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ORE EXTENSIONS OF WEAK ZIP RINGS*

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Abstract. In this paper we introduce the notion of weak zip rings and investigate their properties. We mainly prove that a ring R is right (left) weak zip if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is right (left) weak zip. Let α be an endomorphism and δ an α-derivation of a ring R. Then R is a right (left) weak zip ring if and only if the skew polynomial ring $R[x; \alpha, \delta]$ is a right (left) weak zip ring when R is (α, δ)-compatible and reversible.

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1. Introduction. Throughout this paper R denotes an associative ring with unity, $\alpha : R \longrightarrow R$ is an endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. We denote $S = R[x; \alpha, \delta]$ as the Ore extension whose elements are the polynomials over R; the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. Following Rage and Chhawchharia [**14**], a ring R is said to be Armendariz in that whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. Recall that a ring R is called

$$\text{reduced if } a^2 = 0 \Rightarrow a = 0, \text{ for all } a \in R,$$
$$\text{reversible if } ab = 0 \Rightarrow ba = 0, \text{ for all } a, b \in R,$$
$$\text{semicommutative if } ab = 0 \Rightarrow aRb = 0, \text{ for all } a, b \in R.$$

The following implications hold:

$$\text{Reduced} \Rightarrow \text{Reversible} \Rightarrow \text{Semicommutative}.$$

In general, each of these implications is irreversible (see [**13**]).

According to Krempa [**10**], an endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring R α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by Hong et al. [**7**]. Properties of α-rigid rings have been studied in Krempa [**10**], Hong [**7**] and Hirano [**5**]. Let α be an endomorphism and δ an α-derivation of a ring R. Following Hashemi and Moussavi [**4**], a ring R is

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said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called δ-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, then R is said to be (α, δ)-compatible. A ring R is α-rigid if and only if R is (α, δ)-compatible and reduced (see [**6**]).

For any subset X of a ring R, $r_R(X)$ denotes the right annihilator of X in R. Faith [**2**] called a ring R right zip provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. R is zip if it is right and left zip. The concept of zip rings was initiated by Zelmanowitz [**16**] and appeared in various papers [**1**–**3**]. Zelmanowitz stated that any ring satisfying the descending chain condition on right annihilators is a right zip ring, but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [**1**] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is a zip ring. The authors in [**9**] proved that R is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring when R is an Armendariz ring. In [**15**], Wagner Cortes studied the relationship between right (left) zip property of R and skew polynomial extensions over R by using the skew versions of Armendariz rings and generalised some results of [**9**].

Motivated by the above, in this paper we introduce the notion of weak zip rings and study the relationship between right (left) weak zip property of R and skew polynomial extension $R[x; \alpha, \delta]$ over R. We mainly prove that a ring R is right (left) weak zip if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is right (left) weak zip. Let α be an endomorphism and δ an α-derivation of a ring R. Then R is a right (left) weak zip ring if and only if the skew polynomial ring $R[x; \alpha, \delta]$ is a right (left) weak zip ring when R is (α, δ)-compatible and reversible.

For a ring R, we denote by nil(R) the set of all nilpotent elements of R and by $T_n(R)$ the n-by-n upper triangular matrix ring over R.

2. Weak zip rings. Let R be a ring. A right (left) weak annihilator of a subset X of R is defined by $Nr_R(X) = \{a \in R \mid xa \in \text{nil}(R) \text{ for all } x \in X\}$ $(Nl_R(X) = \{a \in R \mid ax \in \text{nil}(R) \text{ for all } x \in X\})$. We call a ring R right weak zip provided that $Nr_R(X) \subseteq \text{nil}(R)$, where X is a subset of R; then there exists a finite subset $Y \subseteq X$ such that $Nr_R(Y) \subseteq \text{nil}(R)$. We define left weak zip rings similarly. If a ring is both left and right weak zip, we say that the ring is a weak zip ring. Obviously, if a ring R is reduced, then R is a zip ring if and only if R is a weak zip ring.

Let R be a ring. Then by C. Y. Hong [**8**], there exists an $n \times n$ upper triangular matrix ring over a right zip ring which is not right zip for any $n \geq 2$. But we have the following result:

PROPOSITION 2.1. *Let R be a ring and $n \geq 2$. Then $T_n(R)$ is a right (left) weak zip ring if and only if R is a right (left) weak zip ring.*

Proof. We will show the right case because the left case is similar. □

Assume that R is a right weak zip ring and $X \subseteq T_n(R)$ with $Nr_{T_n(R)}(X) \subseteq \text{nil}(T_n(R))$. Let

$$Y_i = \left\{ a_{ii} \in R, \mid \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in X \right\}, 1 \leq i \leq n.$$

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Then $Y_i \subseteq R,\ 1 \leq i \leq n$. If $b \in Nr_R(Y_i)$, then

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \cdot bE_{ii} \in \text{nil}(T_n(R))$$

for any

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in X,$$

where E_{ii} is the usual matrix unit with 1 in the (i, i)-coordinate and zero elsewhere. Thus, $bE_{ii} \in Nr_{T_n(R)}(X) \subseteq \text{nil}(T_n(R))$ and so $b \in \text{nil}(R)$. Hence $Nr_R(Y_i) \subseteq \text{nil}(R)$, $1 \leq i \leq n$. Since R is a right weak zip ring, there exists a finite subset $Y_i' \subseteq Y_i$ such that $Nr_R(Y_i') \subseteq \text{nil}(R)$, $1 \leq i \leq n$. For each $c \in Y_i'$, there exists

$$A_c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{pmatrix} \in X$$

such that $c_{ii} = c$, $1 \leq i \leq n$. Let X_i' be a minimal subset of X such that $A_c \in X_i'$ for each $c \in Y_i'$. Then X_i' is a finite subset of X. Let $X_0 = \bigcup_{1 \leq i \leq n} X_i'$. Then X_0 is also a finite subset of X. If

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in Nr_{T_n(R)}(X_0),$$

then $A'B \in \text{nil}(T_n(R))$ for all

$$A' = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{pmatrix} \in X_0.$$

Let

$$U_i = \left\{ a'_{ii} \in R \ \middle| \ \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} \end{pmatrix} \in X_0 \right\},\ 1 \leq i \leq n.$$

Clearly, $Y_i' \subseteq U_i$ for all $1 \leq i \leq n$. So $Nr_R(U_i) \subseteq Nr_R(Y_i') \subseteq \text{nil}(R)$ for all $1 \leq i \leq n$. Since $A'B \in \text{nil}(T_n(R))$ implies $a'_{ii}b_{ii} \in \text{nil}(R)$ for all $1 \leq i \leq n$, we obtain

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$b_{ii} \in Nr_R(U_i) \subseteq Nr_R(Y_i') \subseteq \mathrm{nil}(R)$. Thus $b_{ii} \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$, and hence $B \in \mathrm{nil}(T_n(R))$. Therefore $Nr_{T_n(R)}(X_0) \subseteq \mathrm{nil}(T_n(R))$, and so $T_n(R)$ is a right weak zip ring.

Conversely, assume that $T_n(R)$ is a right weak zip ring, and $X \subseteq R$ with $Nr_R(X) \subseteq \mathrm{nil}(R)$. Let $Y = \{aI | a \in X\} \subseteq T_n(R)$, where I is the $n \times n$ identity matrix. If

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in Nr_{T_n(R)}(Y),$$

then $aI \cdot B \in \mathrm{nil}(T_n(R))$ for all $a \in X$. Thus $ab_{ii} \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$ and all $a \in X$. Therefore $b_{ii} \in Nr_R(X)$, and so $b_{ii} \in \mathrm{nil}(R)$ for all $1 \leq i \leq n$. Hence $B \in \mathrm{nil}(T_n(R))$, and so $Nr_{T_n(R)}(Y) \subseteq \mathrm{nil}(T_n(R))$. Since $T_n(R)$ is a right weak zip ring, there exists a finite subset $Y_0 = \{a_1 I, a_2 I, \ldots, a_m I\} \subseteq Y$ such that $Nr_{T_n(R)}(Y_0) \subseteq \mathrm{nil}(T_n(R))$. Let $X_0 = \{a_1, a_2, \ldots, a_m\} \subseteq X$. If $c \in Nr_R(X_0)$, then $a_k I \cdot cE_{11} \in \mathrm{nil}(T_n(R))$ for all $k = 1, 2, \ldots, m$. Thus, $cE_{11} \in Nr_{T_n(R)}(Y_0) \subseteq \mathrm{nil}(T_n(R))$ and so $c \in \mathrm{nil}(R)$. Therefore, $Nr_R(X_0) \subseteq \mathrm{nil}(R)$ and so R is right weak zip.

EXAMPLE 2.2. Let R be a domain; then R is a weak zip ring by definition. Based on Proposition 2.1, any $n \times n$ upper triangular matrix ring over a domain is a weak zip ring.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

COROLLARY 2.3. *$T(R, R)$ is right (left) weak zip if and only if R is right (left) weak zip.*

Proof. The proof is similar to that of Proposition 2.1. □

LEMMA 2.4 ([**12**]). *Let R be a semicommutative ring. The nil(R) is an ideal of R.*

LEMMA 2.5. *Let R be semicommutative. Then $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ is a nilpotent element of $R[x]$ if and only if $a_i \in$ nil(R) for all $0 \leq i \leq n$.*

Proof. It is an immediate consequence of [**12**, Proposition 3.3] and [**12**, Lemma 3.7]. □

In [**1**], it is shown that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is zip. As to weak zip rings, we have the following:

PROPOSITION 2.6. *Let R be a semicommutative ring. Then R is right (left) weak zip if and only if $R[x]$ is right (left) weak zip.*

Proof. Suppose that $R[x]$ is right weak zip. Let $Y \subseteq R$ with $Nr_R(Y) \subseteq \mathrm{nil}(R)$. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in Nr_{R[x]}(Y)$, then $bf(x) = ba_0 + ba_1 x + \cdots + ba_n x^n \in$

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nil($R[x]$) for any $b \in Y$. Thus $ba_i \in$ nil(R) by Lemma 2.5, and so $a_i \in Nr_R(Y)$ for all $0 \leq i \leq n$, and hence $a_i \in$ nil(R) for all $0 \leq i \leq n$. Therefore $f(x) \in$ nil($R[x]$) by Lemma 2.5. So $Nr_{R[x]}(Y) \subseteq$ nil($R[x]$). Since $R[x]$ is right weak zip, there exists a finite subset $Y_0 \subseteq Y$ such that $Nr_{R[x]}(Y_0) \subseteq$ nil($R[x]$). Therefore $Nr_R(Y_0) = Nr_{R[x]}(Y_0) \cap R \subseteq$ nil(R), and hence R is right weak zip. □

Conversely, assume that R is right weak zip. Let $X \subseteq R[x]$ with $Nr_{R[x]}(X) \subseteq$ nil($R[x]$). Now let Y be the set of all coefficients of elements in X. Then $Y \subseteq R$. If $a \in Nr_R(Y)$, then $ba \in$ nil(R) for any $b \in Y$. So for any $f(x) = r_0 + r_1x + \cdots + r_nx^n \in X$, we have $r_ia \in$ nil(R) for all $0 \leq i \leq n$. Hence $f(x)a \in$ nil($R[x]$) by Lemma 2.5 and so $a \in Nr_{R[x]}(X) \subseteq$ nil($R[x]$). Thus $a \in$ nil(R) and so $Nr_R(Y) \subseteq$ nil(R). Since R is a right weak zip ring, there exists a finite subset $Y_0 \subseteq Y$ such that $Nr_R(Y_0) \subseteq$ nil(R). For each $a \in Y_0$, there exists $g_a(x) \in X$ such that some of the coefficients of $g_a(x)$ is a. Let X_0 be a minimal subset of X such that $g_a(x) \in X_0$ for each $a \in Y_0$. Then X_0 is a finite subset of X. Let Y_1 be the set of all coefficients of elements of X_0. Then $Y_0 \subseteq Y_1$, and so $Nr_R(Y_1) \subseteq Nr_R(Y_0) \subseteq$ nil(R). If $g(x) = b_0 + b_1x + \cdots + b_kx^k \in Nr_{R[x]}(X_0)$, then $f(x)g(x) \in$ nil($R[x]$) for any $f(x) = a_0 + a_1x + \cdots + a_tx^t \in X_0$. Since

$$f(x)g(x) = \left(\sum_{i=0}^{t} a_ix^i\right)\left(\sum_{j=0}^{k} b_jx^j\right) = \sum_{s=0}^{t+k}\left(\sum_{i+j=s} a_ib_j\right)x^s \in \text{nil}(R[x]),$$

we have the following system of equations by Lemma 2.5:

$$\Delta_s = \sum_{i+j=s} a_ib_j \in \text{nil}(R), \quad s = 0, 1, \ldots, t+k.$$

We will show that $a_ib_j \in$ nil(R) by induction on $i + j$.

If $i + j = 0$, then $a_0b_0 \in$ nil(R), $b_0a_0 \in$ nil(R).

Now suppose that s is a positive integer such that $a_ib_j \in$ nil(R) when $i + j < s$. We will show that $a_ib_j \in$ nil(R) when $i + j = s$. Consider the following equation:

$$(*): \ \Delta_s = a_0b_s + a_1b_{s-1} + \cdots + a_sb_0 \in \text{nil}(R).$$

Multiplying $(*)$ by b_0 from left, we have $b_0a_sb_0 = b_0\Delta_s - (b_0a_0)b_s - (b_0a_1)b_{s-1} - \cdots - (b_0a_{s-1})b_1$. By induction hypothesis, $a_ib_0 \in$ nil(R) for all $0 \leq i < s$, and so $b_0a_i \in$ nil(R) for all $0 \leq i < s$. Thus $b_0a_sb_0 \in$ nil(R) and so $b_0a_s \in$ nil(R), $a_sb_0 \in$ nil(R). Multiplying $(*)$ by $b_1, b_2, \ldots, b_{s-1}$ from left side, respectively, yields $a_{s-1}b_1 \in$ nil(R), $a_{s-2}b_2 \in$ nil(R), $\ldots$, $a_0b_s \in$ nil(R) in turn. This means that $a_ib_j \in$ nil(R) when $i + j = s$. Therefore by induction, we obtain $a_ib_j \in$ nil(R) for each i, j. Thus $b_j \in Nr_R(Y_1) \subseteq$ nil(R) for all $0 \leq j \leq k$, and so $g(x) \in$ nil($R[x]$) by Lemma 2.5. Hence $Nr_{R[x]}(X_0) \subseteq$ nil($R[x]$). Therefore $R[x]$ is a right weak zip ring.

Similarly, we can show that if R is semicommutative, then R is left weak zip if and only if $R[x]$ is left weak zip.

3. Ore extensions over weak zip rings. Let α be an endomorphism of R and $\delta : R \longrightarrow R$ an additive map of R. The application δ is said to be an α-derivation if $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$. The Ore extension $S = R[x; \alpha, \delta]$ is the set of polynomials $\sum_{i=0}^{m} a_ix^i$ with the usual sum, and the multiplication rule is $xa = \alpha(a)x + \delta(a)$. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x; \alpha, \delta]$. We say that $f(x) \in$ nil(R)[$x; \alpha, \delta$] if and only if

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$a_i \in \text{nil}(R)$ for all $0 \leq i \leq n$. Let I be a subset of R. We denote by $I[x;\alpha,\delta]$ the subset of $R[x;\alpha,\delta]$, where the coefficients of elements in $I[x;\alpha,\delta]$ are in subset I, equivalently, for any skew polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$, $f(x) \in I]x;\alpha,\delta]$ if and only if $a_i \in I$ for all $0 \leq i \leq n$. If $f(x) \in R[x;\alpha,\delta]$ is a nilpotent element of $R[x;\alpha,\delta]$, then we say $f(x) \in \text{nil}(R[x;\alpha,\delta])$. For $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$, we denote by $\{a_0, a_1, \ldots, a_n\}$ the set of coefficients of $f(x)$. Let $a_i \in R$, $1 \leq i \leq n$; we also denote by $a_1a_2, \ldots, a_n$ the product of all a_i, $1 \leq i \leq n$.

Let δ be an α-derivation of R. For integers i, j with $0 \leq i \leq j$, $f_i^j \in \text{End}(R,+)$ will denote the map which is the sum of all possible words in α, δ built with i letters α and $j-i$ letters δ. For instance, $f_0^0 = 1, f_j^j = \alpha^j, f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. The next Lemma appears in [**11**, Lemma 4.1].

LEMMA 3.1. *For any positive integer n and $r \in R$, we have $x^n r = \sum_{i=0}^n f_i^n(r)x^i$ in the ring $R[x;\alpha,\delta]$.*

LEMMA 3.2 ([**2**]). *Let R be an (α,δ)-compatible ring. Then we have the following:*

(1) *If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for all positive integers n.*
(2) *If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.*
(3) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for all positive integers m, n.*

LEMMA 3.3. *Let δ be an α-derivation of R. If R is an (α,δ)-compatible ring, then $ab = 0$ implies $af_i^j(b) = 0$ for all $j \geq i \geq 0$ and $a, b \in R$.*

Proof. If $ab = 0$, then $a\alpha^i(b) = a\delta^j(b) = 0$ for all $i \geq 0$ and $j \geq 0$ because R is (α,δ)-compatible. Then $af_i^j(b) = 0$ for all i, j. □

LEMMA 3.4. *Let δ be an α-derivation of R. If R is (α,δ)-compatible and reversible, then $ab \in nil(R)$ implies $af_i^j(b) \in nil(R)$ for all $j \geq i \geq 0$ and $a, b \in R$.*

Proof. Since $ab \in \text{nil}(R)$, there exists some positive integer k such that $(ab)^k = 0$. $0 = (ab)^k = abab\cdots ab \Rightarrow abab\cdots abaf_i^j(b) = 0 \Rightarrow af_i^j(b)ab\cdots ab = 0 \Rightarrow af_i^j(b)ab\cdots abaf_i^j(b) = 0 \Rightarrow af_i^j(b)af_i^j(b)ab\cdots ab = 0 \Rightarrow \cdots \Rightarrow af_i^j(b) \in \text{nil}(R)$. □

LEMMA 3.5. *Let R be an (α,δ)-compatible ring. If $a\alpha^m(b) \in nil(R)$ for $a, b \in R$, and m is a positive integer, then $ab \in nil(R)$.*

Proof. Since $a\alpha^m(b) \in \text{nil}(R)$, there exists some positive integer n such that $(a\alpha^m(b))^n = 0$. In the following computations, we use freely the condition that R is (α,δ)-compatible.

$$\begin{aligned}
(a\alpha^m(b))^n &= \underbrace{a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)}_{n} = 0 \\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)ab = 0 \\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)\alpha^m(ab) = 0 \\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)a\alpha^m(bab) = 0 \\
&\Rightarrow a\alpha^m(b)a\alpha^m(b)\cdots a\alpha^m(b)abab = 0 \\
&\Rightarrow \cdots \Rightarrow ab \in nil(R).
\end{aligned}$$
□

LEMMA 3.6. *Let R be (α,δ)-compatible. If R is a reversible ring, then $f(x) = a_0 + a_1x + \cdots + a_nx^n \in nil(R[x;\alpha,\delta])$ if and only if $a_i \in nil(R)$ for all $0 \leq i \leq n$.*

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Proof. ($\Longrightarrow$) Suppose $f(x) \in \mathrm{nil}(R[x;\alpha,\delta])$. There exists some positive integer k such that $f(x)^k = (a_0 + a_1x + \cdots + a_nx^n)^k = 0$. Then

$$0 = f(x)^k = \text{`lower terms'} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)x^{nk}.$$

Hence $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$, and α-compatibility and reversibility of R gives $a_n \in \mathrm{nil}(R)$. So by Lemma 3.4, $a_n = 1 \cdot a_n \in \mathrm{nil}(R)$ implies $1 \cdot f_i^j(a_n) = f_i^j(a_n) \in \mathrm{nil}(R)$ for all $0 \le i \le j$. Thus we obtain

$$\begin{aligned}(a_0 + a_1x + \cdots + a_{n-1}x^{n-1})^k &= \text{`lower terms'}\\ &+ a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k}\end{aligned}$$

$\in \mathrm{nil}(R)[x;\alpha,\delta]$ since $\mathrm{nil}(R)$ is an ideal of R. Hence $a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(k-1)(n-1)}(a_{n-1}) \in \mathrm{nil}(R)$ and so $a_{n-1} \in \mathrm{nil}(R)$ by Lemma 3.5. Using induction on n we obtain $a_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$. □

($\Longleftarrow$) Suppose that $a_i^{m_i} = 0$, $i = 0, 1, \ldots, n$. Let $k = \sum_{i=0}^{n} m_i + 1$. We claim that $f(x)^k = (a_0 + a_1x + \cdots + a_nx^n)^k = 0$. From

$$\begin{aligned}
\left(\sum_{i=0}^{n} a_ix^i\right)^2 &= \left(\sum_{i=0}^{n} a_ix^i\right)\left(\sum_{i=0}^{n} a_ix^i\right)\\
&= \left(\sum_{i=0}^{n} a_ix^i\right)a_0 + \left(\sum_{i=0}^{n} a_ix^i\right)a_1x\\
&\quad + \cdots + \left(\sum_{i=0}^{n} a_ix^i\right)a_sx^s + \cdots + \left(\sum_{i=0}^{n} a_ix^i\right)a_nx^n\\
&= \sum_{i=0}^{n} a_if_0^i(a_0) + \left(\sum_{i=1}^{n} a_if_1^i(a_0)\right)x + \cdots + \left(\sum_{i=s}^{n} a_if_s^i(a_0)\right)x^s\\
&\quad + \cdots + \left(\sum_{i=n}^{n} a_if_n^i(a_0)\right)x^n + \left(\sum_{i=0}^{n} a_if_0^i(a_1) + \left(\sum_{i=1}^{n} a_if_1^i(a_1)\right)x\right.\\
&\quad \left. + \cdots + \left(\sum_{i=n}^{n} a_if_n^i(a_1)\right)x^n\right)x + \cdots + \left(\sum_{i=0}^{n} a_if_0^i(a_s) + \left(\sum_{i=1}^{n} a_if_1^i(a_s)\right)x\right.\\
&\quad \left. + \cdots + \left(\sum_{i=n}^{n} a_if_n^i(a_s)\right)x^n\right)x^s + \cdots + \left(\sum_{i=0}^{n} a_if_0^i(a_n) + \left(\sum_{i=1}^{n} a_if_1^i(a_n)\right)x\right.\\
&\quad \left. + \cdots + \left(\sum_{i=n}^{n} a_if_n^i(a_n)\right)x^n\right)x^n\\
&= \sum_{i=0}^{n} a_if_0^i(a_0) + \left(\sum_{i=1}^{n} a_if_1^i(a_0) + \sum_{i=0}^{n} a_if_0^i(a_1)\right)x + \left(\sum_{i=2}^{n} a_if_2^i(a_0) + \sum_{i=1}^{n} a_if_1^i(a_1)\right.\\
&\quad \left. + \sum_{i=0}^{n} a_if_0^i(a_2)\right)x^2 + \cdots + \left(\sum_{s+t=k}\left(\sum_{i=s}^{n} a_if_s^i(a_t)\right)\right)x^k + \cdots + a_n\alpha^n(a_n)x^{2n},
\end{aligned}$$

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it is easy to check that the coefficients of $\left(\sum_{i=0}^{n} a_i x^i\right)^k$ can be written as sums of monomials of length k in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \ldots, a_n\}$ and $v \geq u \geq 0$ are positive integers. Consider each monomial

$$\underbrace{a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})}_{k+1},$$

where $a_{i_1}, a_{i_2}, \cdots a_{i_k} \in \{a_0, a_1, \cdots, a_n\}$, and $t_j, s_j (t_j \geq s_j, 2 \leq j \leq k)$ are non-negative integers. We will show that $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k}) = 0$. If the number of a_0 in $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})$ is greater than m_0, then we can write monomial $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})$ as

$$b_1 \left(f_{s_{01}}^{t_{01}}(a_0)\right)^{j_1} b_2 \left(f_{s_{02}}^{t_{02}}(a_0)\right)^{j_2} \cdots b_v \left(f_{s_{0v}}^{t_{0v}}(a_0)\right)^{j_v} b_{v+1},$$

where $j_1 + j_2 + \cdots + j_v > m_0$, $1 \leq j_1, j_2, \ldots, j_v$ and $b_q (q = 1, 2, \ldots, v+1)$ is a product of some elements choosing from $\{a_{i1}, f_{s_2}^{t_2}(a_{i_2}), \ldots, f_{s_k}^{t_k}(a_{i_k})\}$ or is equal to 1. Since $a_0^{j_1+j_2+\cdots+j_v} = 0$ and R is reversible and (α, δ)-compatible, we have

$$\begin{aligned}
0 = a_0^{j_1+j_2+\cdots+j_v} &= \underbrace{a_0 a_0 \cdots a_0}_{j_1+j_2+\cdots+j_v} \\
\Rightarrow\ & a_0 a_0 \cdots \left(f_{s_{01}}^{t_{01}}(a_0)\right) = 0 \\
\Rightarrow\ & \left(f_{s_{01}}^{t_{01}}(a_0)\right) a_0 \cdots a_0 = 0 \\
\Rightarrow\ & \left(f_{s_{01}}^{t_{01}}(a_0)\right)^{j_1} a_0 \cdots a_0 = 0 \\
\Rightarrow\ & \cdots \\
\Rightarrow\ & \left(f_{s_{01}}^{t_{01}}(a_0)\right)^{j_1} \left(f_{s_{02}}^{t_{02}}(a_0)\right)^{j_2} \cdots \left(f_{s_{0v}}^{t_{0v}}(a_0)\right)^{j_v} = 0 \\
\Rightarrow\ & b_1 \left(f_{s_{01}}^{t_{01}}(a_0)\right)^{j_1} b_2 \left(f_{s_{02}}^{t_{02}}(a_0)\right)^{j_2} \cdots b_v \left(f_{s_{0v}}^{t_{0v}}(a_0)\right)^{j_v} b_{v+1} = 0.
\end{aligned}$$

Thus $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k}) = 0$. If the number of a_i in $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})$ is greater than m_k, then similar discussion yields that $a_{i1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k}) = 0$. Thus each monomial appears in $\left(\sum_{i=0}^{n} a_i x^i\right)^k$ equal to 0. Therefore $\sum_{i=0}^{n} a_i x^i \in R[x; \alpha, \delta]$ is a nilpotent element.

Hirano observed relations between annihilators in a ring R and annihilators in $R[x]$ (see [**6**]). In this note we investigate the relations between right (left) weak annihilators in a ring R and right (left) weak annihilators in skew polynomial ring $S = R[x; \alpha, \delta]$. Given a ring R, we define $Nr\mathrm{Ann}_R(2^R) = \{Nr_R(U) \mid U \subseteq R\}$, $Nr\mathrm{Ann}_S(2^S) = \{Nr_S(V) \mid V \subseteq S\}$, $Nl\mathrm{Ann}_R(2^R) = \{Nl_R(U) \mid U \subseteq R\}$, $Nl\mathrm{Ann}_S(2^S) = \{Nl_S(V) \mid V \subseteq S\}$. Given a skew polynomial $f(x) \in R[x; \alpha, \delta]$, let C_f denote the set of all coefficients of $f(x)$, and for a subset V of $R[x; \alpha, \delta]$, let C_V denote the set $\bigcup_{f \in V} C_f$.

Lemma 3.7. *Let R be a reversible and (α, δ)-compatible ring. Then for any subset $U \subseteq R$, we have the following:*

(1) $Nr_S(U) = Nr_R(U)[x; \alpha, \delta]$.
(2) $Nl_S(U) = Nl_R(U)[x; \alpha, \delta]$.

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Proof. (1) Clearly, $Nr_R(U)[x;\alpha,\delta] \subseteq Nr_S(U)$. For any skew polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Nr_S(U)$, we have $rf(x) = ra_0 + ra_1x + \cdots + ra_nx^n \in \text{nil}(S)$ for any $r \in U$. So $ra_i \in \text{nil}(R)$ for all $0 \le i \le n$ and all $r \in U$ by Lemma 3.6, and hence $a_i \in Nr_R(U)$ for all $0 \le i \le n$. Thus $f(x) \in Nr_R(U)[x;\alpha,\delta]$ and so $Nr_S(U) \subseteq Nr_R(U)[x;\alpha,\delta]$. Therefore we obtain $Nr_S(U) = Nr_R(U)[x;\alpha,\delta]$.

(2) For any $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Nl_R(U)[x;\alpha,\delta]$, $a_ir \in \text{nil}(R)$ for all $0 \le i \le n$ and any $r \in U$. Then $a_if_s^t(r) \in \text{nil}(R)$ for $0 \le i \le n$ and all positive integers s and t with $t \ge s$ by Lemma 3.4. Thus,

$$\begin{aligned} f(x)r &= (a_0 + a_1x + \cdots + a_nx^n)r \\ &= \sum_{i=0}^{m} a_if_0^i(r) + \left(\sum_{i=1}^{m} a_if_1^i(r)\right)x + \cdots + \left(\sum_{i=s}^{m} a_if_s^i(r)\right)x^s + \cdots + a_n\alpha^n(r)x^n \in \text{nil}(S) \end{aligned}$$

by Lemma 3.6, and so $Nl_R(U)[x;\alpha,\delta] \subseteq Nl_S(U)$.

Conversely, assume that $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Nl_S(U)$. Then

$$\begin{aligned} f(x)r &= (a_0 + a_1x + \cdots + a_nx^n)r \\ &= \sum_{i=0}^{m} a_if_0^i(r) + \left(\sum_{i=1}^{m} a_if_1^i(r)\right)x + \cdots + \left(\sum_{i=s}^{m} a_if_s^i(r)\right)x^s + \cdots + a_n\alpha^n(r)x^n \\ &= \Delta_0 + \Delta_1x + \cdots + \Delta_nx^n \in nil(S) \end{aligned}$$

for all $r \in U$. Then we have the following system of equations by Lemma 3.6:

$$\begin{aligned} &(1) \quad \Delta_n = a_n\alpha^n(r) \in \text{nil}(R), \\ &(2) \quad \Delta_{n-1} = a_{n-1}\alpha^{n-1}(r) + a_nf_{n-1}^n(r) \in \text{nil}(R), \\ &\qquad \vdots \\ &(3) \quad \Delta_s = \sum_{i=s}^{m} a_if_s^i(r) \in \text{nil}(R). \end{aligned}$$

From equation (1), we obtain $a_nr \in \text{nil}(R)$ by Lemma 3.5, and so $a_nf_s^t(r) \in \text{nil}(R)$ by Lemma 3.4. From equation (2), we have $a_{n-1}\alpha^{n-1}(r) = \Delta_{n-1} - a_nf_{n-1}^n(r) \in \text{nil}(R)$ and so $a_{n-1}r \in \text{nil}(R)$. Continuing this procedure yields that $a_ir \in \text{nil}(R)$ for all $0 \le i \le n$. Hence $a_i \in Nl_R(U)$ for all $0 \le i \le n$, and so $f(x) \in Nl_R(U)[x;\alpha,\delta]$. Therefore $Nl_S(U) = Nl_R(U)[x;\alpha,\delta]$.

With the above Lemma 3.7, we have maps: $\phi : Nr\text{Ann}_R(2^R) \longrightarrow Nr\text{Ann}_S(2^S)$ defined by $\phi(I) = I[x;\alpha,\delta]$ for every $I \in Nr\text{Ann}_R(2^R)$ and $\psi : Nl\text{Ann}_R(2^R) \longrightarrow Nl\text{Ann}_S(2^S)$ defined by $\psi(J) = J[x;\alpha,\delta]$ for every $J \in Nl\text{Ann}_R(2^R)$. Obviously, ϕ and ψ are injective. □

THEOREM 3.8. *Let R be a reversible and (α,δ)-compatible ring. Then we have the following:*

(1) *$\phi : NrAnn_R(2^R) \longrightarrow NrAnn_S(2^S)$ defined by $\phi(I) = I[x;\alpha,\delta]$ for every $I \in NrAnn_R(2^R)$ is bijective.*
(2) *$\psi : NlAnn_R(2^R) \longrightarrow NlAnn_S(2^S)$ defined by $\psi(J) = J[x;\alpha,\delta]$ for every $J \in NlAnn_R(2^R)$ is bijective.*

Proof. (1) It is only necessary to show that ϕ is surjective. Let $f(x) = \sum_{j=0}^{n} b_jx^j \in Nr_S(V) \in Nr\text{Ann}_S(2^S)$. Then we have $g(x)f(x) \in \text{nil}(S)$ for every

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$g(x) = \sum_{i=0}^{m} a_i x^i \in V$. Since

$$g(x)f(x) = \left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \left(\sum_{i=0}^{m} a_i x^i\right) b_0 + \left(\sum_{i=0}^{m} a_i x^i\right) b_1 x$$

$$+ \cdots + \left(\sum_{i=0}^{m} a_i x^i\right) b_n x^n$$

$$= \sum_{i=0}^{m} a_i f_0^i(b_0) + \left(\sum_{i=1}^{m} a_i f_1^i(b_0)\right) x + \cdots + \left(\sum_{i=s}^{m} a_i f_s^i(b_0)\right) x^s$$

$$+ \cdots + a_m \alpha^m(b_0) x^m + \left(\sum_{i=0}^{m} a_i f_0^i(b_1) + \left(\sum_{i=1}^{m} a_i f_1^i(b_1)\right) x + \cdots\right.$$

$$\left. + \left(\sum_{i=s}^{m} a_i f_s^i(b_0)\right) x^s + \cdots + a_m \alpha^m(b_1) x^m\right) x$$

$$+ \cdots + \left(\sum_{i=0}^{m} a_i f_0^i(b_n) + \left(\sum_{i=1}^{m} a_i f_1^i(b_n)\right) x + \cdots + a_m \alpha^m(b_n) x^m\right) x^n$$

$$= \sum_{i=0}^{m} a_i f_0^i(b_0) + \left(\sum_{i=1}^{m} a_i f_1^i(b_0) + \sum_{i=0}^{m} a_i f_0^i(b_1)\right) x + \cdots$$

$$+ \left(\sum_{s+t=k}\left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right)\right) x^k + \cdots + a_m \alpha^m(b_n) x^{m+n} \in nil(S).$$

□

Then we have the following equations by Lemma 3.6:

(4) $\Delta_{m+n} = a_m \alpha^m(b_n) \in \mathrm{nil}(R)$,
(5) $\Delta_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) \in \mathrm{nil}(R)$,
(6) $\Delta_{m+n-2} = a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_i f_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^{m} a_i f_{m-2}^i(b_n) \in \mathrm{nil}(R)$,

$\vdots$

(7) $\Delta_k = \sum_{s+t=k}\left(\sum_{i=s}^{m} a_i f_s^i(b_t)\right) \in \mathrm{nil}(R)$.

From equation (4) and Lemma 3.5, we obtain $a_m b_n \in \mathrm{nil}(R)$, and so $b_n a_m \in \mathrm{nil}(R)$. Now we show that $a_i b_n \in \mathrm{nil}(R)$ for all $0 \le i \le m$. If we multiply equation (5) on the left side by b_n, then $b_n a_{m-1}\alpha^{m-1}(b_n) = b_n \Delta_{m+n-1} - (b_n a_m \alpha^m(b_{n-1}) + b_n a_m f_{m-1}^m(b_n)) \in \mathrm{nil}(R)$ since the nil(R) of a reversible ring is an ideal. Thus by Lemma 3.5, we obtain $b_n a_{m-1} b_n \in \mathrm{nil}(R)$, and so $b_n a_{m-1} \in \mathrm{nil}(R)$, $a_{m-1} b_n \in \mathrm{nil}(R)$. If we multiply equation (6) on the left side by b_n, then we obtain $b_n a_{m-2} f_{m-2}^{m-2}(b_n) = b_n a_{m-2}\alpha^{m-2}(b_n) = b_n \Delta_{m+n-2} - b_n a_m \alpha^m(b_{n-2}) - b_n a_{m-1} f_{m-1}^{m-1}(b_{n-1}) - b_n a_m f_{m-1}^m(b_{n-1}) - b_n a_{m-1} f_{m-2}^{m-1}(b_n) - b_n a_m f_{m-2}^m(b_n) = b_n \Delta_{m+n-2} - (b_n a_m)\alpha^m(b_{n-2}) - (b_n a_{m-1}) f_{m-1}^{m-1}(b_{n-1}) - (b_n a_m) f_{m-1}^m(b_{n-1}) - (b_n a_{m-1}) f_{m-2}^{m-1}(b_n) - (b_n a_m) f_{m-2}^m (b_n) \in nil(R)$ since nil(R) is an ideal of R. Thus

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we obtain $a_{m-2}b_n \in \mathrm{nil}(R)$ and $b_n a_{m-2} \in \mathrm{nil}(R)$. Continuing this procedure yields that $a_i b_n \in \mathrm{nil}(R)$ for all $0 \le i \le m$, and so $a_i f_s^t(b_n) \in \mathrm{nil}(R)$ for any $t \ge s \ge 0$ and $0 \le i \le m$ by Lemma 3.4. Thus it is easy to verify that $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n-1} b_j x^j) \in \mathrm{nil}(S)$. Applying the preceding method repeatedly, we obtain that $a_i b_j \in \mathrm{nil}(R)$ for all $0 \le i \le m, 0 \le j \le n$. So $b_j \in Nr_R(C_V)$ and $f(x) \in Nr_R(C_V)[x; \alpha, \delta]$, and hence it is easy to see that $Nr_S(V) = Nr_R(C_V)[x; \alpha, \delta] = \phi(Nr_R(C_V))$. Therefore ϕ is surjective.

(2) The proof of (2) is similar.

Corollary 3.9. *Let R be reversible. Then we have the following:*

(1) $\phi : NrAnn_R(2^R) \longrightarrow NrAnn_{R[x]}(2^{R[x]})$ *defined by* $\phi(I) = I[x]$ *for every* $I \in NrAnn_R(2^R)$ *is bijective.*

(2) $\psi : NlAnn_R(2^R) \longrightarrow NlAnn_{R[x]}(2^{R[x]})$ *defined by* $\psi(J) = J[x]$ *for every* $J \in NlAnn_R(2^R)$ *is bijective.*

Proof. Let $\alpha = 1_R$ be the identity endomorphism of R and $\delta = 0$. Then $R[x; \alpha, \delta] \cong R[x]$. Hence we complete the proof by Theorem 3.8.

Actually, as to polynomial ring $R[x]$, the condition that R is reversible in Corollary 3.9 can be replaced by that R is semicommutative. We have the following: □

Corollary 3.10. *Let R be semicommutative. Then we have the following:*

(1) $\phi : NrAnn_R(2^R) \longrightarrow NrAnn_{R[x]}(2^{R[x]})$ *defined by* $\phi(I) = I[x]$ *for every* $I \in NrAnn_R(2^R)$ *is bijective.*

(2) $\psi : NlAnn_R(2^R) \longrightarrow NlAnn_{R[x]}(2^{R[x]})$ *defined by* $\psi(J) = J[x]$ *for every* $J \in NlAnn_R(2^R)$ *is bijective.*

Proof. (1) For any subset $U \subseteq R$, it is easy to see that $Nr_R(U)[x] \subseteq Nr_{R[x]}(U)$. Also for any polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in Nr_{R[x]}(U)$, we have $rf(x) = ra_0 + ra_1 x + \cdots + ra_n x^n \in \mathrm{nil}(R[x])$ for any $r \in U$. Then $ra_i \in \mathrm{nil}(R)$ for all $0 \le i \le n$ by Lemma 2.5, and so $a_i \in Nr_R(U)$ for all $0 \le i \le n$. Thus $f(x) \in Nr_R(U)[x]$ and so $Nr_{R[x]}(U) \subseteq Nr_R(U)[x]$. Therefore $Nr_{R[x]}(U) = Nr_R(U)[x]$, which implies that ϕ is well defined. Obviously, ϕ is injective. So it is necessary to show that ϕ is surjective. Let $f(x) = \sum_{j=0}^{n} b_j x^j \in Nr_{R[x]}(V) \in NrAnn_{R[x]}(2^{R[x]})$. Then we have $g(x)f(x) \in \mathrm{nil}(R[x])$ for every $g(x) = \sum_{i=0}^{m} a_i x^i \in V$. Since

$$g(x)f(x) = \left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i b_j\right) x^k \in \mathrm{nil}(R[x]),$$

similar to the proof of Proposition 2.6, we obtain $a_i b_j \in \mathrm{nil}(R)$ for each i, j. So $b_j \in Nr_R(C_V)$ and $f(x) \in Nr_R(C_V)[x]$, and hence $Nr_{R[x]}(V) = Nr_R(C_V)[x] = \phi(Nr_R(C_V))$. Therefore ϕ is bijective.

(2) Similarly we can proof (2). □

Theorem 3.11. *Let R be (α, δ)-compatible. If R is reversible, then the following statements are equivalent:*

(1) *R is right (left) weak zip.*

(2) *$S = R[x; \alpha, \delta]$ is right (left) weak zip.*

Proof. We will show the right case because the left case is similar. □

(1) $\Longrightarrow$ (2) Suppose that R is right weak zip. Let $X \subseteq S$ such that $Nr_S(X) \subseteq \mathrm{nil}(S)$. For a skew polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in S$, C_f denotes the set of coefficients of $f(x)$,

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and for a subset V of S, C_V denotes the set $\bigcup_{f\in V} C_f$. Then $C_V \subseteq R$. Now we show that $Nr_R(C_X) \subseteq \text{nil}(R)$. If $r \in Nr_R(C_X)$, then $ar \in \text{nil}(R)$ for any $a \in C_X$. So for any skew polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in X$, we obtain $a_i r \in \text{nil}(R)$ and so $a_i f_s^t(r) \in \text{nil}(R)$ by Lemma 3.4. Hence $f(x)r \in \text{nil}(S)$ by Lemma 3.6 and so $r \in Nr_S(X) \subseteq \text{nil}(S)$. Thus $r \in \text{nil}(R)$ and so $Nr_R(C_X) \subseteq \text{nil}(R)$. Since R is right weak zip, there exists a finite subset $Y_0 \subseteq C_X$ such that $Nr_R(Y_0) \subseteq \text{nil}(R)$. For each $a \in Y_0$, there exists $g_a(x) \in X$ such that some of the coefficients of $g_a(x)$ are a. Let X_0 be a minimal subset of X such that $g_a(x) \in X_0$ for each $a \in Y_0$. Then X_0 is a finite subset of X. Let Y_1 be the set of all coefficients of elements of X_0, then $Y_0 \subseteq Y_1$ and so $Nr_R(Y_1) \subseteq Nr_R(Y_0) \subseteq \text{nil}(R)$. If $f(x) = a_0 + a_1 x + \cdots + a_k x^k \in Nr_S(X_0)$, then $g(x)f(x) \in \text{nil}(S)$ for any $g(x) = b_0 + b_1 x + \cdots + b_t x^t \in X_0$. Using the same method in the proof of Theorem 3.8, we obtain $b_i a_j \in \text{nil}(R)$ for each i, j. Thus $a_j \in Nr_R(Y_1) \subseteq \text{nil}(R)$ for $0 \le j \le k$ and so $f(x) \in \text{nil}(S)$ by Lemma 3.6. Hence $Nr_S(X_0) \subseteq \text{nil}(S)$. Therefore $S = R[x; \alpha, \delta]$ is a right weak zip ring.

Conversely, suppose that $S = R[x; \alpha, \delta]$ is right weak zip. Let Y be a subset of R such that $Nr_R(Y) \subseteq \text{nil}(R)$. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in Nr_S(Y)$, then $a_i \in Nr_R(Y) \subseteq \text{nil}(R)$ for all $0 \le i \le n$ by Lemma 3.7, and so $f(x) \in \text{nil}(S)$ by Lemma 3.6. Hence $Nr_S(Y) \subseteq \text{nil}(S)$. Since $S = R[x; \alpha, \delta]$ is right weak zip, there exists a finite set $Y_0 \subseteq Y$ such that $Nr_S(Y_0) \subseteq \text{nil}(S)$. Hence $Nr_R(Y_0) = Nr_S(Y_0) \cap R \subseteq \text{nil}(R)$. Therefore R is a right weak zip ring.

COROLLARY 3.12. *Let R be reversible. Then we have the following:*

(1) *If R is α-compatible, then the skew polynomial ring $R[x; \alpha]$ is right (left) weak zip if and only if R is right (left) weak zip.*

(2) *If R is δ-compatible, then the differential polynomial ring $R[x; \delta]$ is right (left) weak zip if and only if R is right (left) weak zip.*

Proof. By virtue of Theorem 3.9, we complete the proof. □

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