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A Generalized Poisson Transform of an L^p-Function over the Shilov Boundary of the n-Dimensional Lie Ball

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Abstract. Let $\mathcal{D}$ be the n-dimensional Lie ball and let $\mathfrak{B}(S)$ be the space of hyperfunctions on the Shilov boundary S of $\mathcal{D}$. The aim of this paper is to give a necessary and sufficient condition on the generalized Poisson transform $P_{l,\lambda} f$ of an element f in the space $\mathfrak{B}(S)$ for f to be in $L^p(S)$, $1 < p < \infty$. Namely, if F is the Poisson transform of some $f \in \mathfrak{B}(S)$ (*i.e.*, $F = P_{l,\lambda} f$), then for any $l \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ such that $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$, we show that $f \in L^p(S)$ if and only if f satisfies the growth condition

$$\|F\|_{\lambda,p} = \sup_{0 \le r < 1} (1 - r^2)^{\mathcal{R}e[i\lambda] - \frac{n}{2} + l} \Big[\int_S |F(ru)|^p du \Big]^{\frac{1}{p}} < +\infty.$$

1 Introduction and Notations

Let $X = G/K$ be a Hermitian symmetric space of non-compact type. Let (χ_l, K_c) be a holomorphic character of the complexification K_c of K and $E_l = G \times_{\chi_l} \mathbb{C}$ the associated homogenous line bundle over X. Shimeno [7] proved that each eigenfunction of all invariant differential operators on E_l is the Poisson transform of an element f in the space $\mathfrak{B}(G/P_{min}; L_{l,\lambda})$ of hyperfunction sections of the line bundle $L_{l,\lambda}$ over the Furstenberg boundary G/P_{min} of X under certain condition on the parameter λ.

Recently, Ben Said proved a Fatou-type theorem for line bundles [1], and he characterized the range of the Poisson transform of L^p-functions on the maximal boundary of X as a Hardy-type space.

Since the space $\mathfrak{B}(G/P_{\Xi}; s)$ ($s \in \mathbb{C}$) of hyperfunction valued sections of degenerate principal series representations attached to the Shilov boundary $S \simeq G/P_{\Xi}$ of X is a G-submodule of $\mathfrak{B}(G/P_{min}; L_{l,\lambda_s})$ for some $\lambda_s \in \mathbb{C}$, it is natural to investigate under what conditions on the generalized Poisson transform F of f will f be in $L^p(S)$.

To state the main result of this paper, let us introduce some notations. For $l \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, we define the generalized Poisson transform $P_{l,\lambda}$ acting on hyperfunctions $f \in \mathfrak{B}(S)$ by

$$(P_{l,\lambda} f)(z) = \int_S \left(\frac{e^{2i\theta}}{{}^t(u - z)(u - z)} \right)^l \left(\frac{1 - 2\,{}^t\bar{z}z + |{}^t zz|^2}{|{}^t(u - z)(u - z)|^2} \right)^{\frac{\frac{n}{2} - l + i\lambda}{2}} f(u)du, \quad z \in \mathcal{D}.$$

The main result can be stated as follows.

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Theorem 1.1 *Let $l \in \mathbb{Z}, \lambda \in \mathbb{C}$ such that $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$. Then, we have the following.*

(i) *Let $F = P_{l,\lambda} f$, $f \in L^p(S), 1 < p < \infty$. Then*

$$\|F\|_{\lambda,p} = \sup_{0 \le r < 1} (1 - r^2)^{\mathcal{R}e[i\lambda] - \frac{n}{2} + l} \Big(\int_S |F(ru)|^p du \Big)^{\frac{1}{p}} < \infty.$$

(ii) *Let $f \in \mathfrak{B}(S)$.*
For $1 < p < \infty$, if $F = P_{l,\lambda} f$ satisfies $\|P_{l,\lambda} f\|_{\lambda,p} < \infty$, then f is in $L^p(S)$.
Moreover, there exists a positive constant $\gamma_l(\lambda)$ such that for every function $f \in L^p(S)$, we have

$$|C_l(\lambda)| \|f\|_p \le \|P_{l,\lambda} f\|_{\lambda,p} \le \gamma_l(\lambda) \|f\|_p.$$

(iii) *Let $F = P_{l,\lambda} f, f \in L^2(S)$. Then its L^2-boundary value f is given by the following inversion formula:*

$$f(u) = |C_l(\lambda)|^{-2} \lim_{r \longrightarrow 1^-} (1 - r^2)^{2(l + \mathcal{R}e[i\lambda] - \frac{n}{2})} \int_S F(rv) \overline{P_{l,\lambda}(ru, v)} dv, \quad \text{in } L^2(S),$$

where $C_l(\lambda)$ is given by (3.1) *(see Section 3).*

The main tool to obtain our results is the asymptotic behavior of the generalized spherical functions, which is a consequence of the following Fatou-type theorem.

Theorem 1.2 *Let $l \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ such that $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$. Then, we have*

$$\lim_{r \longrightarrow 1^-} (1 - r^2)^{-(\frac{n}{2} - l - i\lambda)} P_{l,\lambda} f(ru) = C_l(\lambda) f(u),$$

(i) *uniformly for f in the space $C(S)$ of all continuous functions on S,*
(ii) *uniformly in $L^p(S)$, if $f \in L^p(S)$, $1 < p < \infty$.*

We now describe the organization of this paper. In Section 2, we define a generalized Poisson transform. In Section 3, we establish a Fatou-type theorem. In Section 4, we give the precise action of the Poisson transform on $L^2(S)$ (Proposition 4.1). In the last section, we prove Theorem 1.1.

Notice that the case $l = 0$ corresponds to our main theorem in [2], which is governed by a Hua system.

This leads to the conjecture that a Hua system depending on l might exist that could characterize the range of the Poisson transform $P_{l,\lambda}$.

2 Poisson Transform

In this section, we consider a Poisson transform for the line bundle E_l.

Let

$$G = SO(n, 2) = \{g \in SL(n + 2, \mathbb{R}), \quad {}^t g I_{n,2} g = I_{n,2}\},$$

where $I_{n,2} = \begin{pmatrix} -I_n & 0 \\ 0 & I_2 \end{pmatrix}$.

The group $K = S(O(n) \times O(2))$ is a maximal compact subgroup of G.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. Let θ denote the corresponding Cartan involution of G and $\mathfrak{g}$. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the -1-eigenspace of θ in $\mathfrak{g}$.

Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. For any subset $\mathfrak{m}$ of $\mathfrak{g}_c$, we denote by $\mathfrak{m}_c$ the complex subspace of $\mathfrak{g}_c$ spanned by $\mathfrak{m}$.

Since the symmetric space G/K is Hermitian, there exist abelian subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ of $\mathfrak{g}_c$ such that $\mathfrak{p}_c = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Let G_c be the complexification of G with the Lie algebra $\mathfrak{g}_c$. We denote by K_c (resp. P_+, P_-) the complex analytic subgroup of G_c corresponding to $\mathfrak{k}_c$ (resp. $\mathfrak{p}_+, \mathfrak{p}_-$). Then G/K is realized as the G-orbit of the origin $U = K_c P_-$ of the generalized flag manifold G_c/U. Thus $P_+ K_c P_-$ is an open subset of G_c, and any element $w \in P_+ K_c P_-$ is uniquely expressed as $w = p_+ k p_-$, with $p_+ \in P_+, k \in K_c, p_- \in P_-$. This is called the Harish–Chandra decomposition. One can prove that $GU \subset P_+ U$ and that there exists a unique bounded domain $\mathcal{D}$ in $\mathfrak{p}_+$ such that $GU = (\exp \mathcal{D})U$. Then there are canonical isomorphisms $G/K \simeq GU/U \simeq \mathcal{D}$ given by $gK \mapsto gU \mapsto g \cdot 0 = z$. For $g \in G$, $z \in \mathcal{D}$, $g \cdot z$ denotes the unique element of $\mathcal{D}$ such that $g(\exp z)U = (\exp g \cdot z)U$. One fixes a point $\mu U \in G_c/U$ such that μU belongs to the boundary of GU/U and the G-orbit of μU is compact. The G-orbit $G\mu U/U$ is the Shilov boundary of the bounded domain $GU/U \cong G/K$, and the isotropic subgroup of the point μU in G_c/U is a maximal parabolic subgroup of G, which will be denoted by P_Ξ.

In our case $\mathfrak{p}_+ \simeq \mathbb{C}^n$,

$$\mathcal{D} = \left\{ z \in \mathbb{C}^n; \quad {}^t\bar{z}z < \tfrac{1}{2}(1 + |{}^t zz|^2) < 1 \right\},$$

and the action of G on $\mathcal{D}$ is given, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, by

$$g \cdot z = \left(Az + B \begin{pmatrix} \frac{i}{2}(1 -{}^t zz) \\ \frac{1}{2}(1 +{}^t zz) \end{pmatrix} \right) \left((-i, 1) \left(Cz + D \begin{pmatrix} \frac{i}{2}(1 -{}^t zz) \\ \frac{1}{2}(1 +{}^t zz \end{pmatrix} \right) \right)^{-1},$$

Put

$$u_\circ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \mu_\circ = \exp(u_\circ) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \gamma,$$

where

$$\gamma = \begin{pmatrix} I_n & 0 \\ 0 & \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \end{pmatrix}$$

Then, clearly, we get $\mu U = \mu_\circ U = (\exp u_\circ)U$, which implies that $G \cap \mu U \mu^{-1} = G \cap \mu_\circ U \mu_\circ^{-1} = P_\Xi$.

Put

$$S = \{ u \in p_+ \, ; \exp uU \in G\mu_\circ U/U \} = \left\{ u = e^{i\theta} x \, ; \quad 0 \le \theta < 2\pi, \quad x \in S^{n-1} \right\},$$

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where

$$S^{n-1} = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n ; \quad \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

Then S is the Shilov boundary of $\mathcal{D}$. let $P_\Xi = M_\Xi A_\Xi N_\Xi^\pm$ be a Langlands decomposition of the maximal parabolic subgroup P_Ξ of G:

$$M_\Xi = \left\{ \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{pmatrix} ; m_1 \in \{-1, 1\}, \quad m_2 \in SO(n-1, 1) \right\},$$

$$A_\Xi = \left\{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_n & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \in G ; \quad t \in \mathbb{R} \right\},$$

$$N_\Xi^\pm = \left\{ \begin{pmatrix} 1 + \frac{1}{2}(\xi^2 -^t \eta\eta) & ^t\eta & \xi & \frac{1}{2}(^t\eta\eta - \xi^2) \\ -\eta & I_{n-1} & 0 & \eta \\ \xi & 0 & 1 & -\xi \\ \frac{1}{2}(\xi^2 -^t \eta\eta) & ^t\eta & \xi & 1 + \frac{1}{2}(^t\eta\eta - \xi^2) \end{pmatrix} ; \xi \in \mathbb{R}, \quad \eta \in \mathbb{R}^{n-1} \right\}.$$

Let $a_\Xi = \mathbb{R}X_\circ$ be the one dimensional Lie algebra of A_Ξ

$$X_\circ = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

On a_Ξ, we define the linear form by $\rho_\circ(X_\circ) = 2$, and, on A_Ξ, we use the coordinate $a_t = e^{tX_\circ}$; $t \in \mathbb{R}$.

For $\lambda \in \mathbb{C}$ and $l \in \mathbb{Z}$, let $\xi_{l,\lambda}$ denote the C^∞-character of P_Ξ given by $\xi_{l,\lambda}(ma_t n) = m_1^l e^{2t(\frac{n}{2} - i\lambda)}$; $a_t = e^{tX_\circ} \in A_\Xi$, $n \in N_\Xi^\pm$ and

$$m = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{pmatrix} \in M_\Xi.$$

Put $\tilde{K}_c = \gamma K_c \gamma^{-1}$, $\quad \tilde{P}_- = \gamma P_- \gamma^{-1}$. Then, $U = K_c P_- = \gamma^{-1} \tilde{K}_c \tilde{P}_- \gamma$

$$\tilde{K}_c = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix} \in SL(n+2, \mathbb{C}); \quad \alpha \in SO(n, \mathbb{C}), \quad \delta \in \mathbb{C}^*) \right\},$$

$$\tilde{P}_- = \left\{ \begin{pmatrix} I_n & w & 0 \\ 0 & 1 & 0 \\ -2^t w & -^t ww & 1 \end{pmatrix} ; \quad w \in \mathbb{C}^n \right\}.$$

For $\lambda \in \mathbb{C}$ and $l \in \mathbb{Z}$, let χ_l denote the one-dimensional representation of U given by

$$\chi_l\colon \quad U = \gamma^{-1}\tilde{K}_c\tilde{P}_-\gamma \longrightarrow \mathbb{C}^*,$$
$$\gamma^{-1}\begin{pmatrix} \alpha & 0_{n,1} & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}\widetilde{P}_-\gamma \longmapsto (\delta)^{-l},$$

and we denote the corresponding representation of K by the same notation. Thus, for any

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K, \qquad k_2 = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

we have $\chi_l(k) = e^{-il\varphi}$.

We denote by E_l the line bundle over G/K associated with χ_l. Then the space of all C^∞-sections of E_l is identified with

$$C^\infty(G/K;\chi_l) = \left\{ f \in C^\infty(G)\,; \quad f(gk) = \chi_l^{-1}(k)f(g)\,; \quad g \in G,\, k \in K \right\}.$$

We denote by $L_{\xi_{l,\lambda}}$ the line bundle on G/P_Ξ associated with $\xi_{l,\lambda}$. Then the space of the hyperfunction sections on $L_{\xi_{l,\lambda}}$ is identified with

$$\mathfrak{B}(G/P_\Xi;\xi_{l,\lambda}) = \Big\{ f \in \mathfrak{B}(G); f(gma_tn) = \xi_{l,\lambda}^{-1}(ma_tn)f(g)$$
$$= e^{2(i\lambda-\frac{n}{2})t}\xi_{l,\lambda}^{-1}(m)f(g)\,; g \in G, m \in M_\Xi, a_t \in A_\Xi, n \in N_\Xi^+ \Big\}.$$

For $\phi \in \mathfrak{B}(G/P_\Xi;\xi_{l,\lambda})$, we define the Poisson integral $\widetilde{P}_{l,\lambda}\phi$ by

$$(\widetilde{P}_{l,\lambda}\phi)(g) = \int_K \chi_l(k)\phi(gk)dk.$$

Here dk denotes the invariant measure on K with total measure 1.

For $g \in G, g = kman\,(k \in K,\ m \in M_\Xi,\ a \in A_\Xi,\ n \in N_\Xi^+)$, we put

$$\kappa(g) = k,\ \widetilde{\kappa}(g) = km,\ H_\Xi(g) = \log a,\ n(g) = n.$$

We define $\omega_l(km) = \chi_l(k)\xi_{l,\lambda}(m) \quad (k \in K, m \in M_\Xi)$.

A straightforward computation shows that (see [6])

$$(2.1) \qquad (\widetilde{P}_{l,\lambda}\phi)(g) = \int_K \omega_l(\widetilde{\kappa}(g^{-1}k))e^{-(i\lambda+\frac{n}{2})\rho_\circ(H_\Xi(g^{-1}k))}\phi(k)dk.$$

Put

$$\mathfrak{B}(G\mu_\circ U/U;\chi_l) = \Big\{ \psi \in \mathfrak{B}(G\mu_\circ U)\,; \quad \psi(wu) = \chi_l^{-1}(u)\psi(w),$$
$$w \in G\mu_\circ U, u \in U \Big\}$$

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and

$$C^\infty(GU/U;\chi_l) = \Big\{ h \in C^\infty(GU)\,; \quad h(wu) = \chi_l^{-1}(u)h(w), \quad w \in GU,\ u \in U \Big\}.$$

Then, we obtain the following four isomorphisms

$$\begin{aligned} &C^\infty(GU/U;\chi_l) \longrightarrow C^\infty(G/K;\chi_l), && C^\infty(GU/U;\chi_l) \longrightarrow C^\infty(\mathcal{D}) \\ &h \longmapsto f, \quad f(g) = h(g),\ g \in G, && h \longmapsto F, \quad F(z) = h(\exp z),\ z \in \mathcal{D}, \end{aligned}$$

$$\begin{aligned} &\mathfrak{B}(G\mu_\circ U/U;\chi_l) \longrightarrow \mathfrak{B}(G/P_\Xi;\xi_{l,\lambda}), && \mathfrak{B}(G\mu_\circ U/U;\chi_l) \longrightarrow \mathfrak{B}(S) \\ &\psi \longmapsto \phi, \quad \phi(g) = \psi(g\mu_\circ),\ g \in G, && \psi \longmapsto \Phi, \quad \Phi(u) = \psi(\exp u),\ u \in S. \end{aligned}$$

Since $GU = (\exp \mathcal{D})U$ we have for any $g \in G$ and $k \in K$

$$\begin{aligned} g &= (\exp g \cdot 0)u(g) = (\exp z)u(g) \\ k\mu_\circ &= k(\exp u_\circ) = (\exp k \cdot u_\circ)u(k) = (\exp u)k. \end{aligned}$$

This implies that

$$\begin{aligned} (\widetilde{P}_{l,\lambda}\phi)(g) &= h((\exp z)u(g)) = \chi_l(u(g))^{-1}h(\exp z) = \chi_l(u(g))^{-1}(P_{l,\lambda}\Phi)(z) \\ \phi(k) &= \psi(k\mu_\circ) = \psi((\exp u)k) = \chi_l^{-1}(k)\psi(\exp u) = \chi_l^{-1}(k)\Phi(u). \end{aligned}$$

Substituting these functions into (2.1), we obtain

$$(P_{l,\lambda}\Phi)(z) = \int_S P_{l,\lambda}(z,u)\Phi(u)du,$$

where $P_{l,\lambda}(z,u)$ is the generalized Poisson kernel of the Lie ball $\mathcal{D}$ with respect to its Shilov boundary S given by

$$P_{l,\lambda}(z,u) = \chi_l(u(g))\chi_l^{-1}(k)\omega_l(\widetilde{\kappa}(g^{-1}k))e^{-(i\lambda+\frac{n}{2})\rho_\circ(H_\Xi(g^{-1}k))}, \quad z = g\cdot 0, \quad u = k\cdot u_\circ.$$

A straightforward computation shows that (see [2, 6])

$$P_{l,\lambda}(z,u) = \left(\frac{e^{2i\theta}}{{}^t(u-z)(u-z)}\right)^l \left(\frac{1-2\,{}^t\bar{z}z + |{}^tzz|^2}{|{}^t(u-z)(u-z)|^2}\right)^{\frac{\frac{n}{2}-l+i\lambda}{2}}, \quad l \in \mathbb{Z}, \quad \lambda \in \mathbb{C}.$$

3 Proof of Theorem 1.2

We begin by showing that the integral giving the c-function $C_l(\lambda)$ is absolutely convergent if $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$.

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Lemma 3.1 *Let $\lambda \in \mathbb{C}$ such that $\Re e[i\lambda] > \frac{n}{2} - 1$. Then, the integral*

$$C_l(\lambda) = 2^{2(i\lambda - \frac{n}{2})} \int_{N_\Xi^-} \omega_l(\widetilde{\kappa}(\bar{n})) e^{-(i\lambda + \frac{n}{2})\rho_\circ H_\Xi(\bar{n})} d\bar{n}.$$

converges absolutely.

Here $N_\Xi^- = \theta(N_\Xi^+)$, where θ is the Cartan involution of $SO(n,2)$ given by

$$\theta(g) = \begin{pmatrix} I_n & 0 \\ 0 & -I_2 \end{pmatrix} g \begin{pmatrix} I_n & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad g \in SO(n,2).$$

$$N_\Xi^- = \left\{ \begin{pmatrix} 1 + \frac{1}{2}(\xi^2 -^t \eta\eta) & ^t\eta & -\xi & \frac{1}{2}(\xi^2 -^t \eta\eta) \\ -\eta & I_{n-1} & 0 & -\eta \\ -\xi & 0 & 1 & -\xi \\ \frac{1}{2}(^t\eta\eta - \xi^2) & -^t\eta & \xi & 1 + \frac{1}{2}(^t\eta\eta - \xi^2) \end{pmatrix} ; \xi \in \mathbb{R},\ \eta \in \mathbb{R}^{n-1}. \right\}$$

To prove this lemma, we need the following lemma.

Lemma 3.2 (see [2,6]) *For any $g^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SO(n,2)$, we have*

$$e^{-\rho_\circ(H_\Xi(g))} = \left(\frac{1 - 2^t\bar{z}z + |^t zz|^2}{\left|^t(u_\circ - z)(u_\circ - z)\right|^2} \right)^{\frac{1}{2}}, \omega_l(\widetilde{\kappa}(g)) = \left(\frac{\left|^t(u_\circ - z)(u_\circ - z)\Delta\right|}{^t(u_\circ - z)(u_\circ - z)\Delta} \right)^l$$

and $|\Delta|^{-2} = 1 - 2^t\bar{z}z + |^t zz|^2$, where $z = g^{-1} \cdot 0$ and $\Delta = \frac{1}{2}(-i,1)D\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)$.

Proof of Lemma 3.1 By using Lemma 3.2, we get

$$\begin{aligned} \left| \omega_l(\widetilde{\kappa}(\bar{n})) e^{-(i\lambda + \frac{n}{2})\rho_\circ H_\Xi(\bar{n})} \right| &= \left(\frac{\left|^t(u_\circ - z)(u_\circ - z)\right|^2}{1 - 2^t\bar{z}z + |^t zz|^2} \right)^{-\frac{\Re e[i\lambda] + \frac{n}{2}}{2}} \\ &= \left(\frac{\left|1 - 2z_1 +^t zz\right|^2}{1 - 2^t\bar{z}z + |^t zz|^2} \right)^{-\frac{\Re e[i\lambda] + \frac{n}{2}}{2}}, \\ z &=^t (z_1, \dots, z_n) = \bar{n}^{-1} \cdot 0. \end{aligned}$$

Thus we assume that $i\lambda$ is real and $l = 0$.

Now, we consider the following function

$$f(x,y) = 16y + 4(1 + x - y)^2 - 4y - (2 + \tfrac{1}{2}(x - y))^2, \quad x \in \mathbb{R}^+, \quad y \in \mathbb{R}^+.$$

For $x \geq y \geq 0$, we get that

$$f(x,y) = 12y + 6(x - y) + \tfrac{15}{4}(x - y)^2 \geq 0.$$

For $0 \leq x \leq y$, to study the sign of $f(x,y)$, we evaluate the sign of $\frac{\partial f}{\partial y}(x,y)$

$$\frac{\partial f}{\partial y}(x,y) = 6 + \tfrac{15}{2}(y - x) > 0,$$

which implies that $f(x,\cdot)$ is increasing. Then $f(x,y) > f(x,x) = 12x \geq 0$.
Henceforth,

$$f({}^t\eta\eta,\xi^2) = 16\xi^2+4(1+{}^t\eta\eta-\xi^2)^2-4\xi^2-(2+\frac{1}{2}({}^t\eta\eta-\xi^2))^2 \geq 0, \eta\in\mathbb{R}^{n-1}, \xi\in\mathbb{R}.$$

Thus for any

$$\bar{n}^{-1} = \begin{pmatrix} 1+\frac{1}{2}(\xi^2-{}^t\eta\eta) & {}^t\eta & -\xi & \frac{1}{2}(\xi^2-{}^t\eta\eta) \\ -\eta & I_{n-1} & 0 & -\eta \\ -\xi & 0 & 1 & -\xi \\ \frac{1}{2}({}^t\eta\eta-\xi^2) & -{}^t\eta & \xi & 1+\frac{1}{2}({}^t\eta\eta-\xi^2) \end{pmatrix} \in N_{\Xi}^-,$$

we have

$$\left|{}^t(u_\circ - z)(u_\circ - z)\right|^2 = \frac{16\xi^2+4(1+{}^t\eta\eta-\xi^2)^2}{4\xi^2+(2+\frac{1}{2}({}^t\eta\eta-\xi^2))^2} \geq 1, \quad z = \bar{n}^{-1}\cdot 0.$$

This assures that the following integral

$$\begin{aligned} I &= \int_{N_{\Xi}^-} \left|\omega_0(\widetilde{\kappa}(\bar{n}))e^{-(i\lambda+\frac{n}{2})\rho_\circ H_{\Xi}(\bar{n})}\right| d\bar{n} \\ &= \int_{N_{\Xi}^-} \left(\frac{\left|{}^t(u_\circ-z)(u_\circ-z)\right|^2}{1-2{}^t\bar{z}z+|{}^tzz|^2}\right)^{-\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}} d\bar{n}, \\ &\leq \int_{N_{\Xi}^-} (1-2{}^t\bar{z}z+|{}^tzz|^2)^{\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}} d\bar{n}, \quad z = \bar{n}^{-1}\cdot 0. \end{aligned}$$

Thus

$$I \leq \int_{SO(n,2)} \left(1-2{}^t\bar{z}z+|{}^tzz|^2\right)^{\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}} dg = \int_{\mathcal{D}} \left(1-2{}^t\bar{z}z+|{}^tzz|^2\right)^{\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}} dz, z = g^{-1}\cdot 0.$$

It is known that (see [5, p. 12])

$$\int_{\mathcal{D}} \left(1-2{}^t\bar{z}z+|{}^tzz|^2\right)^{\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}} dz = \frac{\pi^n\Gamma\left(1+\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}\right)}{2^{n-1}\left(\frac{3n}{2}+\mathcal{R}e[i\lambda]\right)\Gamma\left(n+\frac{\mathcal{R}e[i\lambda]+\frac{n}{2}}{2}\right)} < \infty.$$

This concludes the proof of Lemma 3.1. ∎

Proof of Theorem 1.2 (i) For $\phi \in C(G/P_{\Xi},\xi_{l,\lambda})$, the map $h \to \chi_l(h)\phi(ka_th)$ is a $K\cup M_{\xi}$-invariant function on K. Put $g = \kappa(g)m(g)e^{H_{\Xi}(g)}n(g)$, then by [4, Chpt. I,

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Thm. 5.20], we have

$$\begin{aligned}\widetilde{P}_{l,\lambda}\phi(ka_t) &= \int_K \chi_l(h)\phi(ka_t h)dh\\ &= \int_{N_\Xi^-} \chi_l(\kappa(\bar{n}))\phi(ka_t\kappa(\bar{n}))e^{-n\rho_\circ(H_\Xi(\bar{n}))}d\bar{n}\\ &= \int_{N_\Xi^-} \chi_l(\kappa(\bar{n}))\xi_{l,\lambda}(m(\bar{n}))e^{-(i\lambda+\frac{n}{2})\rho_\circ H_\Xi(\bar{n})}\phi(ka_t\bar{n})d\bar{n}\\ &= e^{t(i\lambda-\frac{n}{2})}\int_{N_\Xi^-} \omega_l(\widetilde{\kappa}(\bar{n}))e^{-(i\lambda+\frac{n}{2})\rho_\circ H_\Xi(\bar{n})}\phi(ka_t\bar{n}a_{-t})d\bar{n}.\end{aligned}$$

Next, since $a_t\bar{n}a_{-t}$ goes to the identity element e of G, as $t \longrightarrow \infty$, we deduce that

$$\lim_{t\longrightarrow\infty} e^{(-i\lambda+\frac{n}{2})t}\widetilde{P}_{l,\lambda}\phi(ka_t) = 2^{2(\frac{n}{2}-i\lambda)}C_l(\lambda)\phi(k). \tag{3.1}$$

To justify the reversal order of the limit and integration, we use the dominated convergence theorem. For this, let

$$\psi_t(\bar{n}) = \omega_l(\widetilde{\kappa}(\bar{n}))e^{-(\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(\bar{n}))}\phi(ka_t\bar{n}a_{-t}).$$

Since $|\omega_l(\widetilde{\kappa}(\bar{n}))| = 1$ and $|\xi_{l,\lambda}(m)| = 1$ for all $m \in M_\Xi$, we have

$$\begin{aligned}|\psi_t(\bar{n})| &= \left| e^{-(\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(\bar{n}))}\phi(ka_t\bar{n}a_{-t})\right|\\ &= \left| e^{-(\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(\bar{n}))}\xi_{l,\lambda}^{-1}(m(a_t\bar{n}a_{-t}))e^{(-\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))}\phi(k\kappa(a_t\bar{n}a_{-t}))\right|\\ &\leq \left| e^{-(\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(\bar{n})+(-\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))}\right| \sup_{\bar{n},t}|\phi(k\kappa(a_t\bar{n}a_{-t}))|\\ &\leq \left| e^{-(\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(\bar{n})+(-\frac{n}{2}+i\lambda)\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))}\right| \sup_{k\in K}|\phi(k)|.\end{aligned}$$

In order to complete the proof, we apply the following lemma.

Lemma 3.3 *Let $t > 0$ and $\bar{n} \in N_\Xi^-$. Then, we have*

(i) $e^{\rho_\circ(H_\Xi(\bar{n}))} \geq 1$.
(ii) $e^{\rho_\circ(H_\Xi(\bar{n}))} \geq e^{\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))}$.
(iii) $e^{\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))} \geq 1$.

The proof will be given at the end of this section.

For the case $-1 < \mathcal{R}e[i\lambda] - \frac{n}{2} \leq 0$, we use (iii) of the above lemma to see that

$$|\psi_t(\bar{n})| \leq \sup_{k\in K}|\phi(k)|e^{-(\mathcal{R}e[i\lambda]+\frac{n}{2})\rho_\circ H_\Xi(\bar{n})},$$

which is an integrable function on N_Ξ^-.

In the case $\mathcal{R}e[i\lambda]-\frac{n}{2}\geq 0$, we use (ii) of the above lemma to see that

$$e^{-(\mathcal{R}e[i\lambda]+\frac{n}{2})\rho_\circ H_\Xi(\bar{n})+(\mathcal{R}e[i\lambda]-\frac{n}{2})\rho_\circ(a_t\bar{n}a_{-t})}\leq e^{-(\mathcal{R}e[i\lambda]+\frac{n}{2})\rho_\circ H_\Xi(\bar{n})+(\mathcal{R}e[i\lambda]-\frac{n}{2})\rho_\circ(\bar{n})}$$
$$=e^{-n\rho_\circ H_\Xi(\bar{n})}.$$

Thus,

$$|\psi_t(\bar{n})|\leq \sup_{k\in K}|\phi(k)|e^{-n\rho_\circ H_\Xi(\bar{n})}.$$

Hence, the result follows, since $\int_{N_\Xi^-}e^{-n\rho_\circ H_\Xi(\bar{n})}d\bar{n}<\infty$.

For any $\phi\in C(G/P_\Xi,\xi_{l,\lambda})$ and $\Phi\in C(S)$, we have

$$\phi(h)=\chi_l(k)^{-1}\Phi(u),\quad u=h\cdot u_\circ,\ h\in K,$$

$$(\widetilde{P}_{l,\lambda}\phi)(ka_t)=\int_K\omega_l(\widetilde{\kappa}((ka_t)^{-1}h))e^{-(i\lambda+\frac{n}{2})\rho_\circ(H_\Xi((ka_t)^{-1}h)))}\phi(h)dh,$$

$$(P_{l,\lambda}\Phi)(z)=\int_S P_{l,\lambda}(z,\tilde{u})\Phi(\tilde{u})d\tilde{u},\quad z=ka_t\cdot 0,$$

where

$$P_{l,\lambda}(z,\tilde{u})=\chi_l(u(ka_t))\chi_l^{-1}(h)\omega_l(\widetilde{\kappa}((ka_t)^{-1}h))e^{-(i\lambda+\frac{n}{2})\rho_\circ(H_\Xi((ka_t)^{-1}h)))},$$

$z=ka_t\cdot 0$, $\tilde{u}=h\cdot u_\circ$.

For $\Phi\in C(S)$, consider the function $\phi\in C(G/P_\Xi,\xi_{l,\lambda})$ such that

$$\phi(h)=\chi_l(k)^{-1}\Phi(u),\ u=h\cdot u_\circ,\ h\in K.$$

Then,

$$\widetilde{P}_{l,\lambda}\phi)(ka_t)=\chi_l(u(ka_t))^{-1}((P_{l,\lambda}\Phi)(z)).$$

Let $r\in[0,1[$ such that $z=ka_t\cdot 0=ru=rk\cdot u_\circ$, which implies that $e^t=\frac{(1+r)^2}{1-r^2}$.

Then, by using formula (3.1), we obtain

$$\lim_{r\longrightarrow 1^-}\Big(\frac{(1+r)^2}{1-r^2}\Big)^{\frac{n}{2}-i\lambda}\chi_l(u(ka_t))^{-1}(P_{l,\lambda}\Phi)(ru)=2^{2(\frac{n}{2}-i\lambda)}C_l(\lambda)\chi_l^{-1}(k)\Phi(u)$$

Thus, since $\chi_l(k)\chi_l^{-1}(u(ka_t))=(1-r^2)^l$ (see [5]), we have

$$\lim_{r\longrightarrow 1^-}(1-r^2)^{i\lambda-\frac{n}{2}}\chi_l(k)\chi_l^{-1}(u(ka_t))(P_{l,\lambda}\Phi)(ru)=\lim_{r\longrightarrow 1^-}(1-r^2)^{i\lambda-\frac{n}{2}+l}(P_{l,\lambda}\Phi)(ru)$$
$$=C_l(\lambda)\Phi(u).$$

Before giving the proof of Theorem 1.2(ii), we recall a result about representations of compact groups.

Let $\widehat{K}$ be the set of equivalence classes of finite-dimensional irreducible representations of K. For $\delta\in\widehat{K}$, let $C(S)(\delta)$ be the linear span of all K-finite functions on S of type δ. Then, by the Stone–Weierstrass theorem, the algebraic sum $\bigoplus_{\delta\in\widehat{K}}C(S)(\delta)$ is dense in $C(S)$ under the topology of uniform convergence. Since S is compact, $C(S)$ is dense in $L^p(S)$ for $1\leq p<\infty$, thus $\bigoplus_{\delta\in\widehat{K}}C(S)(\delta)$ is dense in $L^p(S)$.

For the proof of Theorem 1.2(ii), we need the following lemma.

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Lemma 3.4 *Let $\lambda \in \mathbb{C}$ such that $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$. Then, there exists a positive constant $\gamma_l(\lambda)$ such that for $p \in]1, \infty[$ and $f \in L^p(S)$, we have:*

$$\Big(\int_S |P_{l,\lambda} f(ru)|^p du \Big)^{\frac{1}{p}} \leq \gamma_l(\lambda)(1-r^2)^{-(\mathcal{R}e[i\lambda] - \frac{n}{2} + l)} \|f\|_p.$$

Proof For every $r \in [0, 1[$, we introduce the function $P^r_{l,\lambda}$ on K as follows

$$P^r_{l,\lambda}(k) = P_{l,\lambda}(ru_\circ, k^{-1}u_\circ).$$

Then, the above integral can be written as a convolution over the compact group K,

$$P_{l,\lambda} f(ru) = f * P^r_{l,\lambda}(k), \qquad u = ku_\circ.$$

By the Young–Hausdorff inequality, we have

$$\Big(\int_S |P_{l,\lambda} f(ru)|^p du \Big)^{\frac{1}{p}} \leq \|f\|_p \|P^r_{l,\lambda}\|_1.$$

Next, using the fact that

$$\begin{aligned} \|P^r_{l,\lambda}\|_1 &= \int_S |P_{l,\lambda}(ru_\circ, u)| du \\ &= (1-r^2)^{\frac{n}{2}+l-\mathcal{R}e[i\lambda]} \int_S \Big(\frac{1}{|{}^t(ru_\circ - u)(ru_\circ - u)|} \Big)^{\frac{n}{2}+\mathcal{R}e[i\lambda]} du, \end{aligned}$$

we obtain from the Forelli–Rudin inequality (see [3]) that there exists a positive constant $\gamma_l(\lambda)$ such that

$$\|P^r_{l,\lambda}\|_1 \leq \gamma_l(\lambda)(1-r^2)^{-(\mathcal{R}e[i\lambda] - \frac{n}{2} + l)}.$$

This completes the proof of Lemma 3.4. ∎

Now, let us prove Theorem 1.2(ii). Let $f \in L^p(S)$. Then, for any $\epsilon > 0$, there exists $\Phi \in \bigoplus_{\delta \in \hat{K}} C(S)(\delta)$ such that $\|f - \Phi\|_p < \epsilon$, and one gets

$$\begin{aligned} \|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda}(f) - f\|_p &\leq \|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda}(f - \Phi)\|_p \\ &+ \|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda}\Phi - \Phi\|_p + \|\Phi - f\|_p, \end{aligned}$$

where $P^r_{l,\lambda} f(u) = P_{l,\lambda} f(ru)$. By Lemma 3.4

$$\|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda}(f - \Phi)\|_p \leq \gamma_l(\lambda)|C_l(\lambda)|^{-1}\|\Phi - f\|_p$$

and Theorem 1.2(i), we get

$$\lim_{t \longrightarrow \infty} \|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda}\Phi - \Phi\|_p = 0.$$

Therefore,

$$\lim_{t \longrightarrow \infty} \|C_l(\lambda)^{-1}(1-r^2)^{-(\frac{n}{2}-l-i\lambda)} P^r_{l,\lambda} f - f\|_p \leq \epsilon(\gamma_l(\lambda) + 1),$$

which implies (ii) and the proof of Theorem 1.2 is finished. ∎

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Proof of Lemma 3.3 For any

$$\bar{n}^{-1} = \begin{pmatrix} 1+\frac{1}{2}(\xi^2 -{}^t\,\eta\eta) & {}^t\eta & -\xi & \frac{1}{2}(\xi^2 -{}^t\,\eta\eta) \\ -\eta & I_{n-1} & 0 & -\eta \\ -\xi & 0 & 1 & -\xi \\ \frac{1}{2}({}^t\eta\eta-\xi^2) & -{}^t\eta & \xi & 1+\frac{1}{2}({}^t\eta\eta-\xi^2) \end{pmatrix} \in N_\Xi^-$$

and

$$a_t = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & I_{n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} \in A_\Xi,$$

we have

$$a_t\bar{n}^{-1}a_{-t} = \begin{pmatrix} 1+\frac{1}{2}(\xi^2 -{}^t\,\eta\eta)e^{-2t} & {}^t\eta e^{-t} & -\xi e^{-t} & \frac{1}{2}(\xi^2 -{}^t\,\eta\eta)e^{-2t} \\ -\eta e^{-t} & I_{n-1} & 0 & -\eta e^{-t} \\ -\xi e^{-t} & 0 & 1 & -\xi e^{-t} \\ \frac{1}{2}({}^t\eta\eta-\xi^2)e^{-2t} & -{}^t\eta e^{-t} & \xi e^{-t} & 1+\frac{1}{2}({}^t\eta\eta-\xi^2)e^{-2t} \end{pmatrix}.$$

Thus

$$z = \bar{n}^{-1}\cdot 0 = \frac{1}{2+2i\xi+\frac{1}{2}({}^t\eta\eta-\xi^2)}\begin{pmatrix} -i\xi-\frac{1}{2}({}^t\eta\eta-\xi^2) \\ -\eta \end{pmatrix}$$

and

$$\tilde{z} = a_t\bar{n}^{-1}a_{-t}\cdot 0 = \frac{1}{2+2i\xi e^{-t}+\frac{1}{2}({}^t\eta\eta-\xi^2)e^{-2t}}\begin{pmatrix} -i\xi e^{-t}-\frac{1}{2}({}^t\eta\eta-\xi^2)e^{-2t} \\ -\eta e^{-t} \end{pmatrix}.$$

By using Lemma 3.2, we have

$$\begin{aligned} e^{2\rho_\circ(H_\Xi(\bar{n}))} &= \frac{\left|{}^t(u_\circ - z)(u_\circ - z)\right|^2}{1-2{}^t\bar{z}z+|{}^tzz|^2} = \frac{\left|1-2z_1+{}^t zz\right|^2}{1-2{}^t\bar{z}z+|{}^tzz|^2} \\ &= 1+2({}^t\eta\eta+\xi^2)+({}^t\eta\eta-\xi^2)^2 \geq 1, \quad {}^tz=(z_1,\dots,z_n) \end{aligned}$$

and

$$e^{2\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))} = \frac{\left|{}^t(u_\circ - \tilde{z})(u_\circ - \tilde{z})\right|^2}{1-2{}^t\bar{\tilde{z}}\tilde{z}+|{}^t\tilde{z}\tilde{z}|^2} = 1+2({}^t\eta\eta+\xi^2)e^{-2t}+({}^t\eta\eta-\xi^2)^2e^{-4t} \geq 1.$$

Thus,

$$e^{2\rho_\circ(H_\Xi(\bar{n}))} - e^{2\rho_\circ(H_\Xi(a_t\bar{n}a_{-t}))} = 2({}^t\eta\eta+\xi^2)(1-e^{-2t})+({}^t\eta\eta-\xi^2)^2(1-e^{-4t}) \geq 0. \blacksquare$$

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4 The Precise Action of the Generalized Poisson Transform $P_{l,\lambda}$ on $L^2(S)$

In this section, we have to study the action of the generalized Poisson transform $P_{l,\lambda}$ on $L^2(S)$.

First, recall that the Peter–Weyl decomposition can be stated as

$$L^2(S) = \bigoplus_{m \in \wedge} V_m,$$

where $\wedge$ is the set of all two-tuple, $m = (m_1, m_2) \in \mathbb{Z}^2$ with $m_1 \geq m_2$. The K-irreducible component V_m is the finite linear span $\{\phi_m \circ k, k \in K\}$. Here the function $\phi_m \in V_m$ is the zonal spherical function.

Proposition 4.1 *Let $\lambda \in \mathbb{C}, l \in \mathbb{Z}$ and let $f \in V_m$. Then, we have*

$$(P_{l,\lambda} f)(ru) = \Phi^l_{\lambda,m}(r) f(u),$$

where $\Phi^l_{\lambda,m}(r) = (P_{l,\lambda}\phi_m)(ru_\circ)$.

Proof We introduce the operator $P^r_{l,\lambda} \colon L^2(S) \longrightarrow L^2(S)$:

$$(P^r_{l,\lambda} f)(u) = \int_S P_{l,\lambda}(ru, v) f(v) dv.$$

Since the operator $P^r_{l,\lambda}$ commutes with the K-action, and this action is multiplicity free, it is Scalar on each component V_m. Hence there exists a constant $\Phi^l_{\lambda,m}(r)$ such that

$$P^r_{l,\lambda} = \Phi^l_{\lambda,m}(r) \cdot I \text{ on } V_m, \tag{4.1}$$

where I is identity operator on V_m.

Taking the spherical function ϕ_m in (4.1), we get $\Phi^l_{\lambda,m}(r) = (P^r_{l,\lambda}\phi_m)(u_\circ)$. Thus, from Theorem 1.2, we deduce the following asymptotic behavior of the generalized spherical function $\Phi^l_{\lambda,m}(r)$.

Corollary 4.2 *Let $l \in \mathbb{Z},\ \lambda \in \mathbb{C}$ such that $\mathcal{R}e[i\lambda] > \frac{n}{2} - 1$. Then, for $r \in [0, 1[$, we have*

$$\lim_{r \longrightarrow 1^-} (1 - r^2)^{-(\frac{n}{2} - l - i\lambda)} \Phi^l_{\lambda,m}(r) = C_l(\lambda)$$

uniformly in $m \in \wedge$.

4.1 Proof of Theorem 1.1

(i) Let $F = P_{l,\lambda} f,\ f \in L^p(S)$. By Lemma 3.4, we get the right-hand side of the estimate in Theorem 1.1. Thus, $\|P_{l,\lambda} f\|_{\lambda,p} < \infty$.

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(ii) Let $F = P_{l,\lambda} f$, $f \in A'(S)$ such that $\|F\|_{\lambda,2} < \infty$ and $f = \sum_{m\in\wedge} f_m$ be its K-type decomposition, then using Proposition 4.1, we get

$$F(ru) = \sum_{m\in\wedge} \Phi^l_{\lambda,m}(r) f_m(u) \quad \text{in} \quad C^\infty([0,1[\times S).$$

Since $\|F\|_{\lambda,2} < \infty$, we get

$$(1-r^2)^{-(n-l-\mathcal{R}e[i\lambda])} \Big\{ \sum_{m\in\wedge} |\Phi^l_{\lambda,m}(r)|^2 \|f_m\|_2 \Big\}^{\frac{1}{2}} < \infty$$

for every $r \in [0,1[$.

Let $\wedge_\circ$ be a finite subset of $\wedge$, then we have

$$(1-r^2)^{-(\frac{n}{2}-l-\mathcal{R}e[i\lambda])} \Big\{ \sum_{m\in\wedge_\circ} |\Phi^l_{\lambda,m}(r)|^2 \|f_m\|_2 \Big\}^{\frac{1}{2}} \le \|F\|_{\lambda,2} < \infty$$

for every $r \in [0,1[$.

Next, using the asymptotic behavior of $\Phi^l_{\lambda,m}(r)$ given by Corollary 4.2, we obtain

$$|C_l(\lambda)|^2 \sum_{m\in\wedge_\circ} \|f_m\|_2^2 \le \|F\|^2_{\lambda,2} < \infty,$$

from which we deduce that the left-hand side of the estimate in Theorem 1.1 holds for $p = 2$.

For the case $p \in [2,\infty[$, let F be a $\mathbb{C}$-valued function on $\mathcal{D}$ such that $\|F\|_{\lambda,p} < \infty$.

By using the fact that $\|F\|_{\lambda,2} \le \|F\|_{\lambda,p}$, there exist from Theorem 1.1(iii) a function $f \in L^2(S)$ such that $F = P_\lambda f$ and $f(u) = \lim_{r\longrightarrow 1^-} g_r(u)$ in $L^2(S)$, where

$$g_r(u) = |C_l(\lambda)|^{-2}(1-r^2)^{2(l+\mathcal{R}e[i\lambda]-\frac{n}{2})} \int_S F(rv)\overline{P_{l,\lambda}(ru,v)}dv.$$

Let Φ be a continuous function in S. Then we have

$$\lim_{r\longrightarrow 1^-} \int_S g_r(u)\overline{\Phi(u)}du = \int_S f(u)\overline{\Phi(u)}du.$$

But

$$\begin{aligned}\int_S g_r(u)\overline{\Phi(u)}du &= |C_l(\lambda)^{-2}(1-r^2)|^{2(l+\mathcal{R}e[i\lambda]-\frac{n}{2})} \int_S \Big(\int_S F(rv)\overline{P_{l,\lambda}(ru,v)}dv \Big) \overline{\Phi(u)}du \\ &= |C_l(\lambda)^{-2}(1-r^2)|^{2(l+\mathcal{R}e[i\lambda]-\frac{n}{2})} \int \int_S \overline{P_{l,\lambda}\Phi(rv)} F(rv)dv.\end{aligned}$$

Thus by using the Holder inequality, we obtain

$$\Big| \int_S \overline{P_{l,\lambda}\Phi(rv)} F(rv)dv \Big| \le \Big(\int_S |F(rv)|^p dv \Big)^{\frac{1}{p}} \Big(\int_S |(P_{l,\lambda}\Phi)(rv)|^q dv \Big)^{\frac{1}{q}},$$

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where q is such that $\frac{1}{p}+\frac{1}{q}=1$.

Since $\|F\|_{\lambda,p}<\infty$, we obtain

$$\Big|\int_S g_r(u)\overline{\Phi(u)}du\Big| \le |C_l(\lambda)^{-2}\|(1-r^2)|^{(l+\mathcal{R}e[i\lambda]-\frac{n}{2})}\Big(\int_S |(P_{l,\lambda}\Phi)(rv)|^q dv\Big)^{\frac{1}{q}}\|F\|_{\lambda,p}.$$

Next, Theorem 1.2 shows that, for every $q>1$,

$$\Phi(u)=C_l(\lambda)^{-1}\lim_{r\longrightarrow 1^-}(1-r^2)^{(l+\mathcal{R}e[i\lambda]-\frac{n}{2})}(P_{l,\lambda}\Phi)(ru)\quad\text{in}\quad L^q(S).$$

Hence,

$$\|f\|_p=\sup_{\|\Phi\|\le 1}|\int_S f(u)\overline{\Phi(u)}|du\le C_l(\lambda)^{-1}\|\Phi\|_q\|F\|_{\lambda,p}.$$

Finally, we deduce that $f\in L^p(S)$ and that $|C_l(\lambda)|\|f\|_p\le\|F\|_{\lambda,p}$.

For the case $1<p\le 2$, Let x_n be an approximation of the identity in the space $C(K/K\cap M_\Xi,\chi_l)$ of continuous functions Φ on K satisfying $\Phi(km)=\chi_l^{-1}(m)\Phi(k)$, $m\in K\cap M_\Xi$. That is, $\int_K|\chi_l(k)|x_n(k)dk=1$ and $\lim_{n\to\infty}\int_{K\setminus U}x_n(k)\chi_l(k)dk=0$ for every neighborhood U of the neutral element of K.

For each n, define the function F_n on G/K by

$$F_n(gK)=\int_K x_n(k)F(k^{-1}gK)dk.$$

Then, $\lim_{n\to\infty}F_n=F$ pointwise in G. Since $F=P_{l,\lambda}f,\quad f\in A'(S)$, there exist $f_n\in A'(S)$ such that $F_n=P_{l,\lambda}f_n$.

For each $r\in[0,1[$, define a function F_n^r in S by $F_n^r(u)=F(ru)$. Then,

$$\chi_l(k_\circ)F_n(rk_\circ\cdot e)=\chi_l(k_\circ)F_n^r(k_\circ\cdot e)=(\chi_l x_n*\chi_l F^r)(k_\circ).$$

Therefore,

$$\|\chi_l F_n^r\|_2\le\|\chi_l x_n\|_2\|\chi_l F^r\|_1\le\|\chi_l x_n\|_2\|\chi_l F^r\|_p$$

which implies that $\|F_n\|_{\lambda,2}<\infty$. Thus $f_n\in L^2(S)$.

Let q such that $\frac{1}{p}+\frac{1}{q}=1$ and let L_n be the linear form defined in $L^q(S)$ by

$$L_n(\Phi)=\int_K\chi_{2l}(k)f_n(k)\Phi(k)dk.$$

Since $p\le 2$, we have $f_n\in L^p(S)$ and

$$|L_n\Phi|\le\|\chi_{2l}f_n\|_p\|\Phi\|_q\le\|\chi_{2l}\|_1\|f_n\|_p\|\Phi\|_q.$$

By Theorem 1.2(ii), we known that

$$f_n(u)=\lim_{r\longrightarrow 1^-}|C(\lambda)|^{-1}(1-r^2)^{(i\lambda-\frac{n}{2}+l)}P_{l,\lambda}f_n(ru)\quad\text{in}\quad L^p(S).$$

Hence, there exists a sequence (r_j) with $r_j \to 1^-$ as $j \to \infty$ such that

$$f_n(u) = \lim_{j \longrightarrow \infty} |C(\lambda)|^{-1}(1 - r_j^2)^{(i\lambda - \frac{n}{2} + l)} P_{l,\lambda} f_n(r_j u)$$

almost everywhere in S.

By the classical Fatou lemma, we have

$$\|f_n\|_p \leq |C(\lambda)|^{-1} \sup_j |(1 - r_j^2)|^{(\mathcal{R}e[i\lambda] - \frac{n}{2} + l)} \Big(\int_S |F_n(r_j u)|^p du \Big)^{\frac{1}{p}},$$

which gives

$$\|f_n\|_p \leq |C(\lambda)|^{-1} \|P_{l,\lambda} f_n\|_{\lambda,p}.$$

Hence,

$$|L_n(\Phi)| \leq |C(\lambda)|^{-1} \|F_n\|_{\lambda,p} \|\Phi\|_q.$$

Now, from $\|\chi_l F_n^r\|_p \leq \|\chi_l x_n\|_1 \|\chi_l F^r\|_p = \|\chi_l F^r\|_p$, we deduce that $\|F_n\|_{\lambda,p} \leq \|F\|_{\lambda,p}$ and $|L_n \Phi| \leq |C(\lambda)|^{-1} \|F\|_{\lambda,p} \|\Phi\|_q$.

Therefore the linear functionals L_n are uniformly bounded. By the Banach–Alaoglu theorem, there exists a subsequence $\{L_{n_j}\}$ that converges under the weak* topology to a bounded linear functional L on $L^q(S)$, with $\|L\| \leq |C(\lambda)|^{-1} \|F\|_{\lambda,p}$. By the Riesz representation theorem, there exists a unique function $f \in L^p(S)$ such that

$$L(\Phi) = \int_K \chi_{2l}(k) f(k) \Phi(k) dk.$$

Now, let $\phi_g(k) = \chi_l(u(g)) \chi_l^{-1}(k) \omega_l(\widetilde{\kappa}(g^{-1}k)) e^{-(i\lambda + n)\rho_\circ(H_\Xi(g^{-1}k))}$.

Then, $F_n(gK) = L_n(\chi_{-2l}(k)\phi_g(k))$. Since (F_{n_j}) converge pointwise to F and (L_{n_j}) converge to L under the weak* topology, we have

$$F(gK) = \lim_{n_j \longrightarrow \infty} F_{n_j}(gK) = \lim_{n_j \longrightarrow \infty} L_{n_j}(\chi_{-2l}\phi_g) = L(\chi_{-2l}\phi_g).$$

Therefore, $F(gK) = P_{l,\lambda} f(gK)$.

(iii) Let $F = P_{l,\lambda} f$, $f \in L^2(S)$. Expanding f into its K-type series, $f = \sum_{m \in \wedge} f_m$ and using Proposition 4.1, we get the series expansion of F

$$F(ru) = \sum_{m \in \wedge} \Phi^l_{\lambda,m}(r) f_m(u)$$

in $C^\infty([0, 1[\times S)$, with $\sum_{m \in \wedge} |\Phi^l_{\lambda,m}(r)|^2 \|f_m\|_2^2 < \infty$, for all $r \in [0, 1[$.

Now, for each $r \in [0, 1[$, consider the following $\mathbb{C}$-valued function g_r on the Shilov boundary S given by

$$g_r(u) = (1 - r^2)^{-2(\frac{n}{2} - l - \mathcal{R}e[i\lambda])} \int_S F(rv) \overline{P_{l,\lambda}(ru, v)} dv.$$

Thus,

$$g_r(u) = (1 - r^2)^{-2(\frac{n}{2} - l - \mathcal{R}e[i\lambda])} \int_S \sum_{m \in \wedge} \Phi^l_{\lambda,m}(r) f_m(v) \overline{P_{l,\lambda}(ru, v)} dv.$$

Since, for every fixed $r \in [0, 1[$, the series $\sum_{m \in \wedge} \Phi^l_{\lambda,m}(r) f_m(v)$ uniformly converges on S, we get

$$g_r(u) = (1 - r^2)^{-d(\frac{n}{d} - l - \mathcal{R}e[i\lambda])} \sum_{m \in \wedge} \Phi^l_{\lambda,m}(r) \int_S f_m(v) \overline{P_{l,\lambda}(ru, v)} dv,$$

and by Proposition 4.1, we have

$$g_r(u) = (1 - r^2)^{-2(\frac{n}{2} - l - \mathcal{R}e[i\lambda])} \sum_{m \in \wedge} |\Phi^l_{\lambda,m}(r)|^2 f_m(u),$$

noticing that

$$\| |C_l(\lambda)|^{-2} g_r - f \|_2^2 = \sum_{m \in \wedge} \Big[|C_l(\lambda)|^{-2} (1 - r^2)^{-2(\frac{n}{2} - l - \mathcal{R}e[i\lambda])} |\Phi^l_{\lambda,m}(r)|^2 - 1 \Big]^2 \times \|f_m\|_2^2$$

and, using the limit of the generalized spherical function $\Phi^l_{\lambda,m}(r)$ (which uniformly in $m \in \wedge$) given by Corollary 4.2, we see that

$$\lim_{r \to 1^-} \|C_l(\lambda))^{-2} g_r - f\|_2^2 = 0,$$

which gives the desired result. ■

Remark 4.3 Note that, to prove the Theorems 1.1 and Theorem 1.2 in the case n even one can proceed also by computing explicitly $\Phi^l_{\lambda,m}(r)$ and its asymptotic behavior [2].

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