EXACT DISTRIBUTION OF THE QUOTIENT OF INDEPENDENT GENERALIZED GAMMA VARIABLES

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1. Introduction. Let X be a random variable whose frequency function is

$$(1.1) \qquad f(x;a,d,p) = \frac{p}{\Gamma(\frac{d}{p})a^d} x^{d-1} e^{-(\frac{x}{a})^p} \qquad x>0;a,d,p>0 .$$

Form (1.1) is Stacy's [3] generalization of the gamma distribution. The familiar gamma, chi, chi-squared, exponential and Weibull variates are special cases, as are certain functions of normal variate - viz., its positive even powers, its modulus, and all positive powers of its modulus. Form (1.1) is also a special case of a function introduced by L. Amoroso [1] in analyzing the distribution of income. This note derives the exact distribution of the quotient of two independent generalized gamma variables.

2. Distribution of the quotient. Let X and Y be independently distributed with respective frequency functions $f(x;a_1,d_1,p)$ and $f(y;a_2,d_2,p)$. Let $U = \log X - \log Y$ where X and Y are defined as above. The characteristic function of the distribution of U is given by [2]

$$(2.1) \qquad \phi(t) = \frac{p^2}{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})a_1^{d_1}a_2^{d_2}} \int_0^\infty e^{-(\frac{x}{a_1})^p} x^{d_1-1+it} dx \int_0^\infty e^{-(\frac{y}{a_2})^p} y^{d_2-1+it} dy$$

$$(2.2) \qquad = \frac{a_1^{it} a_2^{-it} \Gamma(\frac{d_1}{p} + \frac{it}{p})\Gamma(\frac{d_2}{p} - \frac{it}{p})}{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})} .$$

The distribution of U is given by [2]

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(2.3)

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itu} a_1^{it} a_2^{-it} \Gamma(\frac{d_1}{p} + \frac{it}{p}) \Gamma(\frac{d_2}{p} - \frac{it}{p}) dt}{\Gamma(\frac{d_1}{p}) \Gamma(\frac{d_2}{p})} .$$

Let $\frac{d_2}{p} - \frac{it}{p} = -z$, so that

(2.4)

$$f(u) = \frac{p\ e^{-d_2(u - \log \frac{a_1}{a_2})}}{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})\, 2\pi i} \int_{-\frac{d_2}{p} - i\infty}^{-\frac{d_2}{p} + i\infty} e^{-zp(u - \log \frac{a_1}{a_2})} \Gamma(\frac{d_1}{p} + \frac{d_2}{p} + z) \Gamma(-z) dz .$$

It may be shown that [4]

$$\frac{1}{2\pi i} \int_{-\frac{d_2}{p} - i\infty}^{-\frac{d_2}{p} + i\infty} e^{-zp(u - \log \frac{a_1}{a_2})} \Gamma(\frac{d_1}{p} + \frac{d_2}{p} + z) \Gamma(-z) dz$$

$$= \Gamma(\frac{d_1}{p} + \frac{d_2}{p}) [1 + e^{-p(u - \log \frac{a_1}{a_2})}]^{-(\frac{d_1}{p} + \frac{d_2}{p})} \qquad u - \log \frac{a_1}{a_2} > 0 ,$$

so that

(2.5)

$$f(u) = \frac{\Gamma(\frac{d_1}{p} + \frac{d_2}{p}) p e^{-d_2(u - \log \frac{a_1}{a_2})}}{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})[1 + e^{-p(u - \log \frac{a_1}{a_2})}]^{\frac{d_1}{p} + \frac{d_2}{p}}} , \quad u - \log \frac{a_1}{a_2} > 0 .$$

Since $e^U = X/Y = W$, we obtain the exact distribution of the quotient

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(2.6)
$$g(w) = \frac{\Gamma(\frac{d_1}{p}+\frac{d_2}{p})\, p e^{d_2(\log \frac{a_1}{a_2})}\, w^{-d_2-1}}{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})[1+w^{-p} e^{(\log\frac{a_1}{a_2})}]^{\frac{d_1}{p}+\frac{d_2}{p}}}, \qquad w > 0 .$$

Setting $d_1 = d_2 = d$ and $a_1 = a_2 = a$, we have the special case

(2.7)
$$g(w) = \frac{\Gamma(\frac{2d}{p})p w^{d-1}}{[\Gamma(\frac{d}{p})]^2 (1+w^p)^{2d/p}}, \qquad w > 0 .$$

(2.7) is the distribution of the quotient $W = X/Y$ where X and Y have the same generalized gamma distribution.

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4. Whittaker and Watson, Modern Analysis. Second edition, p.283.

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