Steinberg in [2]. This note is mainly concerned with explaining how his criterion for G being a pseudo-reflection group can be translated into one involving Poincaré duality.

It is a standard fact from invariant theory that, for any G, R is finitely generated as an algebra and the extension $R \subset S$ is finite. In other words, S is a finite R module or, equivalently, S/I is a finite dimensional algebra. A finite dimensional graded $\mathbb{F}$ algebra A is said to satisfy *Poincaré duality* if there exists a positive integer N such that

(1.3) $A^N = \mathbb{F}$ while $A^i = 0$ for $i > N$

(1.4) the pairing $\begin{matrix} A^i \otimes A^{N-i} \longrightarrow A^N = \mathbb{F} \\ x \otimes y \longmapsto x \cdot y \end{matrix}$ is nonsingular for each $0 \leq i \leq N$.

The rest of this paper will be devoted to proving

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THEOREM 1.5. *$G \subset \mathrm{GL}(V)$ is a pseudo-reflection group if and only if S/I satisfies Poincaré duality.*

In the next two sections we will study the harmonic elements of $G \subset \mathrm{GL}(V)$. In particular, in §3, we will use the harmonic polynomials to prove the above theorem. Throughout this paper we will assume that $\mathbb{F}$ is a field of characteristic 0, V is a finite dimensional vector space over $\mathbb{F}$, and $G \subset \mathrm{GL}(V)$ is a finite group.

2. **Differential operators.** Let V^* be the dual vector space of V. In order to define the harmonic elements of $S = S(V)$, we also need to introduce the symmetric algebra $S^* = S(V^*)$ of V^* and consider its relation to S. That will be done in this section. The harmonics will be defined and studied in §3. We are going to think of both S and S^* as dual graded Hopf algebras. Regarding the algebra structure, both S and S^* are polynomial algebras. If $\{t_1, \ldots, t_n\}$ is a basis of V then the monomials in $\{t_1, \ldots, t_n\}$ are an $\mathbb{F}$ basis of S, and we then write $S = \mathbb{F}[t_1, \ldots, t_n]$. Similarly, for any basis $\{\alpha_1, \ldots, \alpha_n\}$ of V^*, we can write $S^* = \mathbb{F}[\alpha_1, \ldots, \alpha_n]$. The gradings on S and S^* are determined by the stipulation that the elements of V and V^* have degree 1. The coalgebra structures $\Delta\colon S \longrightarrow S \otimes S$ and $\Delta^*\colon S^* \longrightarrow S^* \otimes S^*$ are determined by stipulating that the elements of V and V^* are primitive *i.e.*

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \text{for all } x \in V$$
$$\Delta^*(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha \quad \text{for all } \alpha \in V^*.$$

So S and S^* are primitively generated Hopf algebras. Moreover they are dual Hopf algebras. The Kronecker pairing

$$\langle\ ,\ \rangle\colon V^* \otimes V \longrightarrow \mathbb{F}$$

extends to a pairing

$$\langle\ ,\ \rangle\colon S^* \otimes S \longrightarrow \mathbb{F}$$

which relates the Hopf algebra structure of S^* and S. Notably

(2.1) $$\langle \alpha, x \cdot y \rangle = \langle \Delta^*(\alpha), x \otimes y \rangle$$

for any $\alpha \in S^*$ and $x, y \in S$.

Besides thinking of S^* as the dual Hopf algebra of S, we can also interpret S^* as a Hopf algebra of differential operators acting on S. For any $\alpha \in S^*$ we will use $D_\alpha\colon S \longrightarrow S$ to denote the corresponding linear operator. We begin with a relatively informal description of the action $S^* \otimes S \longrightarrow S$. For any $\alpha \in V^*$ the operator D_α is a derivation determined by the rules:

$$D_\alpha(x) = \langle \alpha, x \rangle \quad \text{for } x \in V$$
$$D_\alpha(xy) = D_\alpha(x)y + xD_\alpha(y).$$

For an arbitrary $\alpha \in S^* = \mathbb{F}[\alpha_1, \ldots, \alpha_n]$ one then defines D_α by replacing $\{\alpha_1, \ldots, \alpha_n\}$ in $\alpha = f(\alpha_1, \ldots, \alpha_n)$ by the derivatives $\{D_{\alpha_1}, \ldots, D_{\alpha_n}\}$. In other words, multiplication

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in S^* corresponds to composition of the associated differential operators. More formally, the action $S^* \otimes S \longrightarrow S$ is determined by the two requirements that:

(2.2) for any $\alpha \in V^*$ and $x \in V$, $D_\alpha(x) = \langle \alpha, x \rangle$

(2.3) for any $\alpha \in S^*$ and $x, y \in S$ if $\Delta^*(\alpha) = \sum \alpha_i' \otimes \alpha_i''$ then

$$D_\alpha(x \cdot y) = \sum D_{\alpha_i'}(x) \cdot D_{\alpha_i''}(y).$$

This definition of D_α agrees with the previous one. First of all, for $\alpha \in V^*$ we have $\Delta^*(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$ so it follows from property (2.3) that D_α is a derivation. Secondly, it follows from property (2.3) plus the identity $\Delta^*(\alpha \cdot \beta) = \Delta^*(\alpha)\Delta^*(\beta)$ that $D_\alpha D_\beta = D_{\alpha\beta}$ for any $\alpha, \beta \in S^*$.

The action of S^* on S incorporates the pairing $\langle\ ,\ \rangle \colon S^* \otimes S \longrightarrow \mathbb{F}$. For it follows from (2.1), (2.2) and (2.3) that

(2.4) for any $\alpha \in S^{*k}$ and $x \in S^k$, $D_\alpha(x) = \langle \alpha, x \rangle$.

There is a third way as well to define the operations D_α. For we can use the above defining properties to deduce the following formula for D_α.

LEMMA 2.5. *Given $\alpha \in S^*$ and $x \in S$ if $\Delta(x) = \sum x_i' \otimes x_i''$ then $D_\alpha(x) = \sum \langle \alpha, x_i' \rangle x_i''$.*

PROOF. Our proof is by induction on the degree of α. First of all, suppose that $\alpha \in V^*$ *i.e.* $\deg(\alpha) = 1$. Write $S = \mathbb{F}[t_1, \ldots, t_n]$. It suffices to verify the formula for any monomial $t^E = t_1^{e_1} \cdots t_n^{e_n}$. By property 2.4 we have $D_\alpha(t_i) = \langle \alpha, t_i \rangle$. By the derivation property of D_α we then have

$$D_\alpha(t^E) = \sum_i e_i t^{\hat{E}_i} \langle \alpha, t_i \rangle \quad \text{where } \hat{E}_i = (e_1, e_2, \ldots, e_i - 1, e_{i+1}, \ldots).$$

The coproduct $\Delta \colon S \longrightarrow S \otimes S$ satisfies

$$\Delta(t^E) = \sum_{F+G=E} (F, G) t^F \otimes t^G$$

where

$$(F, G) = \prod_s \frac{(f_s + g_s)!}{f_s!\, g_s!}.$$

One can reformulate the above identity as

$$D_\alpha(t^E) = \sum_{F+G=E} (F, G) \langle \alpha, t^F \rangle t^G$$

which is the desired formula.

Next pick $k \geq 2$ and suppose that the lemma holds in degree $< k$. In proving the lemma in degree k, one can reduce to monomials. Every monomial α of degree k can be decomposed $\alpha = \alpha' \alpha''$, where $\deg(\alpha'), \deg(\alpha'') < k$. In particular, the lemma holds for $D_{\alpha'}$ and $D_{\alpha''}$. Pick $x \in S$ and write $\Delta(x) = \sum x_i' \otimes x_i''$. Since $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ we can write

$$(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x) = \sum x_i' \otimes x_{ij} \otimes x_j''$$

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where x_i' and x_j'' are as above. We now have the identities

$$\begin{aligned} D_\alpha(x) &= D_{\alpha'}D_{\alpha''}(x) \\ &= D_{\alpha'}\Big[\sum\langle\alpha'', x_i'\rangle x_i''\Big] \\ &= \sum\langle\alpha'', x_i'\rangle\langle\alpha', x_{ij}\rangle x_j'' \\ &= \sum\langle\alpha''\otimes\alpha', \Delta(x_j')\rangle x_j'' \\ &= \sum\langle\alpha''.\alpha', x_j'\rangle x_j'' \\ &= \sum\langle\alpha, x_j'\rangle x_j''. \end{aligned}$$

∎

Lastly, we want to observe that it follows from the above lemma that the action $S^*\otimes S \longrightarrow S$ can be dualized in an appropriate sense so as to be equivalent to the product $S^*\otimes S^* \longrightarrow S^*$. The relationship is given by the following lemma.

LEMMA 2.6. *For any $\alpha,\beta \in S^*$, $x \in S$, $\langle\alpha, D_\beta(x)\rangle = \langle\alpha\cdot\beta, x\rangle$.*

PROOF.

$$\begin{aligned} \langle\alpha\cdot\beta, x\rangle &= \langle\alpha\otimes\beta, \Delta^*(x)\rangle \\ &= \Big\langle\alpha\otimes\beta, \sum x_j'\otimes x_j''\Big\rangle \\ &= \Big\langle\alpha, \sum x_j'\langle\beta, x_j''\rangle\Big\rangle \\ &= \langle\alpha, D_\beta(x)\rangle. \end{aligned}$$

The last equality follows from the previous lemma. ∎

3. **Harmonic elements.** The invariants of S were introduced in §1. One can also look at the invariants of S^*. The action of G on V induces an action on V^* by the rule

$$\langle\varphi\cdot\alpha, x\rangle = \langle\alpha, \varphi^{-1}\cdot x\rangle$$

for any $\alpha \in V^*$, $x \in V$ and $\varphi \in G$. The action of G on V^* extends to an action on S^* and so one can consider $R^* = S^{*^G}$.

DEFINITION. An element $x \in S$ is said to be *harmonic* if $D_\alpha(x) = 0$ for all $\alpha \in R^*$.

REMARK. The term harmonic arises from the case of the Coxeter groups $W(A_{n-1}) = \Sigma_n$, $W(B_n) = \Sigma_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ and $W(D_n) = \Sigma_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$. When interpreted as reflection groups, these groups act by permuting $\{t_1,\ldots,t_n\}$ as well as by changing their signs. It follows that $t_1^2+\cdots+t_n^2$ is an invariant of each of them. Consequently, a harmonic element in each of these cases must satisfy

$$\Delta(x) = 0$$

where $\Delta = \frac{\partial^2}{\partial t_1^2}+\cdots+\frac{\partial^2}{\partial t_n^2}$ is the usual Laplacian. This equation is the usual definition of a harmonic function. The concept of a harmonic element is traditionally ascribed to Borel. They were studied by Steinberg in [2] although he did not use the term harmonic.

We will let $H \subset S$ denote the subspace of harmonic elements. Analogous to the ideal $I \subset S$ defined in §1, one can form the ideal $I^* \subset S^*$ generated by R_+^*. Observe that the definition of H can be strengthened to assert

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LEMMA 3.1. *$x \in H$ if and only if $D_\alpha(x) = 0$ for all $\alpha \in I^*$.*

There is also another approach to harmonics which will be useful in what follows. Namely,

LEMMA 3.2. *The submodule $H \subset S$ is dual to the quotient module $S^* \longrightarrow S^*/I^*$ for any $\alpha \in S^*$, $\beta \in R^*$. By Lemma 2.6 we have the identity*

$$\langle \alpha \cdot \beta, x \rangle = \langle \alpha, D_\beta(x) \rangle \quad \textit{for any } x \in S.$$

Moreover

$$\langle \alpha, D_\beta(x) \rangle = 0 \quad \textit{for all } \alpha \textit{ and } \beta \textit{ as above if and only if } x \in H.$$

As a final comment concerning Lemma 3.2, observe that since S^*/I^* is a quotient algebra it follows that $H \subset S$ is a subcoalgebra.

Next we can consider H as a S^* module. For the action of S^* on S via the maps $D_\alpha \colon S \longrightarrow S$ leaves H invariant. For if $x \in H$ then for any $\alpha \in S^*$, $\beta \in R^*$, $D_\beta D_\alpha(x) = D_\alpha D_\beta(x) = D_\alpha(0) = 0$. So $D_\alpha(x) \in H$. Recall that a S^* module is *cyclic* if it is generated by a single element. As a preliminary to proving Theorem 1.5, we next explain how an extension of the arguments in [2] can be used to prove

THEOREM 3.3 (STEINBERG). *$G \subset \mathrm{GL}(V)$ is a pseudo-reflection group if and only if H is a cyclic S^* module.*

Steinberg's paper actually deals with a slightly different situation. First of all, his arguments deal with the case where the roles of S and S^* are interchanged. He considers S as an algebra of differential operators on S^* rather than vice versa. So, from his vantage point, the harmonic elements are located in S^*. Secondly, he only works with $\mathbb{F} = \mathbb{C}$, the complex numbers, and S^* (= the polynomial functions V) is expanded to the larger algebra $\hat{S}^*$ of entire functions on V. The action of S on S^* is extended to an action on $\hat{S}^*$, and analogues in $\hat{S}^*$ of the harmonics are then studied.

In this paper we are dealing with the action of S^* on S. In analogy with Steinberg's strategy we will expand S to a larger algebra $\hat{S}$. We pass from the polynomials $S = S(V)$ to the formal power series $S((V))$. Actually it suffices (and is convenient) to work with a subalgebra $\hat{S} \subset S((V))$. For each $x \in V \subset S$ we have

$$e^x = \sum_{n \geq 0} x^n/n!$$

in $S((V))$. Let

$$\hat{S} = \text{ the algebra of } S((V)) \text{ generated by } S \text{ and } \{e^x \mid x \in V\}.$$

The differential action of S^* on S extends to an action of S^* on $\hat{S}$. Namely, we differentiate e^x in the usual manner. For any $\alpha \in S^*$ we have

$$D_\alpha(e^x) = \alpha(x)e^x.$$

Here we are identifying S^* with the polynomial functions on V so that $\alpha(x)$ denotes the value of the polynomial $\alpha \in S^*$ on $x \in V$. With the above alterations the proof of Theorem 3.3 is now directly analogous to the arguments appearing in [2]. We will briefly outline our version of these arguments. For more details consult [2].

First of all, assume that $G \subset \mathrm{GL}(V)$ is a pseudo-reflection group. Each pseudo-reflection $s\colon V \longrightarrow V$ has a unique (up to scalar multiple) eigenvector $a \in V$ where $s(a) = \zeta \cdot a$ for a primitive n-th root of unity $\zeta = \zeta_n$ $(n \geq 2)$. We can form the element $\Omega = \prod_s a_s$ in S. It is straightforward to show that $\Omega \in H$ and, hence, $D\Omega = \{D_\alpha(\Omega) \mid \alpha \in S^*\}$ satisfies $D\Omega \subset H$. If one combines Lemma 3.1 with

LEMMA 3.4. *Given $\alpha \in S^*$ of degree > 0 then $D_\alpha(\Omega) = 0$ if and only if $\alpha \in I^*$.*

then one obtains the equality $D\Omega = H$ and, hence, H is cyclic. For a proof of Lemma 3.4 consult Steinberg's proof of Theorem 1.3(b) in [2].

Secondly, assume that H is a cyclic S^* module. For each $x \in V$ we can define an analogue of the harmonic polynomials.

DEFINITION. $H_x = \{h \in \hat{S} \mid D_\alpha(h) = \alpha(x)h$ for all $\alpha \in R^*\}$. Let $d_x = \dim_{\mathbb{F}} H_x$.

A basic fact of invariant theory is that $\dim_{\mathbb{F}} S/I \geq |G|$ and $\dim_{\mathbb{F}} S/I = |G|$ if and only if G is a pseudo-reflection group. So, to prove the proposition, it suffices to prove $\dim_{\mathbb{F}} S/I = |G|$. One can prove

LEMMA 3.5. *For any x, $d_x \geq \dim_{\mathbb{F}} S/I$.*

LEMMA 3.6. *If the isotropy group G_x of H_x is trivial then $d_x \leq |G|$.*

Putting together these lemmas we have, for the appropriate x,

$$|G| \leq \dim_{\mathbb{F}} S/I \leq d_x \leq |G|.$$

Thus $\dim_{\mathbb{F}} S/I = |G|$ as desired. The proof of Lemma 3.5 is analogous to that of Lemma 4.3 in [2]. The proof of Lemma 3.6 is analogous to that of Theorem 1.2(b) in [2].

As a remark, the hypothesis of H being cyclic is used in the proof of Lemma 3.5. If one chooses a cyclic generator $P \in H$ then the element $P_x = e^x P$ satisfies $P_x \in H_x$. One can show that $D_\alpha(P_x) \in H_x$ for all $\alpha \in S^*$ and that $D_\alpha(P_x) \neq 0$ if $\alpha \neq 0$ in S^*/I^*.

We now set about using Theorem 3.3 to prove Theorem 1.5.

PROOF OF THEOREM 1.5. As explained at the beginning of this section, the inclusion $G \subset \mathrm{GL}(V)$ induces, via duality, an inclusion $G \subset \mathrm{GL}(V^*)$. We will prove the dual assertion that

(3.7) $G \subset \mathrm{GL}(V^*)$ is a pseudo-reflection group if and only if S^*/I^* satisfies Poincaré duality.

To prove this equivalence it suffices to prove that

(3.8) H is a cyclic S^* module if and only if S^*/I^* satisfies Poincaré duality.

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For it is straightforward that $G \subset \mathrm{GL}(V^*)$ is a pseudo-reflection group if and only if $G \subset \mathrm{GL}(V)$ is a pseudo-reflection group (pseudo-reflections dualize to pseudo-reflections). So (3.7) follows from (3.8) by applying Theorem 3.3.

We now set about proving (3.8). It follows from Lemma 3.1 that we have an action $S^*/I^* \otimes H \longrightarrow H$. This action is dual to the product map $S^*/I^* \otimes S^*/I^* \longrightarrow S^*/I^*$. For, by Lemma 2.6 plus the duality established between S^*/I^* and H in Lemma 3.2, we have the identity

$$(3.9) \qquad \langle \alpha \cdot \beta, x \rangle = \langle \alpha, D_\beta(x) \rangle \quad \text{for any } \alpha, \beta \in S^*/I^*,\ x \in H.$$

This identity enables us to relate the two conditions appearing in (3.8). First, asserting that H is cyclic with generator Ω is equivalent to asserting that for every $\alpha \in S^*/I^*$ we can find $\beta \in S^*/I^*$ such that $\langle \alpha, D_\beta(\Omega) \rangle \neq 0$. Here we are also using the duality between H and S^*/I^*. Secondly, asserting that S^*/I^* satisfies Poincaré duality is equivalent to asserting that there exists Ω so that, for every $\alpha \in S^*/I^*$, we can find $\beta \in S^*/I^*$ such that $\langle \alpha \cdot \beta, \Omega \rangle \neq 0$. ∎

References

1. R. Stanley, *Invariants of Finite Groups and their Application to Combinatorics*, Bull. Amer. Math. Soc. **1**(1979), 475–511.
2. R. Steinberg, *Differential Equations Invariant under Finite Reflection Groups*, Trans. Amer. Math. Soc. **112**(1964), 392–400.

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