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APPLICATION OF A SELECTION THEOREM TO HYPERSPACE CONTRACTIBILITY

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1. Introduction. For X a metric continuum, 2^X denotes the hyperspace of all nonempty subcompacta, with the topology induced by the Hausdorff metric H, and $C(X) \subset 2^X$ the hyperspace of subcontinua. These hyperspaces are continua, in fact are arcwise-connected, since there exist order arcs between each hyperspace element and the element X. They also have trivial shape, i.e., maps of the hyperspaces into ANRs are homotopic to constant maps. For a detailed discussion of these and other general hyperspace properties, we refer the reader to Nadler's monograph [**4**].

The question of hyperspace contractibility was first considered by Wojdyslawski [**8**], who showed that 2^X and $C(X)$ are contractible if X is locally connected. Kelley [**2**] gave a more general condition (now called property K) which is sufficient, but not necessary, for hyperspace contractibility. The continuum X has *property* K if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every pair of points x, y with $d(x, y) < \delta$ and every subcontinuum M containing x, there exists a subcontinuum N containing y with $H(M, N) < \epsilon$. Kelley also observed that 2^X and $C(X)$ are contractible if and only if the inclusion map of $F_1(X) = \{ \{x\}:x \in X\}$ in 2^X is homotopic to a constant map.

Let $\Lambda(X) \subset C(C(X))$ denote the space of maximal order arcs in $C(X)$. Thus each $\alpha \in \Lambda(X)$ is an order arc with a singleton as one endpoint and X as the other. Let $\omega:C(X) \to [0, 1]$ be a Whitney map. It is easily seen that the continuum $\Lambda(X)$ is homeomorphic to the function space of ω-parametrized segments

$$\{\alpha:[0, 1] \to C(X)|\alpha(s) \subset \alpha(t) \text{ if } s < t, \text{ and } \omega(\alpha(t)) = t \text{ for all } t\},$$

topologized by the sup metric. We will freely identify these spaces, writing $\alpha(t) = \alpha \cap \omega^{-1}(t)$ for $\alpha \in \Lambda(X)$ and $0 \leqq t \leqq 1$.

Let $e:\Lambda(X) \to X$ be the endpoint evaluation map defined by $e(\alpha) = \alpha(0)$. Clearly, $C(X)$ is contractible if and only if e has a right inverse map $f:X \to \Lambda(X)$. We will show that if there exists a lower semi-continuous set-valued function $\Phi:X \to \Lambda(X)$, with $\Phi(x) \subset e^{-1}(x)$ for each x, then there exists a right inverse f for e, and therefore $C(X)$ is contractible. Note that while the total inverse function e^{-1} is always upper

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semi-continuous, it is lower semi-continuous if and only if X has property K (see Chapter XVI of [**4**]). The inverse map f is obtained by application of a general selection theorem which is formulated with respect to a convex structure on $\Lambda(X)$ in which each $e^{-1}(x)$ is a convex set. The selection theorem is identical in its formal statement to a theorem of Michael [**3**], but our hypotheses on the convex structure are somewhat less restrictive.

It is tempting to look for a simplified characterization of hyperspace contractibility. Call an element $\alpha \in \Lambda(X)$ *admissible* if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every $x \in X$ with $d(x, e(\alpha)) < \delta$, there exists $\alpha' \in e^{-1}(x)$ with $H(\alpha, \alpha') < \epsilon$. The existence of an admissible order arc over each point of X is an obvious necessary condition for contractibility of $C(X)$. However, examples given in Section 6 show that this admissibility condition is not sufficient for hyperspace contractibility.

2. A selection theorem. Let (Y, d) be a metric space, and for each positive integer n let

$$P_n = \left\{ (t_i) \in [0, 1]^n : \sum_1^n t_i = 1 \right\}.$$

2.1. *Definition.* A *convex structure* on (Y, d) is a sequence of subsets $M_n \subset Y^n$ and maps $k_n : M_n \times P_n \rightarrow Y$ satisfying the following conditions:

1) $k_n(y, \dots, y; t_1, \dots, t_n) = y$;

2) $k_n(y_1, \dots, y_n; t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$
$$= k_{n-1}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n);$$

3) for every $\epsilon > 0$ there exists $\delta > 0$ such that for every n and $(t_i) \in P_n$,

$$d(k_n((y_i); (t_i)), k_n((y'_i); (t_i))) < \epsilon$$

if $d(y_i, y'_i) < \delta$ for each i. A subset $C \subset Y$ is *convex* with respect to this convex structure if for each n, $C^n \subset M_n$ and $k_n(C^n \times P_n) \subset C$.

The equi-uniform continuity condition 3) is crucial; this is the convex structure analogue of local convexity in a linear space. The above type of convex structure is slightly different from the one used by Michael [**3**]. For Y a compactum (which will be the case in our application to the space of maximal order arcs), Michael's convex structure satisfies the above conditions. However, Michael includes an extra condition which we do not require:

$$c) \quad k_n((y_i); (t_i)) = k_{n-1}((y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n); (t_1, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_n)) \quad \text{if } y_i = y_{i+1}.$$

This condition is not satisfied by the convex structure which we will define on $\Lambda(X)$ in Section 4.

2.2. THEOREM. *Let X be paracompact, (Y, d) a metric space with a convex structure, and $\Phi: X \to Y$ a lower semi-continuous set-valued function, with each $\Phi(x)$ a complete (with respect to d), convex subset of Y. Then Φ admits a continuous selection $f: X \to Y$.*

As in Michael's proof, f is obtained as the uniform limit of a sequence of Φ-selections which are not necessarily continuous. For $\delta > 0$, we say that a function $g: X \to Y$ is *δ-continuous* if there exists an open cover $\mathcal{U}$ of X such that diam $g(U) < \delta$ for every $U \in \mathcal{U}$. A sequence of Φ-selections $f_n: X \to Y$ is inductively constructed such that:
i) f_n is 2^{-n}-continuous;
ii) $d(f_n, f_{n+1}) < 2^{-n}$.
Then $f = \lim f_n$ is a continuous selection. This construction is accomplished by the following lemma (which was suggested by Professor Michael as a simplification of the author's original construction). Let $\delta = \delta(\epsilon)$ be as provided by condition 3) of Definition 2.1.

2.3. LEMMA. *Let $\epsilon > 0$ and a $\delta(\epsilon)$-continuous selection f for Φ be given. Then for every $\gamma > 0$ there exists a γ-continuous selection g for Φ such that $d(f, g) < \epsilon$.*

Proof. By the $\delta(\epsilon)$-continuity hypothesis, there exists for each $x \in X$ an open neighborhood $U(x)$ such that $f(U(x))$ lies in a neighborhood $V(f(x))$ with diameter less than $\delta(\epsilon)$. Since Φ is lower semi-continuous, we may assume also that

$$\Phi(x') \cap W(f(x)) \neq \emptyset \quad \text{for each } x' \in U(x),$$

where $W(f(x)) \subset V(f(x))$ is a neighborhood with diameter less than $\delta(\gamma/5)$. Let $\{U_\alpha\}$ be a locally finite open refinement of the cover $\{U(x): x \in X\}$. For each α choose x such that $U_\alpha \subset U(x)$, and set

$$V_\alpha = V(f(x)) \quad \text{and} \quad W_\alpha = W(f(x)).$$

Thus

$$f(U_\alpha) \cup W_\alpha \subset V_\alpha,$$
$$U_\alpha \subset \Phi^{-1}(W_\alpha) = \{x \in X: \Phi(x) \cap W_\alpha \neq \emptyset\},$$
$$\text{diam } V_\alpha < \delta(\epsilon), \quad \text{and}$$
$$\text{diam } W_\alpha < \delta(\gamma/5).$$

Let $\{p_\alpha\}$ be a partition of unity on X such that for each α, the support

$$X_\alpha = \text{cl}\{x: p_\alpha(x) > 0\} \subset U_\alpha.$$

Linearly order the index set A for $\{p_\alpha\}$, and for each $x \in X$ set

$$A(x) = \{\alpha \in A : x \in X_\alpha\} = \{\alpha_1, \dots, \alpha_n\},$$

with the ordering inherited from A.

For each x and each $\alpha \in A(x)$, pick $y_\alpha(x) \in \Phi(x) \cap W_\alpha$. The desired selection $g:X \to Y$ is then defined by the formula

$$g(x) = k_n(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x), \dots, p_{\alpha_n}(x)).$$

Clearly, g is a Φ-selection. Since for each $i = 1, \dots, n, f(x) \in f(U_{\alpha_i})$ and $y_{\alpha_i}(x) \in W_{\alpha_i}$, we have

$$\{f(x), y_{\alpha_i}(x)\} \subset V_{\alpha_i} \quad \text{and} \quad d(f(x), y_{\alpha_i}(x)) < \delta(\epsilon).$$

Thus

$$d(f(x), g(x)) = d(k_n(f(x), \dots, f(x);$$
$$p_{\alpha_1}(x), \dots, p_{\alpha_n}(x)), k_n(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x);$$
$$p_{\alpha_1}(x), \dots, p_{\alpha_n}(x))) < \epsilon.$$

We verify that g is γ-continuous. Given $x \in X$, choose a neighborhood $N(x)$ of x disjoint from each X_α not containing x. Thus, $A(x') \subset A(x)$ for each $x' \in N(x)$. By continuity of k_n, we may assume also that

$$d(k_n(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x), \dots, p_{\alpha_n}(x)),$$
$$k_n(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_n}(x'))) < \gamma/5$$

for each $x' \in N(x)$. We claim that

$$\text{diam } g(N(x)) < \gamma.$$

For $x' \in N(x)$ we have

$$A(x') = \{\alpha_{j_1}, \dots, \alpha_{j_m}\},$$

where $1 \leqq j_1 < \dots < j_m \leqq n$. We may assume for convenience of notation that $j_i = i$ for each i. Then

$$d(k_m(y_{\alpha_1}(x), \dots, y_{\alpha_m}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x')),$$
$$k_m(y_{\alpha_1}(x'), \dots, y_{\alpha_m}(x'); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x'))) < \gamma/5,$$

since

$$\{y_{\alpha_i}(x), y_{\alpha_i}(x')\} \subset W_{\alpha_i} \quad \text{and}$$
$$\text{diam } W_{\alpha_i} < \delta(\gamma/5) \quad \text{for each } i = 1, \dots, m.$$

And by the boundary condition 2),

$$k_m(y_{\alpha_1}(x), \dots, y_{\alpha_m}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x'))$$
$$= k_n(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_n}(x')).$$

Thus

$$d(g(x'), g(x)) < \gamma/5 + \gamma/5 = 2\gamma/5,$$

and

$$\text{diam } g(N(x)) < \gamma.$$

Lemma 2.3 provides both the initial and inductive steps for the construction of a sequence $\{f_n\}$ of Φ-selections satisfying the conditions i) and ii). Regarding condition i), we require in fact that each f_n be $\delta(2^{-n})$-continuous, which leads in turn to the condition that

$$d(f_n, f_{n+1}) < 2^{-n}.$$

And clearly, the lemma provides an initial selection f_1 which is $\delta(1/2)$-continuous.

3. Inductively open maps. A surjection $g: Y \to X$ is said to be *inductively open* if there exists a subspace $Y_0 \subset Y$ for which the restriction $g|Y_0: Y_0 \to X$ is an open surjection [**1**]. Equivalently, g is inductively open if there exists a lower semi-continuous set-valued function $\Phi: X \to Y$ such that $\Phi(x) \subset g^{-1}(x)$ for every x. If so, the union of all such functions is the largest function Φ, and each $\Phi(x)$ is a closed subset of $g^{-1}(x)$.

A surjection $g: Y \to X$ is *almost open* [**1**] if for every $x \in X$ there exists $y \in g^{-1}(x)$ such that g maps every neighborhood of y onto a neighborhood of x, i.e., y is approximable by points of the fibers $g^{-1}(x')$, for x' sufficiently close to x. Thus, every inductively open surjection is almost open.

For a continuum X, consider the surjection $e: \Lambda(X) \to X$. As previously noted, X has property K if and only if e is open. In this section and the next, we apply the selection theorem to show that $C(X)$ is contractible if and only if e is inductively open. Examples given in Section 6 show that this condition is strictly stronger than e being almost open.

3.1. PROPOSITION. *Let (Y, d) be a metric space with a convex structure, X a paracompact space, and $g: Y \to X$ a surjection such that each point-inverse $g^{-1}(x)$ is complete and convex. Then g has a right inverse $f: X \to Y$ if and only if g is inductively open.*

Proof. If g is inductively open, let $\Phi: X \to Y$ be the largest lower semi-continuous set-valued function such that $\Phi(x) \subset g^{-1}(x)$ for every x. Then $\Phi(x)$ is closed in $g^{-1}(x)$, hence complete, and by continuity of the convex structure maps k_n, $\Phi(x)$ must also be convex. Thus Φ admits a continuous selection $f: X \to Y$, and f is a right inverse for g. The converse is trivial.

4. A convex structure on $\Lambda(X)$. The following construction, together with (3.1), will complete the proof that $C(X)$ is contractible if and only if the evaluation map $e:\Lambda(X) \to X$ is inductively open.

4.1 PROPOSITION. *For every continuum X, there exists a convex structure on the space* $\Lambda(X)$ *of maximal order arcs in* $C(X)$ *such that each point-inverse* $e^{-1}(x)$ *is a convex subset.*

Proof. Consider $\alpha_1, \ldots, \alpha_n \in e^{-1}(x)$, for some $x \in X$, and $(t_1, \ldots, t_n) \in P_n$, and suppose first that each $t_i > 0$. Let

$$\tau_i = t_i/(t_i + \ldots + t_n), \quad 1 \leqq i \leqq n.$$

We define

$$k_n(\alpha_1, \ldots, \alpha_n; t_1, \ldots, t_n) = \alpha,$$

where

$$\alpha = \{\alpha_1(\tau_1) \cup \ldots \cup \alpha_{i-1}(\tau_{i-1}) \cup \alpha_i(t): \\ 1 \leqq i \leqq n, 0 \leqq t \leqq \tau_i\} \subset C(X).$$

Since $\alpha_1(0) = \{x\}$ and $\alpha_n(\tau_n) = \alpha_n(1) = X$, and since $\alpha \in C(C(X))$ has the linear ordering property that for each $M, N \in \alpha$, either $M \subset N$ or $N \subset M$, α is an order arc in $C(X)$ between $\{x\}$ and X. In the case that $t_i = 0$ for some i, $k_n(\alpha_1, \ldots, \alpha_n; t_1, \ldots, t_n)$ is defined by the boundary condition 2. It is easily seen that $\{k_n\}$ satisfies all the conditions for a convex structure.

5. Fiber functions. In this section we consider certain types of set-valued functions from X to $C(X)$, and their relationship to hyperspace contractibility.

5.1 *Definition.* A *fiber function* for a continuum X is a set-valued function $F:X \to C(X)$ such that

$$\{\{x\}, X\} \subset F(x) \subset \{M \in C(X):x \in M\} \quad \text{for each } x \in X.$$

Let T denote the *total* fiber function, defined by

$$T(x) = \{M \in C(X):x \in M\}.$$

An element M of $T(x)$ is *admissible at* x if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for each $y \in X$ with $d(x, y) < \delta$, there exists $N \in T(y)$ with $H(M, N) < \epsilon$. Let A denote the *admissible* fiber function, defined by

$$A(x) = \{M \in T(x):M \text{ is admissible at } x\}.$$

And let L denote the *largest lower semi-continuous* fiber function. Thus for each x, $L(x) \subset A(x) \subset T(x)$. The fibers $T(x)$, $A(x)$, and $L(x)$ are compact, and are closed under unions. The function T is always upper semi-continuous, but lower semi-continuous only if X has property K. In general, A is neither upper nor lower semi-continuous, and L is not upper semi-continuous.

5.2. *Definitions.* A fiber function F is *path-connected* if each fiber $F(x)$ is path-connected. We say F is *monotone-connected* if, for each element M of a fiber $F(x)$, there exists a path in $F(x) \cap C(M)$ between $\{x\}$ and M. A monotone-connected, lower semi-continuous fiber function is called a c-*function*.

Rhee [**6**] observed that if X admits a continuous c-function, then $C(X)$ is contractible (and conversely), and he asked whether the continuity hypothesis is essential. The following result shows that it is not.

5.3 PROPOSITION. *The evaluation map $e:\Lambda(X) \to X$ is inductively open if and only if X admits a c-function.*

Proof. Suppose that X admits a c-function, and let $K:X \to C(X)$ denote the largest c-function. We show first that K has the same closure properties as L, i.e., each fiber $K(x)$ is compact and is closed under unions. Define a fiber function $\widetilde{K}$ by

$$\widetilde{K}(x) = \{M_1 \cup M_2 : M_1, M_2 \in K(x)\}.$$

Clearly, $\widetilde{K}$ is lower semi-continuous. Consider an element $M_1 \cup M_2$ of $\widetilde{K}(x)$. Let

$$f_i : I \to K(x) \cap C(M_i)$$

be a path between $\{x\}$ and M_i, $i = 1, 2$. Then $g:I \to C(X)$, defined by

$$g(t) = f_1(t) \cup M_2,$$

is a path in $\widetilde{K}(x) \cap C(M_1 \cup M_2)$ between M_2 and $M_1 \cup M_2$. Adjoining g to f_2, we obtain a path in $\widetilde{K}(x) \cap C(M_1 \cup M_2)$ between $\{x\}$ and $M_1 \cup M_2$. Thus, $\widetilde{K}$ is also monotone-connected. Hence, $\widetilde{K} = K$ and each fiber $K(x)$ is closed under finite unions.

Similarly, define a fiber function $\bar{K}$ by $\bar{K}(x) = \overline{K(x)}$, Again, $\bar{K}$ is lower semi-continuous. Consider an element M of $\bar{K}(x)$. Then $M = \lim M_i$, for some sequence $\{M_i\}$ in $K(x)$. Let $f_i:I \to K(x) \cap C(M_i)$ be a path between $\{x\} = f_i(0)$ and $M_i = f_i(1)$, $i = 1, 2, \ldots$. Define paths g_i by

$$g_i(t) = \cup\{f_i(s) : 0 \leqq s \leqq t\}.$$

Since $K(x)$ is closed under finite unions, each $g_i(t)$ is an element of $\overline{K(x)}$. Thus, each g_i is a path in $\bar{K}(x) \cap C(M_i)$ between $\{x\}$ and M_i, with image $g_i(I)$ an order-arc. By compactness of $C(C(X))$, there is a convergent subsequence of $\{g_i(I)\}$ whose limit is an order-arc in $\bar{K}(x)$ between $\{x\}$ and M. Thus, $\bar{K}$ is also monotone-connected. Hence, $\bar{K} = K$ and each fiber $K(x)$ is compact. This shows also that there exist order arcs in $K(x)$ between $\{x\}$ and elements of $K(x)$.

We now define the required set-valued function $\Phi:X \to \Lambda(X)$ by setting

$$\Phi(x) = \{\alpha:\alpha \text{ is an order arc in } K(x) \text{ between } \{x\} \text{ and } X\}.$$

Given $\alpha \in \Phi(x)$ and $\epsilon > 0$, choose points $M_1 = \{x\}, M_2, \ldots, M_k = X$ of α, with $M_i \subset M_{i+1}$ and $H(M_i, M_{i+1}) < \epsilon/2$ for each i. By the lower semi-continuity of K, there exists a neighborhood V of x such that for each $y \in V$, the fiber $K(y)$ has elements $N_1 = \{y\}, \ldots, N_k = X$ with

$$H(M_i, N_i) < \epsilon/2 \quad \text{for each } i.$$

Replacing each N_i by $N_1 \cup \ldots \cup N_i$, we may assume that $N_i \subset N_{i+1}$ for each i. By the previous paragraph, there exists for each $I < k$ an order-arc β_i in $K(y)$ between $\{y\}$ and N_{i+1}. Then

$$\beta = \{N_i \cup N:N \in \beta_i, i = 1, \ldots, k-1\}$$

is an order-arc in $K(y)$ between $\{y\}$ and X, i.e., $\beta \in \Phi(y)$. It is easily verified that $H(\alpha, \beta) < \epsilon$. Thus Φ is lower semi-continuous, and e is inductively open.

The converse is immediate. If $\Phi:X \to \Lambda(X)$ is a lower semi-continuous set-valued function with each $\Phi(x) \subset e^{-1}(x)$, then the fiber function $F:X \to C(X)$ defined by

$$F(x) = \{M \in \alpha:\alpha \in \Phi(x)\}$$

is monotone-connected and lower semi-continuous.

5.4. THEOREM. *For X a continuum, there exist the following implications between properties of the fiber functions T, A, and L, the evaluation map e, and hyperspace contractibility*:

$T = L \iff e$ *open* $\iff X$ *has property K*

$\Downarrow$

L monotone-connected

$\Downarrow$

X admits a c-function $\iff$ *e inductively open* $\iff$ *$C(X)$ contractible*

$\Downarrow$

L path-connected

$\Downarrow$

A path-connected $\iff$ *e almost open.*

Furthermore, none of the downward implications is reversible.

Proof. Note that since the fibers $L(x)$ and $A(x)$ are closed under unions, the existence of a path between $\{x\}$ and X in either of these fibers will imply that the fiber is path-connected. Now suppose there exists a path-connected, lower semi-continuous fiber function F. Then for each x there is a path in $F(x) \subset L(x)$ between $\{x\}$ and X, thus $L(x)$ is path-connected. Similarly, if $L(x) \subset A(x)$ is path-connected, then so is $A(x)$.

Suppose A is path-connected. For $x \in X$ let $f:I \to A(x)$ be a path between $\{x\} = f(0)$ and $X = f(1)$. Define a path $g:I \to T(x)$ by

$$g(t) = \cup\{f(s):0 \leqq s \leqq t\}.$$

The image $g(I)$ is an order-arc between $\{x\}$ and X, and the closure properties of the fiber $A(x)$ imply that $g(I) \subset A(x)$. Then the proof of (5.3) shows that $g(I)$ is an admissible order-arc, thus e is almost open.

The remaining implications are either obvious or have already been established. The counterexamples in the next section will show that the downward implications are not reversible.

6. Counterexamples.

6.1. *Example* [**2**]. The continuum shown in Figure 1 does not have property K, but the fiber function L is monotone-connected. The admissible fiber function A is lower semi-continuous, thus $L = A$. The fiber $A(p)$ is a proper subset of the total fiber $T(p)$.

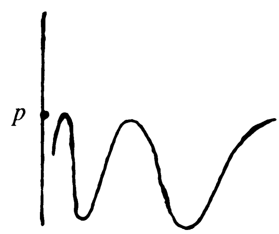

Figure 1

6.2. *Example.* The non-planar continuum in Figure 2 admits a c-function, but L is not monotone-connected. $L = A$, and the fiber $A(p)$ contains the bottom limit arc, but does not contain any proper nondegenerate subarc.

6.3. *Example.* The non-planar continuum in Figure 3 does not admit a c-function, but L is path-connected. Again, $L = A$. The fiber $A(p)$ contains all subarcs of the bottom limit arc containing p, and every path in $A(p)$ from $\{p\}$ to X must go through these subarcs, but none of the nondegenerate subarcs can be approximated by elements of monotone-connected admissible fibers over the points p_i.

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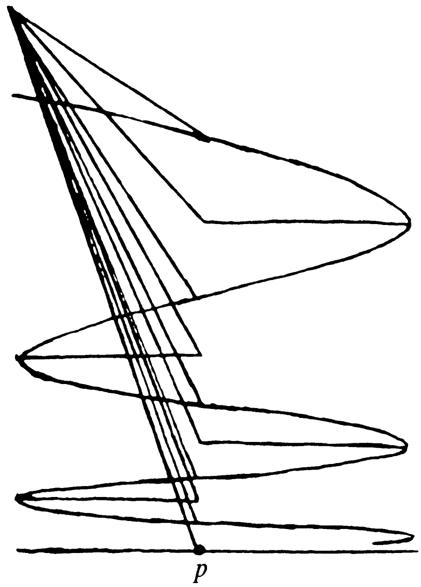

Figure 2

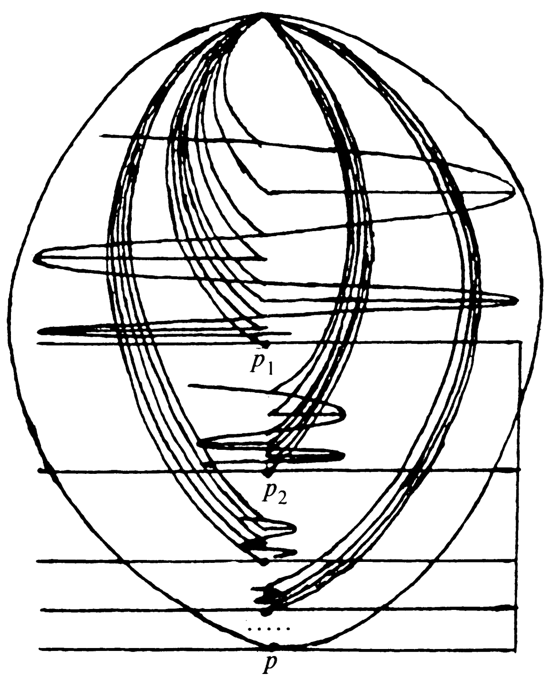

Figure 3

6.4. *Example.* For the continuum in Figure 4, A is path-connected, but L is not. The limit continua in this example are the triods with branch points at $p_1, p_2, \dots$, together with the bottom limit arc containing p. (The sequence of triods converges to this arc.) Paths from $\{p_i\}$ in the fibers $A(p_i)$ are obtained only by expanding southwest from p_i, whereas paths from $\{p\}$ in $A(p)$ are obtained only by expanding east from p. Thus, $\{p\}$ is a component of $L(p)$.

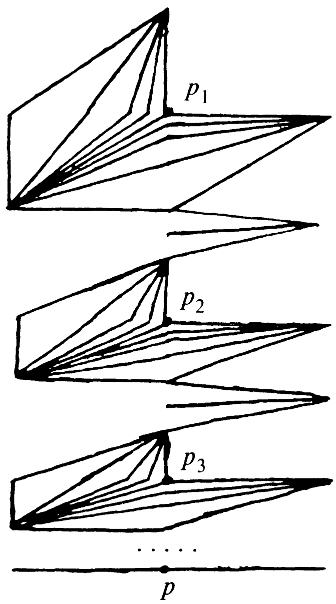

Figure 4

6.5. *Example.* [**2**]. For the continuum in Figure 5, A is not path-connected. The element $\{p\}$ is a component of $A(p)$.

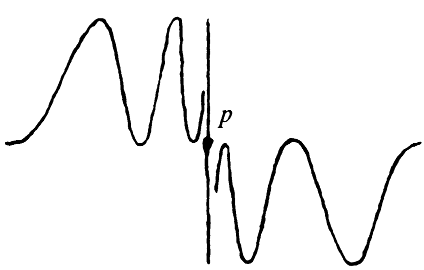

Figure 5

Rhee [**6**] called a continuum X *admissible* if, for some Whitney map ω on $C(X)$, every admissible fiber $A(x)$ intersects every Whitney level $\omega^{-1}(t)$. He observed that this condition, which is not satisfied by the Kelley

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example 6.5, is necessary for hyperspace contractibility, and asked whether it is sufficient. Since a continuum is admissible if its admissible fiber function is path-connected, the examples 6.3 and 6.4 show that admissibility is not a sufficient condition for hyperspace contractibility.

7. Maps preserving hyperspace contractibility. Wardle [7] has shown that confluent images of continua with property K also have property K. However, hyperspace contractibility is not preserved by confluent maps, or even by monotone maps. For example, let S_i, $i = 1, 2$, be disjoint copies of the $\sin(1/x)$-continuum, and let J be an arc meeting each S_i at an endpoint of its limit arc. Take $Y = S_1 \cup J \cup S_2$, and let $q:Y \to Y/J$ be the quotient map. Then $C(Y)$ is contractible, but $C(Y/J)$ is not.

Nadler [4] has shown that hyperspace contractibility is preserved by open surjections, and Nishiura and Rhee [5] have observed that it is preserved by maps with right homotopy inverses. Using the selection theorem, we may extend these results to a larger class of maps which includes all inductively open surjections.

7.1 PROPOSITION. *Let $g:Y \to X$ be a map between continua, and suppose there exists a lower semi-continuous set-valued function $\Phi:X \to Y$ such that $g \circ \Phi:X \to X$ is a single-valued function (therefore continuous) which is homotopic to* id_X. *Then if $C(Y)$ is contractible, so is $C(X)$.*

Proof. Let $\tilde{g}:\Lambda(Y) \to \Lambda(X)$ be the induced map. Since $C(Y)$ is contractible, there exists a right inverse $\alpha:Y \to \Lambda(Y)$ for the evaluation map

$$e_Y:\Lambda(Y) \to Y.$$

Then $\tilde{g} \circ \alpha \circ \Phi:X \to \Lambda(X)$ is a lower semi-continuous set-valued function such that for each $x \in X$,

$$\tilde{g} \circ \alpha \circ \Phi(x) \subset e_X^{-1}(g \circ \Phi(x)).$$

Let $B:X \to \Lambda(X)$ be the largest lower semi-continuous set-valued function such that

$$B(x) \subset e_X^{-1}(g \circ \Phi(x)) \quad \text{for each } x.$$

Each $B(x)$ must be compact and convex with respect to the convex structure on $\Lambda(X)$. By (2.2), B admits a continuous selection $\beta:X \to \Lambda(X)$. Let $h:X \times [0, 1] \to X$ be a homotopy with

$$h(x, 0) = x \text{ and } h(x, 1) = g \circ \Phi(x) \text{ for each } x.$$

Then considering the elements of $\Lambda(X)$ to be ω-parametrized segments, we may define a homotopy

$$H:X \times [0, 1] \to C(X),$$

with $H(x, 0) = \{x\}$ and $H(x, 1) = X$ for each x, as follows:

$$H(x, t) = \begin{cases} \{h(x, 2t)\}, & 0 \leqq t \leqq 1/2 \\ \beta(x)(2t - 1), & 1/2 \leqq t \leqq 1. \end{cases}$$

Thus $C(X)$ is contractible.

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