Canad. Math. Bull. Vol. 58 (2), 2015 pp. 281–284
http://dx.doi.org/10.4153/CMB-2015-010-9

On the Relation of Real and Complex Lie Supergroups

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Abstract. A complex Lie supergroup can be described as a real Lie supergroup with integrable almost complex structure. The necessary and sufficient conditions on an almost complex structure on a real Lie supergroup for defining a complex Lie supergroup are deduced. The classification of real Lie supergroups with such almost complex structures yields a new approach to the known classification of complex Lie supergroups by complex Harish-Chandra superpairs. A universal complexification of a real Lie supergroup is constructed.

1 Introduction

The local differential operators on a real Lie supergroup have the structure of a Lie–Hopf superalgebra that can be algebraically constructed from a real Harish-Chandra superpair (see [6]). Conversely starting from a real Harish-Chandra superpair, Kostant [6] constructed a sheaf of superfunctions by dualising the associated Lie–Hopf superalgebra. This yields a real Lie supergroup and hence an equivalence of categories from real Harish-Chandra superpairs to real Lie supergroups. The described construction highly depends on the softness of the sheaf of smooth functions and cannot be transported directly to the complex setting.

A construction of complex Lie supergroups from complex Harish-Chandra superpairs using analytic continuation on Grassmann variables was given by Berezin (see [1]). Vishnyakova gave a rigorous proof of the equivalence of categories of complex Harish-Chandra superpairs and complex Lie supergroups (see [9]).

In this article complex Lie supergroups and complex Harish-Chandra superpairs are analyzed as real objects with integrable almost complex structure J (see [2]). Starting from a real Lie supergroup, we deduce the conditions that an integrable almost complex structure J has to satisfy in order to define a complex Lie supergroup. The correspondence of complex Lie supergroups and Harish-Chandra superpairs then follows from the real case in [6]. Existence of a universal complexification of a real Lie supergroup with underlying real analytic Lie group is finally derived in the language of Harish-Chandra superpairs. More details can be found in [4].

Received by the editors August 29, 2014.
Published electronically March 2, 2015.
Research supported by the SFB/TR 12, Symmetry and Universality in Mesoscopic Systems, of the Deutsche Forschungsgemeinschaft.
AMS subject classification: **32C11**, 58A50.
Keywords: Lie supergroup, almost complex structure, Harish–Chandra pair, universal complexification.

2 Real Lie Supergroups with Almost Complex Structure

Let $(G,\mathfrak{g})$ be a real Harish-Chandra superpair; *i.e.*, G is a real Lie group and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a real Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$ and the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ integrates to a representation of G. Further, let $\mathcal{G} = (G, \mathcal{C}^{\mathbb{R}}_{\mathcal{G}})$ denote the real Lie supergroup associated with $(G,\mathfrak{g})$ by Kostant's construction (see [6]). The morphisms of multiplication, inverse, and unity induce the structure of a Lie–Hopf superalgebra on the local differential operators on $\mathcal{C}^{\mathbb{R}}_{\mathcal{G}}$. This Lie–Hopf superalgebra is isomorphic to $\mathbb{R}(G)\#E(\mathfrak{g})$ (see [6]). The first factor is the group ring of G, the second is the universal enveloping algebra of $\mathfrak{g}$, and # denotes the semidirect-type product defined via the adjoint action of G on $E(\mathfrak{g})$. An element $g\#X$ in $\mathbb{R}(G)\#E(\mathfrak{g})$ represents the left-invariant operator X followed by evaluation at g. Now let J be an integrable almost complex structure on the supermanifold $\mathcal{G}$. For $X \in \mathfrak{g}$ regarded as a left-invariant derivation on $\mathcal{C}^{\mathbb{R}}_{\mathcal{G}}$, we obtain for any $g \in G$ a well-defined element $J_g(X) \in \mathfrak{g}$ such that $g\#J_g(X) = J(g\#X)$ as derivations. The condition that the multiplication map (m, m^*) on $\mathcal{G}$ is a morphism of complex supermanifolds, *i.e.*, preserves J, is translated to the Lie–Hopf superalgebra as $(h\#1)\cdot J(g\#X) = J((h\#1)\cdot(g\#X))$ for all $g,h \in G$ and $X \in \mathfrak{g}$. This yields for $h = g^{-1}$ that $J_g(X) = J_e(X)$. So J is supposed to map left-invariant derivations to left-invariant derivations and hence to restrict to a map $J_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{g}$. Furthermore, for homogeneous $X_{\pm}$ in the $\pm i$ eigenspace of J, use on the left-hand side of

$$(e\#X_+)\cdot(e\#X_-) - (-1)^{|X_+||X_-|}(e\#X_-)\cdot(e\#X_+) = e\#[X_+, X_-]$$

the compatibility with J first in the first, then in the second arguments of the products. This yields identical results with different signs. So $[X_+, X_-]$ vanishes. Now by the graded Newlander–Nierenberg theorem the supercommutator on vector fields continued $\mathbb{C}$-linearly to $\mathrm{Der}(\mathcal{C}^{\mathbb{R}}_{\mathcal{G}} \otimes_{\mathbb{R}} \mathbb{C})$ preserves the eigenspaces of J (see [7],[8] and arguments parallel to [5, chap. IX.2]). Altogether we obtain J-linearity in both arguments of the superbracket. These conditions are already sufficient.

Theorem 2.1 *A real Lie supergroup $\mathcal{G}$ with almost complex structure J induces a complex Lie supergroup if and only if J preserves left-invariance of superderivations and the Lie superbracket is J-linear in both arguments;* i.e., *J comes from a complex structure on the Lie superalgebra $\mathfrak{g}$.*

Proof If J satisfies the given conditions, then it is integrable due to the graded version of the Newlander–Nierenberg theorem. Furthermore, J is compatible with the adjoint action of G on $\mathfrak{g}$, so J can be continued to $\mathbb{R}(G)\#E(\mathfrak{g})$ compatible with multiplication, inverse $(g\#X \mapsto -g^{-1}\#Ad(g)(X)$ for $g \in G$ and $X \in \mathfrak{g})$ and unity. Hence, the corresponding morphisms are morphisms of complex supermanifolds. ■

Corollary 2.2 ([9]) *The category of complex Lie supergroups is equivalent to the category of complex Harish-Chandra superpairs.*

https://doi.org/10.4153/CMB-2015-010-9 Published online by Cambridge University Press

3 Universal Complexification of Real Lie Supergroups

A universal complexification of a real Lie supergroup $\mathcal{G}$ is a complex Lie supergroup $\mathcal{G}^{\mathbb{C}}$ and a morphism of real Lie supergroups $\Gamma: \mathcal{G} \to \mathcal{G}^{\mathbb{C}}$ with the following universal property. For any morphism of real Lie supergroups $\Phi: \mathcal{G} \to \mathcal{H}$ into a complex Lie supergroup $\mathcal{H}$ there exists a unique morphism of complex Lie supergroups $\Phi^{\mathbb{C}}: \mathcal{G}^{\mathbb{C}} \to \mathcal{H}$ such that $\Phi^{\mathbb{C}} \circ \Gamma = \Phi$. Note that existence of a universal complexification includes uniqueness up to isomorphisms of complex Lie supergroups.

From now on, let G be a real analytic group and set $\mathfrak{g}_{\bar{0}} := \mathrm{Lie}(G)$. Existence of a universal complexification $\gamma: G \to G^{\mathbb{C}}$ of G is stated in [3]. In detail there is an ideal $\mathfrak{p} \subset \mathfrak{g}_{\bar{0}}$ such that the complex Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} := \mathrm{Lie}(G^{\mathbb{C}})$ is isomorphic to $(\mathfrak{g}_{\bar{0}} \otimes \mathbb{C})/(\mathfrak{p} \otimes \mathbb{C})$. The map γ is given on Lie algebra level by the real embedding $\mathrm{emb}_{\bar{0}}: \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}} \otimes \mathbb{C}$ followed by projection. Approaching a universal complexification for Lie supergroups, we find the following lemma.

Lemma 3.1 *Let $(G, \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}})$ be a real Harish-Chandra superpair, let $\gamma: G \to G^{\mathbb{C}}$ be the universal complexification of G and set $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} := \mathrm{Lie}(G^{\mathbb{C}})$. Then $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g}_{\bar{0}}^{\mathbb{C}} \oplus (\mathfrak{g}_{\bar{1}} \otimes \mathbb{C})$ is a complex Lie superalgebra with respect to the inherited Lie superbracket. In particular, $(G^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ is a complex Harish-Chandra superpair.*

Proof The adjoint representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ integrates to a Lie group action $G \to GL_{\mathbb{C}}(\mathfrak{g}_{\bar{1}} \otimes \mathbb{C})$. The universal complexification yields $G^{\mathbb{C}} \to GL_{\mathbb{C}}(\mathfrak{g}_{\bar{1}} \otimes \mathbb{C})$. So on Lie superalgebra level, $\mathfrak{p} \otimes \mathbb{C}$ acts trivially on $\mathfrak{g}_{\bar{1}} \otimes \mathbb{C}$; *i.e.,* it is an ideal in $\mathfrak{g} \otimes \mathbb{C}$. ■

Theorem 3.2 *Let $\mathcal{G}$ be a real Lie supergroup associated with the Harish-Chandra superpair $(G, \mathfrak{g})$ and let $\gamma: G \to G^{\mathbb{C}}$ be the universal complexification of G. Then the complex Harish-Chandra superpair $(G^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ together with the morphism (γ, Γ_*), $\Gamma_* := D_e\gamma \oplus \mathrm{emb}_{\bar{1}}$ is associated with a universal complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$.*

Proof Let $(H, \mathfrak{h})$ be a complex Harish-Chandra superpair and let the pair of maps $(\phi, \Phi_*): (G, \mathfrak{g}) \to (H, \mathfrak{h})$ be a morphism of real Harish-Chandra superpairs. Let $\phi^{\mathbb{C}}: G^{\mathbb{C}} \to H$ be the underlying complexification and set $\Phi_*^{\mathbb{C}} := D_e\phi^{\mathbb{C}} \oplus \sigma$, where $\sigma: \mathfrak{g}_{\bar{1}} \otimes \mathbb{C} \to \mathfrak{h}_{\bar{1}}$ is the complex linear continuation of $\Phi_*|_{\mathfrak{g}_{\bar{1}}}$. Then $(\phi^{\mathbb{C}}, \Phi_*^{\mathbb{C}})$ is unique with the required properties. ■

Acknowledgments The author wishes to thank A. Huckleberry for his advice and discussions.

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