BULL. AUSTRAL. MATH. SOC. 20D10
VOL. 7 (1972), 101-104.

On complemented chief factors of finite soluble groups

D.W. Barnes

Let $G = H_0 > H_1 > \ldots > H_r = 1$ and $G = K_0 > K_1 > \ldots > K_r = 1$ be two chief series of the finite soluble group G. Suppose M_i complements H_i/H_{i+1}. Then M_i also complements precisely one factor K_j/K_{j+1} of the second series, and this K_j/K_{j+1} is G-isomorphic to H_i/H_{i+1}. It is shown that complements M_i can be chosen for the complemented factors H_i/H_{i+1} of the first series in such a way that distinct M_i complement distinct factors of the second series, thus establishing a one-to-one correspondence between the complemented factors of the two series. It is also shown that there is a one-to-one correspondence between the factors of the two series (but not in general constructible in the above manner), such that corresponding factors are G-isomorphic and have the same number of complements.

Throughout this note, G is a finite soluble group. Let A be an irreducible G-module which occurs as a complemented chief factor of G, and let $C = C_G(A)$. Let $R = R(A)$ be the intersection of all normal subgroups D of G such that $D < C$ and C/D is isomorphic to A (as G-module). Clearly C/R is isomorphic to the direct sum of d copies of A for some d. In [3], Gaschütz proves that the number of complemented factors isomorphic to A in a chief series of G is d. This follows at

Received 6 March 1972. This note elaborates arguments of Gaschütz; see [2, Satz 3], [3, Satz 4.1], and also Carter, Fischer, and Hawkes [1, Lemma 2.6].

once from

LEMMA 1. *Let* H/K *be a chief factor of* G . *Then* H/K *is complemented and isomorphic to* A *if and only if* $C \geq HR > KR$.

Proof. Suppose $H/K \simeq A$ and let M be a complement to H/K in G . Then $H \cap \text{Core}_G(M) = K$ and $H.\text{Core}_G(M) = C_G(H/K) = C$. Thus $C \geq HR \geq KR$, $C/\text{Core}_G(M) \simeq H/K \simeq A$, and $\text{Core}_G(M) \geq R$. If $KR = HR$, then

$$\text{Core}_G(M) = KR.\text{Core}_G(M) = H.\text{Core}_G(M) = C ,$$

and we have $K = H \cap \text{Core}_G(M) = H$. Therefore $HR > KR$.

Suppose conversely that $C \geq HR > KR$. Then $H/K \simeq HR/KR$ which, being a chief factor of G between C and R , is isomorphic to A . Since C/R is a completely reducible module, there exists a maximal submodule D/R of C/R such that $D \cap HR = KR$. Since $C_{G/D}(C/D) = C/D$, G/D splits over C/D and all complements to C/D in G/D are conjugate. If M/D is a complement to C/D , then M complements H/K . Thus H/K is complemented.

We have seen that H/K has one conjugacy class of complements for each maximal submodule D/KR of C/KR not containing HR/KR . We now determine explicitly the number of these.

LEMMA 2. *Let* A *be an irreducible* G*-module and let* F *be the endomorphism ring of* A . *Let* V *be the direct sum of* d *copies of* A *and let* H *be a minimal submodule of* V . *Then the number of maximal submodules of* V *not containing* H *is* $|F|^{d-1}$.

Proof. By Schur's Lemma, F is a field (commutative since finite). $V = \{(a_1, \ldots, a_d) \mid a_i \in A\}$ and we have maps $\varepsilon_i : A \to V$ defined by $\varepsilon_i(a) = (0, 0, \ldots, a, 0, \ldots)$ where the non-zero entry is in the ith place. Let M be any maximal submodule of V . Then there is a homomorphism $\alpha : V \to A$ with $\ker\alpha = M$. For each i , we have $\lambda_i \in F$ defined by $\lambda_i = \alpha\varepsilon_i : A \to A$, and $(a_1, \ldots, a_d) \in M$ if and only if

https://doi.org/10.1017/S0004972700044853 Published online by Cambridge University Press

$\sum_{i=1}^{d} \lambda_i(a_i) = 0$. Conversely, for any $\lambda_1, \ldots, \lambda_d \in F$, not all zero, putting $\alpha(a_1, \ldots, a_d) = \sum_{i=1}^{d} \lambda_i(a_i)$ defines an epimorphism $\alpha : V \to A$ whose kernel is a maximal submodule of V . For $\lambda \in F$, $\lambda \neq 0$, $(\lambda\lambda_1, \ldots, \lambda\lambda_d)$ defines the same maximal submodule as $(\lambda_1, \ldots, \lambda_d)$. Thus the number of maximal submodules is $\frac{q^d-1}{q-1}$, where $q = |F|$. The number containing H is $\frac{q^{d-1}-1}{q-1}$ and the result follows.

Since the number of complements in a conjugacy class is $|A|$ provided $C \neq G$, by pairing complemented factors H_i/H_{i+1} and K_j/K_{j+1} isomorphic to A , for which H_iR/R and K_jR/R appear at the same level in the lattice of submodules of C/R , we have

THEOREM 1. *Let* $G = H_0 > H_1 > \ldots > H_r = 1$ *and* $G = K_0 > K_1 > \ldots > K_r = 1$ *be two chief series of the finite soluble group* G . *Then there exists a one-to-one correspondence between the factors of the two series, such that corresponding factors are* G*-isomorphic and have the same number of complements.*

We now prove

THEOREM 2. *Let* $G = H_0 > H_1 > \ldots > H_r = 1$ *and* $G = K_0 > K_1 > \ldots > K_r = 1$ *be two chief series of the finite soluble group* G . *Then complements* M_i *can be chosen for the complemented factors* H_i/H_{i+1} *of the first series, in such a way that distinct* M_i *complement distinct factors of the second series.*

By the discussion above, it is sufficient to prove

LEMMA 3. *Let* $V = U_0 > U_1 > \ldots > U_d = 0$ *and* $V = V_0 > V_1 > \ldots > V_d = 0$ *be composition series of the completely reducible module* V . *Then there exist maximal submodules* $W_1, W_2, \ldots, W_d$ *of* V *such that*

https://doi.org/10.1017/S0004972700044853 Published online by Cambridge University Press

$$U_i = W_1 \cap W_2 \cap \ldots \cap W_i \text{ and } V_i = W_{\alpha_1} \cap W_{\alpha_2} \cap \ldots \cap W_{\alpha_i}$$

for some permutation $\alpha_1, \ldots, \alpha_d$ *of* $1, 2, \ldots, d$.

Proof. The result clearly holds for $d = 1$. We use induction over d . For some k , $V_k \nleq U_1$ but $V_{k+1} \leq U_1$. By the complete reducibility, there exists a minimal submodule Z such that $V_k = Z + V_{k+1}$. By induction, there exist maximal submodules $T_2, \ldots, T_d$ of U_1 such that $U_i = T_2 \cap \ldots \cap T_i$ and $V_i \cap U_1 = T_{\beta_2} \cap \ldots \cap T_{\beta_j}$ for some $\beta_2, \ldots, \beta_j$. Put $W_1 = U_1$, $W_i = T_i + Z$ $(i = 2, \ldots, d)$. Since $V/Z \simeq U_1$ and $T_i = W_i \cap U_1$, these $W_1, \ldots, W_d$ satisfy the requirements.

References

[1] Roger Carter, Bernd Fischer, Trevor Hawkes, "Extreme classes of finite soluble groups", *J. Algebra* 9 (1968), 285-313.

[2] Wolfgang Gaschütz, "Die Eulersche Funktion endlicher auflösbarer Gruppen", *Illinois J. Math.* 3 (1959), 469-476.

[3] Wolfgang Gaschütz, "Praefrattinigruppen", *Arch. Math.* 13 (1962), 418-426.

Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales.

https://doi.org/10.1017/S0004972700044853 Published online by Cambridge University Press