ROOT CLOSURE IN INTEGRAL DOMAINS, II

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In this note, we give an elementary procedure for constructing n-root closed integral domains. We then use this construction to give two interesting examples. First, we give an example of a root closed integral domain which is not quasinormal. Secondly, we show that for any subset S of odd positive primes there is a one-dimensional affine domain which is p-root closed for a prime p if and only if $p \in S$.

For the convenience of the reader, we first recall a few definitions. For a positive integer n, an integral domain R with quotient field K is said to be *n-root closed* if whenever $x^n \in R$ for some $x \in K$, then $x \in R$. If R is n-root closed for each positive integer n, then R is *root-closed*. Note that for relatively prime positive integers m and n, R is mn-root closed if and only if R is both m- and n-root closed. Hence we often restrict ourselves to the case in which R is p-root closed for a prime p. The domain R is *seminormal* if whenever $x^2, x^3 \in R$ for some $x \in K$, then $x \in R$. Clearly an integrally closed domain is root closed, and for each $n \geq 2$, an n-root closed domain is seminormal. In general, though, neither implication is reversible. However, examples of n-root closed domains which are not integrally closed do not seem to be too common (cf. [**2**] and [**3**]). Here we show how to construct such a family of examples easily. Root closure has also been investigated in [**1**], [**4**], [**6**], and [**11**].

For any integral domain R with quotient field K, we let A denote the subring $\{f(X) \in R[X] : f(0) = f(1)\}$ of $R[X]$. Also, let $\varphi_n : R \to R$ be the mapping defined by $\varphi_n(x) = x^n$ for each $x \in R$. We first record some observations about the domain A.

PROPOSITION 1. (a) $A = R[X^2 - X, X^3 - X^2] = R + X(X-1)R[X]$.
(b) *A has quotient field $K(X)$ and is not integrally closed.*
(c) *A is seminormal if and only if R is seminormal.*
(d) *A is n-root closed if and only if R is n-root closed and φ_n is injective.*

Proof. (*a*) is easily verified by induction on $\deg f$ for $f \in A$, and (b) is an immediate consequence of (a).

(c) Certainly R is seminormal if A is seminormal. Conversely, let R be seminormal. Suppose that $f^2, f^3 \in A$ for some $f \in K(X)$. Then $f \in R[X]$ since $R[X]$ is seminormal [**6**, Theorem 2] (cf. [**7**, Theorem 1.6] and [**5**, Theorem 1]). Hence $[f(0)]^2 = f^2(0) = f^2(1) = [f(1)]^2$ and $[f(0)]^3 = [f(1)]^3$ yield $f(0) = f(1)$. Thus $f \in A$ and hence A is seminormal.

(d) First, suppose that A is n-root closed. Clearly R is then also n-root closed. To show that φ_n is injective, suppose that $a^n = b^n$ for some $a, b \in R$. Define $f(X) = (b-a)X + a$. Then $f^n \in A$ since $f(0) = a$ and $f(1) = b$. Hence $f \in A$; thus $a = b$ and so φ_n is injective. Conversely, suppose that R is n-root closed and φ_n is injective. If $f^n \in A$ for some $f \in K(X)$, then $f \in R[X]$ since $R[X]$ is n-root closed [**6**, Theorem 2]. Hence $[f(0)]^n = [f(1)]^n$, and thus $f(0) = f(1)$ since φ_n is injective. Thus $f \in A$ and hence A is n-root closed.

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We remark that a special case of our construction has been used in the proof of [**2**, Theorem 2.4]. Parts (c) and (d) of Proposition 1 may also be proved by reducing modulo the conductor $X(X-1)R[X]$ (cf. [**2**, Propositions 2.1 and 2.2]).

We next give some other criteria for φ_n to be injective. The elementary proofs will be omitted.

PROPOSITION 2. *The following statements are equivalent for an integral domain R with quotient field K.*

(a) $\varphi_n : R \to R$ *is injective.*

(b) $\varphi_n : K \to K$ *is injective.*

(c) *If* $x^n = 1$ *for some* $x \in K$, *then* $x = 1$.

For the remainder of this paper, we will restrict ourselves to the case in which R is itself a field. In this case, $A = K[X^2 - X, X^3 - X^2]$ is a one-dimensional seminormal affine domain which is n-root closed if and only if φ_n is injective—i.e., if and only if 1 is the only nth root of unity in K. For example: $\mathbb{Z}/2\mathbb{Z}[X^2 - X, X^3 - X^2]$ is root closed, $\mathbb{Q}[X^2 - X, X^3 - X^2]$ and $\mathbb{R}[X^2 - X, X^3 - X^2]$ are each n-root closed if and only if n is odd, and $\mathbb{C}[X^2 - X, X^3 - X^2]$ is seminormal but not n-root closed for any $n \geq 2$.

We may also localize A. Let M be the maximal ideal $(X^2 - X, X^3 - X^2) = \{f \in K(X) : f(0) = f(1) = 0\}$ of A. Then A_M is n-root closed if and only if A is n-root closed. We prove this in our next theorem, which also collects several earlier observations about the domain A.

THEOREM 3. *Let K be a field,* $A = K[X^2 - X, X^3 - X^2]$, *and* $M = (X^2 - X, X^3 - X^2)$.

(a) *A is a one-dimensional seminormal affine domain which is n-root closed if and only if* 1 *is the only n-th root of unity in K.*

(b) A_M *is a one-dimensional seminormal local domain which is n-root closed if and only if* 1 *is the only n-th root of unity in K.*

Proof. We have already observed that (a) holds. It is well known that a localization of a seminormal (resp. n-root closed) integral domain is also seminormal (resp. n-root closed). Hence we need only show that A is n-root closed whenever A_M is n-root closed. Suppose that $a^n \in A$ for some $a \in K(X)$. Then a is in both $K[X]$ and A_M. Write $a = f/g$ with $f, g \in A$ and $g \notin M$. Then $f = ag$ and $g(0) = g(1) \neq 0$ yield $a(0) = a(1)$. Hence $a \in A$, so A is n-root closed.

Next we give a few specific cases in which we can determine whether A is n-root closed (cf. [**3**, Theorems 1, 2, and 3]). The proofs, which involve only elementary field theory, will be omitted.

PROPOSITION 4. *Let K be a field and* $A = K[X^2 - X, X^3 - X^2]$.

(a) *A is* 2*-root closed if and only if char* $K = 2$.

(b) *If char* $K = p \geq 2$, *then A is n-root closed if and only if* $(|F| - 1, n) = 1$ *for each finite subfield F of K. In particular, A is p-root closed if char* $K = p$.

(c) *A is root closed if and only if char* $K = 2$ *and each element of* $K - \mathbb{Z}/2\mathbb{Z}$ *is transcendental over* $\mathbb{Z}/2\mathbb{Z}$.

(d) *If K is algebraically closed, then A is p-root closed for a prime p if and only char $K = p$. In particular, if char $K = 0$, then A is not n-root closed for any $n \geq 2$; if char $K = p$, then A is n-root closed if and only if n is a p-power.*

We end this paper with two specific applications of the earlier theory. Our first example is a root closed integral domain which is not quasinormal. We recall that a domain R is seminormal if and only if $\text{Pic}(R) = \text{Pic}(R[X])$ and that R is said to be *quasinormal* if $\text{Pic}(R) = \text{Pic}(R[X, X^{-1}])$. It is well known that an integrally closed domain is quasinormal, a quasinormal domain is seminormal, and that in general neither implication is reversible. We show that root closure neither implies nor is implied by quasinormality. This is particularly interesting because in [**9**, Theorem 2.15] it was shown that an n-root closed noetherian domain R is quasinormal if R contains a field which has a nontrivial nth root of unity. Our example shows that this last hypothesis is essential. Finally, recall that an integral domain R is said to be *u-closed* if whenever $x^2 - x$, $x^3 - x^2 \in R$ for some $x \in K$, then $x \in R$. A one-dimensional domain R is quasinormal if and only if R is seminormal and u-closed [**9**, Corollary 1.14].

Example 5. Let $A = \mathbb{Z}/2\mathbb{Z}[X^2 - X, X^3 - X^2]$. We have already observed that A is a one-dimensional root closed affine domain. However, A is not quasinormal since it is not u-closed. We may also localize A at its maximal ideal $M = (X^2 - X, X^3 - X^2)$ to obtain a one-dimensional root closed local domain which is not u-closed and hence not quasinormal.

Thus a root closed integral domain need not be quasinormal. For the other direction, $R = \mathbb{R} + X\mathbb{C}[[X]]$ is a one-dimensional quasinormal local domain which is not n-root closed for any $n \geq 2$ [**8**, Example (a)].

Since an integral domain R is mn-root closed for relatively prime positive integers m and n if and only if R is both m- and n-root closed, $\mathscr{C}(R) = \{n \in \mathbb{N} : R \text{ is } n\text{-root closed}\}$ is a (multiplicative) submonoid of $\mathbb{N}$ generated by positive primes. Moreover, in [**1**, Theorem 2.7] we showed that any (multiplicative) submonoid of $\mathbb{N}$ generated by primes can be realized as $\mathscr{C}(R)$ for some integral domain R. That construction used monoid domains over an arbitrary field and R was usually quite large ($\dim R = 2\,|\{p : p \text{ is prime and } R \text{ is not } p\text{-root closed}\}|$ and R was noetherian if and only if $\dim R$ was finite). The construction here allows R to be a one-dimensional noetherian domain (as long as $p \neq 2$). We state this as a theorem.

Theorem 6. *Let S be a set of odd positive primes. Then there is a one-dimensional seminormal affine domain A such that $\mathscr{C}(A)$ is generated by S. The integral domain A may also be chosen to be a one-dimensional seminormal local domain.*

Proof. By Theorem 3(a), we need only construct a field K such that for each prime p, K contains a primitive pth root of unity if and only if $p \notin S$. Let $T = \{p : p \text{ is prime and } p \notin S\}$ and $K = \mathbb{Q}(\{\zeta_p : p \in T\})$, where ζ_p is a primitive pth root of unity. We need only show that for a prime q, $\zeta_q \in K$ implies $q \in T$. Note that always $\zeta_2 = -1 \in K$ and $2 \in T$. For $q > 2$, if $\zeta_q \in K$, then $\zeta_q \in \mathbb{Q}(\zeta_{p_1}, \ldots \zeta_{p_n}) = \mathbb{Q}(\zeta_{p_1 \ldots p_n})$ for distinct $p_1, \ldots, p_n \in T$. Then

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$(q, p_1 \ldots p_n) \neq 1$ by [**10**, Corollary, page 204], and hence $q \in T$. The last statement in the theorem now follows from Theorem 3(b).

The above construction does not extend to the case in which $2 \in S$. In this case, K would necessarily have char 2 by Proposition 4(a). For example, $K = \mathbb{Z}/2\mathbb{Z}(\zeta_5)$ has 16 elements and hence also $\zeta_3 \in K$. Thus for our construction, if A is both 2- and 3-root closed, then A is also 5-root closed. It would be interesting to know if Theorem 6 is true for any subset S of positive primes.

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