BULL. AUSTRAL. MATH. SOC. MOS 46E35
VOL. 3 (1970), 369-373.

On the sharpness of a limiting case of the Sobolev imbedding theorem

J. A. Hempel, G. R. Morris and N. S. Trudinger

A refinement of the Sobolev imbedding theorem, due to Trudinger, is shown to be optimal in a natural sense.

Let Ω be a bounded domain in Euclidean n space, E^n. The Sobolev spaces $W_p^k(\Omega)$, where k is a non-negative integer and $p \geq 1$, consist of those functions in $L_p(\Omega)$ whose distributional derivatives of orders up to and including k are also in $L_p(\Omega)$ and are Banach spaces under the norm

$$\|u\|_{k,p} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L_p(\Omega)} . \tag{1}$$

The Sobolev imbedding theorem (for the case $k = 1$) asserts that if Ω satisfies a cone condition and $p < n$, the space $W_p^1(\Omega)$ may be continuously imbedded in $L_q(\Omega)$ where $q = np/(n-p)$. If $p > n$, the functions in $W_p^1(\Omega)$ are continuous (after possible redefinition on a set of measure zero). A refinement, proved by Trudinger in [2], shows that the space $W_n^1(\Omega)$ may be continuously imbedded in the Orlicz space $L_\phi(\Omega)$ with defining N function

$$\phi(t) = e^{|t|^{\frac{n}{n-1}}} - 1 . \tag{2}$$

Received 10 August 1970.

The purpose of this note is to show that this result is optimal in the sense that the space $L_\phi(\Omega)$ above cannot be replaced by any smaller Orlicz space. We let $\overset{o}{W}{}_p^k(\Omega) \subset W_p^k(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W_p^k(\Omega)$. The reader is referred to [2] for any other relevant definitions and notation.

THEOREM 1. *The space* $\overset{o}{W}{}_n^1(\Omega)$ *may not be continuously imbedded in any Orlicz space* $L_\psi(\Omega)$ *whose defining function* ψ *increases strictly more rapidly than the function* ϕ *given by* (2).

We remark here that ψ increases strictly more rapidly than ϕ if for every $\alpha > 0$, $\psi(\alpha\lambda)/\phi(\lambda) \to \infty$ as $\lambda \to \infty$. This happens if and only if $L_\psi(\Omega) \underset{\neq}{\subset} L_\phi(\Omega)$. Theorem 1 is a consequence of the following two lemmas:

LEMMA 1. *Let* ϕ *and* ψ *be* N *functions, with* ψ *increasing strictly more rapidly than* ϕ, *and suppose that there exists a continuous imbedding of a normed, linear space* X *into* $L_\psi(\Omega)$. *Then* $J(u) = \int_\Omega \phi(u)dx$ *is bounded on bounded subsets of* X.

Proof. We may identify X and its image $L_\psi(\Omega)$ under the imbedding map. Since this mapping is continuous, there is a constant $K \geq 0$ so that $\|u\|_\psi \leq K\|u\|_X$ for all u in X, where $\|u\|_\psi$ denotes the Luxemburg norm of u, that is

$$\|u\|_\psi = \inf\left\{k > 0 \; ; \; \int_\Omega \psi\left(\frac{u}{k}\right)dx \leq 1\right\} . \tag{3}$$

Since ψ increases strictly more rapidly than ϕ, there exists a non-decreasing function of $\lambda \geq 0$, $N(\lambda)$, satisfying

$$\phi(t) \leq \psi\left(\frac{t}{K\lambda}\right) \quad \text{for} \quad t \geq N(\lambda) . \tag{4}$$

Hence, for any u in X and $\lambda = \|u\|_X$,

https://doi.org/10.1017/S0004972700046074 Published online by Cambridge University Press

$$(5) \qquad \int_\Omega \phi(u)dx \le \int_\Omega \phi(N(\lambda))dx + \int_\Omega \psi\left(\frac{u}{\lambda K}\right)dx$$
$$\le \phi(N(\lambda))|\Omega| + 1 .$$

The lemma is thus proved. //

LEMMA 2. *Let* $A = A(A)$ *denote the set of Lipschitz continuous functions on the interval* $[0, 1]$ *, vanishing at* $x = 1$ *and satisfying* $\int_0^1 x^{n-1}|u'|^n dx \le A$. *Then* $J(u) = \int_0^1 x^{n-1}e^{|u|^{\frac{n}{n-1}}}dx$ *is unbounded on* A *when* $A > n^{n-1}$.

Proof. Let ρ satisfy $0 < \rho < 1$ and consider a sequence of piecewise linear functions, $u_k \in A$, $k = 1, 2, \ldots$, satisfying $u_k'(x) = a_{kj} > 0$ for $x \in (\rho^j, \rho^{j-1})$, $j = 1, 2, \ldots, k$ and $u_k'(x) = 0$ for $x \in (0, \rho^k)$. Then

$$(6) \qquad \int_0^1 x^{n-1}|u_k'|^n dx = \frac{\rho^{-n}-1}{n} \sum_{j=1}^k \rho^{jn}a_{kj}^n$$
$$= \frac{\upsilon^n-1}{n} \sum_{j=1}^k \alpha_{kj}^n$$

where $\upsilon = \rho^{-1}$ and $\alpha_{kj} = \rho^j a_{kj}$.

Also for $x \le \rho^k$, we have

$$(7) \qquad u_k(x) = u_k(\rho^k) = -(\rho^{-1}-1) \sum_{j=1}^k \rho^j a_{kj}$$
$$= -(\upsilon-1) \sum_{j=1}^k \alpha_{kj} .$$

We now choose α_{kj} so that $\sum \alpha_{kj}$ is maximised subject to the constraint $(\upsilon^n-1) \sum \alpha_{kj}^n = An$; that is, we choose

$$\alpha_{k1} = \alpha_{k2} = \ldots = \alpha_{kk} = \left(\frac{An}{\upsilon^n-1}\right)^{\frac{1}{n}} k^{-\frac{1}{n}} \text{, so that}$$

https://doi.org/10.1017/S0004972700046074 Published online by Cambridge University Press

(8) $$u_k(\rho^k) = -(\upsilon-1)\left(\frac{An}{\upsilon^n-1}\right)^{\frac{1}{n}} k^{\frac{n-1}{n}} .$$

Therefore, by (7) and (8),

$$J(u_k) \ge \int_0^{\rho^k} x^{n-1} e^{|u|^{\frac{n}{n-1}}} dx$$
$$= \frac{1}{n} e^{\beta k}$$

where $\beta = \beta(\upsilon) = (\upsilon-1)^{\frac{n}{n-1}}\left(\frac{An}{\upsilon^n-1}\right)^{1/(n-1)} - n\log\upsilon$.

Since $\beta(\upsilon)/(\upsilon-1)$ approaches $A^{\frac{1}{n-1}} - n > 0$ as $\upsilon \to 1$, it is possible to choose $\upsilon > 1$ to guarantee $\beta > 0$. It then follows that $J(u_k)$ is unbounded. //

We remark here that if in the statement of Lemma 2, we assume $A < n^{n-1}$, then $J(u)$ is bounded on A . For then we have

$$|u(x)| \le \int_x^1 |u'(t)|dt$$

$$\le \left(\int_x^1 \frac{dt}{t}\right)^{1-\frac{1}{n}} \left(\int_0^1 t^{n-1}|u'(t)|^n dt\right)^{\frac{1}{n}} \text{ by Hölder's inequality,}$$

$$\le A^{\frac{1}{n}}\left(\log \frac{1}{x}\right)^{1-\frac{1}{n}}$$

and consequently

$$J(u) \le \int_0^1 x^{n-1-A^{\frac{1}{n-1}}} dx = \left(n-A^{\frac{1}{n-1}}\right)^{-1} .$$

To complete the proof of Theorem 1, we may without loss of generality take Ω as the unit sphere in E^n and consider spherically symmetric

https://doi.org/10.1017/S0004972700046074 Published online by Cambridge University Press

functions, $u = u(r)$, only. Then

$$\int_\Omega |Du|^n dx = \omega_n \int_0^1 r^{n-1}|u_r|^n dr \ , \quad \int_\Omega e^{|u|^{\frac{n}{n-1}}} dx = \omega_n \int_0^1 r^{n-1} e^{|u|^{\frac{n}{n-1}}} dr \ ,$$

and Theorem 1 consequently follows from Lemmas 1 and 2. As a consequence of Lemma 2, we also have

THEOREM 2. *The space* $\overset{\circ}{W}{}_n^1(\Omega)$ *may not be compactly imbedded in the Orlicz space* $L_\phi(\Omega)$ *where* ϕ *is given by* (2).

Note that Theorem 1 is also a consequence of Theorem 2. In view of Theorem 2 and Theorem 3 of [2], it would be of interest to study the unique solvability in $\overset{\circ}{W}{}_2^1(\Omega)$ of differential equations such as

$$\Delta u + ue^{u^2} = 0$$

in two dimensions. Hempel [1] has made some progress in this direction.

References

[1] J.A. Hempel, "Superlinear variational boundary value problems and non-uniqueness", Doctoral thesis, University of New England, 1970.

[2] N.S. Trudinger, "On imbedding into Orlicz spaces and some applications", *J. Math. Mech.* 17 (1967), 473-484.

University of New England,
Armidale, New South Wales;

University of New England,
Armidale, New South Wales;

University of Queensland,
St Lucia, Queensland.

https://doi.org/10.1017/S0004972700046074 Published online by Cambridge University Press