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COLLAPSING RIEMANNIAN METRICS TO CARNOT-CARATHEODORY METRICS AND LAPLACIANS TO SUB-LAPLACIANS

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ABSTRACT We study the asymptotic behavior of the Laplacian on functions when the underlying Riemannian metric is collapsed to a Carnot-Carathéodory metric We obtain a uniform short time asymptotics for the trace of the heat kernel in the case when the limit Carnot-Carathéodory metric is almost Heisenberg, the limit of which is the result of Beal-Greiner-Stanton, and Stanton-Tartakoff

0. **Introduction.** In this paper we will study the asymptotic behavior of a Laplacian when the underlying Riemannian metric is collapsed to a Carnot-Carathéodory metric.

Let M be a compact manifold with a Riemannian metric g, H a smooth distribution on M, $H^{\perp}$ the distribution orthogonal to H. Write

$$g = g_H \oplus g_{H^{\perp}},$$

where $g_H, g_{H^{\perp}}$ are the restriction of g to $H, H^{\perp}$ respectively. Define a one-parameter family of Riemannian metrics by setting for $\lambda > 0$,

$$g_{\lambda} = g_H \oplus \lambda^2 g_{H^{\perp}}.$$

Let d_{λ} be the distance of g_{λ}, $\triangle_{\lambda}$ the Laplacian associated with g_{λ}. We are interested in the behavior of $\triangle_{\lambda}$ as $\lambda \longrightarrow \infty$. Of course, in general $\triangle_{\lambda}$ can be very wild when $\lambda \longrightarrow \infty$. For example, if H is integrable, *i.e.* H induces a foliation, then the limit of $\triangle_{\lambda}$ is just the Laplacian along the leave of the foliation, which is not well-posed. Thus we will restrict ourselves to the case where H is not integrable; in fact, we require that H satisfies Hörmander's condition, *i.e.* H generates TM under the Lie bracket of vector fields.

We first study the underlying geometry. It turns out that if H satisfies Hörmander's condition, then the metric space (M, d_{λ}) converges to a metric space as $\lambda \longrightarrow \infty$. The limit distance, d_c can be described as follows. For $x, y \in M$, let

$$d_c(x,y) = \inf_{\gamma \in \Omega_H M(x,y)} \left(\int_0^1 g_H(\dot{\gamma}, \dot{\gamma})\, dt \right)^{1/2}.$$

where $\Omega_H(x, y)$ is the space of absolutely continuous paths which are tangent to H almost everywhere and join x to y. d_c is usually called a *Carnot-Carathéodory* metric on M. So, as $\lambda \longrightarrow \infty$, the geometry of H will become dominate, as d_c only depends on the restriction of g to H, g_H.

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We then consider the Laplacian $\triangle_\lambda$. Fukaya [4] observed that $\triangle_\lambda$ converges to a Hörmander's sum of square of vector fields,

$$\triangle_H = -\sum e_i^2,$$

where e_i is an orthonormal basis for H. Moreover, Fukaya proved that each eigenvalue of $\triangle_\lambda$ will converge to those of $\triangle_H$ as $\lambda \longrightarrow \infty$.

However, we can not expect that the convergence of the eigenvalues to be uniform, since the traces of the corresponding heat kernels have different short time asymptotics: in the Riemannian case the first term of asymptotics is like $t^{-n/2} \cdot$ const, where n is the dimension of the manifold, but in the limit case the first term is like $t^{-n_H/2} \cdot$ const, where n_H is the Hausdorff dimension of the metric space (M, d_c), and $n_H > n$. For example, if M is a 3-dimensional manifold and H a rank 2 distribution, the asymptotics of the trace of the heat kernel of $\triangle_H$ is $t^{-2} \cdot$ const, while that of $\triangle_\lambda$ is $t^{-3/2} \cdot$ const.

We will focus on a special case, namely when the limit Carnot-Carathéodory metric is almost Heisenberg in the sense of Getzler [10], see the definition in §3.2. In this case we obtain a uniform short time asymptotics for the trace of the heat kernel. Our main result is

THEOREM 1. *Suppose g_H is almost Heisenberg. Denote* $\mathrm{Tr}\big(\exp(-s\triangle_\lambda)\big)$ *the trace of the heat kernel. Then, for $a = \lambda^{-2} \neq 0$,*

$$\mathrm{Tr}\big(\exp(-s\triangle_\lambda)\big) = c_a(s)a^{-2n+1/2} + a^{-2n+1/2}C_{1,a}(s), \tag{0.1}$$

where

$$c_a(s) = \frac{1}{(2\pi s)^{n+1}} \int_M \int_{-\infty}^{\infty} \big(b(x,0)\big)^{-1/2} \Big(\frac{2\tau}{\operatorname{sh} 2\tau}\Big)^n \exp\Big(-\frac{a\tau^2}{2b(x,0)s}\Big)\, d\tau\, dv(x), \tag{0.2}$$

and

$$|C_{1,a}(s)| \leq \begin{cases} Cs^{-n+\beta}, & s \leq a; \\ Cs^{-n}, & s \geq a; \end{cases}$$

where $\beta = 1/2 - \ln a / 2 \ln s$, $b(x,0)$ is as in (3.6), C is independent of $a \in (0,1]$.

Note that for fixed $\lambda \neq 0$, then by the principle of stationary phase, the right hand side of (0.1) as $s \longrightarrow 0$ has a singularity of the form $s^{-n-1/2}$, thus there is no contradiction; while if $\lambda \longrightarrow \infty$, the right hand side of (0.1) is

$$\mathrm{vol}(M) \frac{1}{(2\pi s)^{n+1}} \bigg(\int_{-\infty}^{\infty} \Big(\frac{2\tau}{\operatorname{sh}(2\tau)}\Big)^n d\tau + O(s) \bigg).$$

The latter is just the result of Beals-Greiner-Stanton [1], Stanton-Tartakoff [20]. In general, if $\lambda^2 s << 1$, then the trace of the heat kernel behaves like that of $\triangle_1$ associated with g, while if $\lambda^2 s >> 1$, then it behaves like that of $\triangle_H$.

This paper is organized as follows. We first study the underlying geometry. In §1 we prove that (M, d_λ) converges to (M, d_c) as $\lambda \longrightarrow \infty$. We propose the so called "partial connection", in which the covariant derivative is only defined for vectors tangent to H,

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as a candidate for the limit of the Levi-Civita connections (*cf.* § 1.3.). In particular, the partial connection is uniquely determined by g_H and the splitting $TM = H \oplus H^{\perp}$.

In §2 we study the limit of $\triangle_\lambda$.

In §3 we study the short-time asymptotics of the heat kernel on an almost Heisenberg manifold. We will approximate the Laplacian by a left-invariant operator on the Heisenberg group at each point. This will yield an integral equation for the heat kernel. Iterating the integral equation, we obtain the exact fundamental solution. Much more difficult is to obtain uniform estimates for this solution. To do this, we have to introduce a dilation depending on λ, the limit of which as $\lambda \longrightarrow 1$ (resp. $\lambda \longrightarrow \infty$) is the usual dilation on R^n (resp. the Heisenberg dilation).

Finally we remark that in physics a process such as the limit of the (M, g_λ) as $\lambda \longrightarrow \infty$ is in general called an "adiabatic limit ", while in control theory it is called a "penalty limit ".

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1. **The limit of** (M, d_λ).

1.1 *Preliminaries.* In this subsection we first recall some preliminary facts about Carnot-Carathéodory metrics.

Let M be a connected manifold. A smooth distribution on M, H satisfies Hörmander's condition at a given point $x \in M$ if there are smooth vector fields $v_1, \ldots, v_m$ with values in H (m may depend on x), such that $v_1(x), \ldots, v_m(x)$ are linearly independent and span H_x, and T_xM is spanned by

$$v_1(x), \ldots, v_m(x), [v_1, v_2](x), \ldots, \Big[v_{t_1}, \Big[v_{t_2}, \ldots, [v_{t_{r-1}}, v_{t_r}] \cdots\Big]\Big](x). \tag{1.1}$$

We say that H is s-step bracket generating at x if s is the smallest number such that r in (1.1) can be chosen $r < s$.

A classical result of Chow says that if H satisfies Hörmander's condition, then any two points can be joined by an absolutely continuous path tangent to H almost everywhere. Thus, the Carnot-Carathéodory distance d_c is finite.

From now on we assume that H is a smooth distribution and satisfies Hörmander's condition.

1.2 *Limit of Riemannian metrics.*

THEOREM 1.1. *As $\lambda \longrightarrow \infty$, (M, d_λ) converges to (M, d_c) in the sense of Hausdorff.*

PROOF. If the lemma is not true, then there exist $\{x_\lambda\} \longrightarrow x_0$, $\{y_\lambda\} \longrightarrow y_0$ as $\lambda \longrightarrow \infty$ and a positive number ϵ_0 such that

$$|d_\lambda(x_\lambda, y_\lambda) - d_c(x_\lambda, y_\lambda)| \geq \epsilon_0.$$

Since $d_\lambda(x, y)$ is an increasing function of λ for fixed x, y, and $d_\lambda(x, y) \leq d_c(x, y)$, we have

$$d_\lambda(x_\lambda, y_\lambda) - d_c(x_\lambda, y_\lambda) \leq -\epsilon_0. \tag{1.2}$$

Suppose γ_λ is a minimizing geodesic for g_λ which joins x_λ to y_λ. Embed (M,g) into R^N isometrically for some N, $\theta\colon M \longrightarrow R^N$. Let $H^1([0,1],R^N)$ be the space of H^1 mappings with the inner product

$$\big((f,g)\big) = \int_0^1 \Big(\frac{df}{dt}, \frac{dg}{dt}\Big)\,dt,$$

where $(\cdot,\cdot)$ is the standard inner product on R^N. By the weak compactness of the unit ball in $H^1([0,1],R^N)$, there is $\beta \in H^1([0,1],R^N)$ such that

1. $\theta \circ \gamma_\lambda \longrightarrow \beta$ uniformly in C^0 as $\lambda \longrightarrow \infty$ (this implies that β can be written as $\beta = \theta \cdot \gamma_0$ for some path γ_0 on M);

2.

$$\lim_{\lambda\to\infty}\big((\theta\circ\gamma_\lambda, V)\big) = \big((\beta, V)\big), \quad V \in H^1([0,1],R^N). \tag{1.3}$$

In (1.3) take $V=\beta$, then by the Schwartz inequality,

$$\big((\beta,\beta)\big) \le \lim_{\lambda\to\infty} d_\lambda(x_\lambda, y_\lambda).$$

For $x \in M$, let $P_x\colon T_{\theta(x)}R^N \longrightarrow T\theta_x(H_x^\perp)$ be the orthogonal projection. Note that P_x depends continuously on x. By the splitting $TM = H \oplus H^\perp$, we can write

$$\dot\gamma = (\dot\gamma)_H + (\dot\gamma)_{H^\perp}, \tag{1.4}$$

where $(\dot\gamma)_H$ (resp. $(\dot\gamma)_{H^\perp}$) is the projection of $\dot\gamma$ to H (resp. $H^\perp$).

In (1.3) take V such that $\dot V(t) = P_{\gamma_0}\cdot\beta(t)$. So $\dot V(t) = P_{\gamma_0}\cdot T\theta\dot\gamma_0$. We will prove that $\dot V(t)=0$, which implies that γ_0 is horizontal. Now using the orthogonal decomposition (1.4) and the Schwartz inequality, we have

$$\begin{aligned}\big((V,V)\big) &\le \lim_{\lambda\to\infty}\int_0^1 \big(P_{\gamma_0}\cdot T\theta\dot\gamma_\lambda(t), P_{\gamma_0}\cdot T\theta\dot\gamma_\lambda(t)\big)\,dt\\ &= \lim_{\lambda\to\infty}\int_0^1 |P_{\gamma_0}\cdot T\theta(\dot\gamma_\lambda)_H|^2\,dt + \int_0^1 |P_{\gamma_0(t)}T\theta(\dot\gamma_\lambda)_{H^\perp}|^2\,dt.\end{aligned} \tag{1.5}$$

The last term in (1.5) is bounded, as P,θ are smooth, by

$$C_0\int_0^1 g\big((\dot\gamma_\lambda)_{H^\perp}, (\dot\gamma_\lambda)_{H^\perp}\big) \le C_0\lambda^{-2}d_\lambda(x_\lambda,y_\lambda) \longrightarrow 0, \text{ as } \lambda\longrightarrow\infty.$$

Since $P_{\gamma_\lambda}\cdot T\theta(\dot\gamma_\lambda)_H = 0$, the first term in (1.5) is equal to

$$\int_0^1 |(P_{\gamma_\lambda} - P_{\gamma_0})(T\theta\dot\gamma_\lambda)_H|^2.$$

Since $\gamma_\lambda \longrightarrow \gamma_0$ uniformly in C^0, $(P_{\gamma_\lambda(t)} - P_{\gamma_0(t)}) \longrightarrow 0$ as $\lambda\longrightarrow\infty$, and hence the above term converges to zero as $\lambda\longrightarrow\infty$. So we have $\dot V = 0$. Thus γ_0 is horizontal. So

$$d_c(x,y) \le \big(E(\gamma_0)\big)^{1/2} \le \lim_\lambda E(\gamma_\lambda)^{1/2} = \lim_{\lambda\to\infty} d_\lambda(x_\lambda,y_\lambda),$$

where E is the energy functional. This contradicts (1.2).

1.3 *Partial connection.* In this subsection we propose the so-called "partial connection" as the limit of the Levi-Civita connection of (M, g_λ) as $\lambda \longrightarrow \infty$.

Let $\pi_1 \colon TM \longrightarrow H$ be the orthogonal projection corresponding to the decomposition $TM = H \oplus H^\perp$.

DEFINITION 1.1. We say that a bilinear map

$$H_x \times C^\infty(H) \longrightarrow H_x, \quad (v_0, V) \longrightarrow D^H_{v_0} V,$$

depending smoothly on $x \in M$, is a partial connection if

$$D^H_{v_0}(fV) = fD^H_{v_0} V + (v_0 f)V, \quad f \in C^\infty(M);$$

$$D^H_{V_1} V_2 - D^H_{V_2} V_1 = \pi_1[V_1, V_2], \quad V_1, V_2 \in C^\infty(H);$$

$$V_0\langle V_1, V_2\rangle = \langle D^H_{V_0} V_1, V_2\rangle + \langle V_1, D^H_{V_0} V_2\rangle. \tag{1.6}$$

Now let D be the Levi-Civita connection of (M, g). The relation between D and the partial connection is given by

LEMMA 1.2. *The bilinear map $(v_0, V) \in H_x \times C^\infty(H) \longrightarrow \pi_1 D_{v_0} V \in C^\infty(H)$ is a partial connection.*

PROOF. By a direct computation, we verify that the bilinear map satisfies (1.6).

LEMMA 1.3. *Suppose that $X_1, \ldots, X_m$ is an orthonormal basis for H, then for $x \in M$ fixed there is another orthonormal basis $V_1, \ldots, V_m$ for H such that $D^H_{V_i} V_j(x) = 0$ and $V_i(x) = X_i(x)$.*

PROOF. The proof is the same as in Riemannian geometry.

COROLLARY 1.4. *Given g_H and a splitting $TM = H \oplus H^\perp$, then the partial connection is uniquely determined.*

PROOF. Let D^H be the partial connection constructed in Lemma 1.2. We fix a point $x \in M$, and let V_i be the orthonormal frame constructed in Lemma 1.3. Now $\pi_1[V_i, V_j](x) = (D^H_{V_i} V_j - D^H_{V_j} V_i)(x) = 0$. Suppose $\bar{D}^H$ is another partial connection. Write

$$\bar{D}^H_{V_i} V_j = \sum_{l=1}^{m} \Gamma^l_{ij} V_l, \tag{1.7}$$

then at x we have $\Gamma^k_{ij} = \Gamma^k_{ji}$. Insert (1.7) into (1.6), then we see that Γ^l_{ij} at x is uniquely determined by g_c, $H^\perp$.

REMARK 1. Thus the partial connection only depends on the Carnot-Carathéodory metric g_H, and the splitting $TM = H \oplus H^\perp$ (but not on $g_{H^\perp}$).

REMARK 2. There is a corresponding theory of characteristic classes for partial connections, relating the curvature of a partial connection to the global geometry of the distribution H, *cf.* Ge[8].

In the end of this section we make a remark on the volume form of g_λ. Let dv_λ be the volume form associated with g_λ. Then by a direct computation,

$$dv_\lambda = \lambda^{2k}\, dv.$$

2. **Limits of the eigenvalues.**

2.1 *Limit of laplacians.* Let $\triangle_\lambda$ be the Laplacian acting on functions associated with g_λ.

We first specify the limit of $\triangle_\lambda$.

LEMMA 2.1. *The limit of the Laplacian as $\lambda \longrightarrow \infty$ is a second order sub-elliptic operator*

$$\triangle_H = -\sum e_i^2, \tag{2.1}$$

where e_i is an orthonormal frame for H. Moreover, $\triangle_H$ is self-adjoint with respect to

$$(f,h)_0 = \int fh\,dv,$$

where dv is the volume form of g.

PROOF. Let e_i (resp. b_j) be an orthonormal basis for H (resp. $H^\perp$) with respect to g, then

$$\triangle_\lambda = -\sum e_i^2 + \lambda^{-2}\sum b_j^2.$$

So the limit is (2.1). The fact that $\triangle_H$ is self-adjoint follows from the fact that each $\triangle_\lambda$ is self-adjoint with respect to

$$\int(\cdot,\cdot)\,dv_\lambda = \lambda^{2m}\int(\cdot,\cdot)\,dv,$$

where m is the rank of H.

2.2 *Limits of the eigenvalues.* We will need the weighted Sobolev space H^1_w, which is the completion of $C^\infty(M)$ under the norm

$$(f,g)_{w1} = \int\sum\big(e_i(f),e_i(g)\big)\,dv + (f,g)_0.$$

LEMMA 2.2. *Let*

$$\mu_1(\lambda) \le \mu_2(\lambda) \le \cdots, \quad \mu_1 \le \mu_2 \le \mu_3 \le \cdots$$

be the eigenvalues of $\triangle_\lambda, \triangle_H$ respectively. Then, each $\mu_i(\lambda)$ is a decreasing function of λ, converges to μ_i as $\lambda \longrightarrow \infty$.

PROOF. This is essentially Fukaya's result [4]. We will give a slightly different proof.

We first prove that $\mu_i(\lambda)$ is decreasing. Let $\lambda_1 > \lambda_2$, and $f_1, f_2, \ldots$, be the eigenvectors of $\triangle_{\lambda_1}$,

$$\triangle_{\lambda_1} f_i = \mu_i(\lambda_1) f_i.$$

By the max-min principle,

$$\mu_{k+1}(\lambda_1) = \min_{f\perp f_i, i\le k}\frac{(\triangle_{\lambda_1} f,f)_0}{(f,f)_0}.$$

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Since $\triangle_{\lambda_1} \leq \triangle_{\lambda_2}$, so

$$\min_{f \perp f_\imath, \imath \leq k} \frac{(\triangle_{\lambda_1} f, f)_0}{(f, f)_0} \geq \mu_{k+1}(\lambda_2).$$

Hence

$$\mu_{k+1}(\lambda_2) = \max_V \min_{f \perp V} \frac{(\triangle_{\lambda_2} f, f)_0}{(f, f)_0} \geq \mu_{k+1}(\lambda_1)$$

where V runs over all k-dimensional subspace in H^1_w.

Next we prove that $\mu_\imath(\lambda)$ converges to μ_j for some j. Let $\lim_{\lambda \to \infty} \mu_\imath(\lambda) = a_\imath$. Normalize the eigenvector, $f_\imath(\lambda)$, such that its L^2-norm is one. Then its weighted Sobolev's H^1_w norm is bounded by a constant. So a subsequence of $f_\imath(\lambda)$ converges to a function x weakly in H^1_w (strongly in L^2). Now, for any smooth function g,

$$\begin{aligned} 0 &= \Big(\big(\triangle_\lambda - \mu_\imath(\lambda) \big) f_\imath(\lambda), g \Big)_0 \\ &= \Big(f_\imath(\lambda), \big(\triangle_\lambda - \mu_\imath(\lambda) \big) g \Big)_0 \longrightarrow \big(x, (\triangle_H - a_\imath) g \big)_0 = ((\triangle_H - a_\imath)x,)g)_0. \end{aligned}$$

So

$$\triangle_H x = a_\imath x$$

in the sense of distribution. By sub-elliptic estimates, $x_\imath$ is smooth, so $x_\imath$ is an eigenvector of $\triangle_H$.

It remains to prove that for any $\epsilon > 0$, $\mu_\imath(\lambda) < \mu_\imath + \epsilon$ for λ big enough. Again, this can be proved by the min-max principle as above.

3. **Uniform short time asymptotics of heat kernels.** Let g_H be almost Heisenberg. We will approximate $\triangle_\lambda$ at any point by a left-invariant operator on the Heisenberg group in a neighborhood of that point. So we will first study the sub-Laplacian on the Heisenberg group.

3.1 *The case of Heisenberg group.* Let $N_n = R^{2n} \times R$ be the $(2n+1)$-dimensional Heisenberg group; the multiplication is

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2z_1^\top J_n z_2),$$

where $(z, t) \in R^{2n} \times R$, and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let h_n be its Lie algebra, h_n^* the dual of h_n. The distribution H is the left-translation of the subspace $\{(\delta z, 0)\} \subset h_n$. Then the algebra of left-invariant pseudo-differential operators on N_n can be identified with the algebra of smooth functions on h_n^*, and there is a calculus of differential operators on N_n, *cf.* Beals *et al.* [1]. However, we will not use this calculus here. Instead, we will use the method of [20].

Sometimes it is convenient to consider N_n as a homogeneous bundle

(3. 1) $$R \longrightarrow N_n \longrightarrow C^n, \quad (z, t) \longrightarrow z,$$

with a left invariant connection $TM = H \oplus H^{\perp}$, obtained from the decomposition $h_n = \{(\delta z, 0)\} \oplus \{(0, \delta t)\}$ at 0. On N_n there is a family of left-invariant Riemannian metrics. At $(0,0,0) \in N_n$ this metric can be written as

$$g_\lambda = (\delta z_1)^2 + \cdots + (\delta z_{2n})^2 + \lambda^2(\delta t)^2.$$

The limit Carnot-Carathéodory metric g_{N_H} is,

$$g_{N_H} = \sum \big(\delta z_i - 2(J_n z_i)\delta t\big)^2. \tag{3.2}$$

This is just the horizontal lift of the metric on C^n via the connection $TM = H \oplus H^{\perp}$.

Let T_r be the Heisenberg dilation on $N_n \times [0, \infty)$ (*cf.* [1], [20])

$$T_r\big((z,t), a\big) = \big((rz, r^2 t), r^2 a\big), \tag{3.3}$$

and let R_+ denote the interval $[0, \infty)$.

DEFINITION 3.1. A smooth function $f\colon N_n \times R_+ \longrightarrow R$ is weighted homogeneous of degree k if $f \circ T_r = r^k f$. A smooth function $g\colon N_n \times R_+ \times R_+ \longrightarrow R$ is almost weighted homogeneous of degree k if there is another smooth function, $f_1\colon N_n \times R_+ \times R_+ \longrightarrow R$ such that

$$g\big((rz, r^2 t), r^2 s, r^2 a\big) = r^k f_1\big((z,t), s, a, r\big).$$

DEFINITION 3.2. A differential operator L_a on N_n (with parameter $a \in R_+$) is almost weighted homogeneous of degree l if for every almost homogeneous f of degree l_1, $L_a(f)$ is almost homogeneous of degree $l_1 - l$.

For example, $\partial/\partial z_i$ is of degree 1, and $\partial/\partial t$ is of degree 2.

Consider the λ-Laplacian on N_n

$$\bar{\Delta}_\lambda = \sum \Big(\frac{\partial}{\partial x_i} - 2y_i \frac{\partial}{\partial t}\Big)^2 + \Big(\frac{\partial}{\partial y_i} + 2x_i \frac{\partial}{\partial t}\Big)^2 + \lambda^{-2}\Big(\frac{\partial}{\partial t}\Big)^2, \tag{3.4}$$

and the sub-Laplacian

$$\bar{\Delta}_H = \sum \Big(\frac{\partial}{\partial x_i} - 2y_i \frac{\partial}{\partial t}\Big)^2 + \Big(\frac{\partial}{\partial y_i} + 2x_i \frac{\partial}{\partial t}\Big)^2.$$

Both operators are homogeneous of weight 2, if in (3.3) we take $a = \lambda^{-2}$.

The kernel p_H^s of the heat equation

$$\frac{\partial}{\partial s} - \bar{\Delta}_H$$

is

$$p_H^s\big(0,(z,t)\big) = \frac{1}{(2\pi s)^{n+1}} \int \Big(\frac{2\tau}{\operatorname{sh} 2\tau}\Big)^n \exp\Big(\frac{i\tau t}{s} - \Big(\sum_{i=1}^n |z_i|^2/2s\Big)\frac{2\tau}{\operatorname{th} 2\tau}\Big)\, d\tau.$$

Since $\partial/\partial t$ commutes with $\bar{\triangle}_H$, so the heat kernel of $\bar{\triangle}_\lambda$ is the convolution

$$p_\lambda\big(0,(z,t),s\big) = \lambda\int\Big(\frac{1}{2\pi s}\Big)^{n+3/2}\Big(\frac{2\tau}{\operatorname{sh}2\tau}\Big)^n \exp\Big(\frac{-(t-t_1)^2\lambda^2}{2s}\Big)$$
$$\exp\Big(\frac{i\tau t_1}{s}-\Big(\sum_{i=1}^n |z_i|^2/2s\Big)\frac{2\tau}{\operatorname{th}2\tau}\Big)\,d\tau\,dt_1.$$

By abusing notations, we rewrite the above formula as

$$\bar{p}\big(0,(z,t),s,a\big) = \frac{1}{a^{1/2}}\int\Big(\frac{1}{2\pi s}\Big)^{n+3/2}\Big(\frac{2\tau}{\operatorname{sh}2\tau}\Big)^n$$
$$\exp\Big(\frac{-(t-t_1)^2}{2as}\Big)\exp\Big(\frac{i\tau t_1}{s}-\Big(\sum_{i=1}^n |z_i|^2/2s\Big)\frac{2\tau}{\operatorname{th}2\tau}\Big)\,d\tau\,dt_1.$$

Using the fact that the Fourier transformation of $\exp(-t^2/2)$ is $\pi^{1/2}\exp(-t^2/2)$, we have

$$(3.5)\quad \bar{p}\big(0,(z,t),s,a\big) = \Big(\frac{1}{2\pi s}\Big)^{n+1}\int\Big(\frac{2\tau}{\operatorname{sh}2\tau}\Big)^n\exp\Big(-\frac{\tau^2 a}{s}-\frac{i\tau t}{s}-\frac{\sum|z|^2}{2s}\frac{2\tau}{\operatorname{th}2\tau}\Big)\,d\tau.$$

It is easy to see that the fundamental solution is weighted homogeneous.

3.2 *Almost Heisenberg manifolds.* We say that g_H (the restriction of g to H) is *almost Heisenberg* if H is a contact distribution, and at any point we can choose local coordinates $\{x_i, y_i, t\}$ such that in these coordinates,

$$g_H = \sum(dx_i - 2y_i\,dt)^2 + (dy_i + 2x_i\,dt)^2 + O(1),$$

where $O(1)$ denotes a term of higher order.

An equivalent definition is as follows.

Recall that if H satisfies Hörmander's condition, then we can define a simply connected nilpotent group at each point. Given a point $x\in M$, the Lie algebra of this nilpotent group is given by

$$H_x \oplus (H_1/H)_x \oplus (H_2/H_1)_x \oplus\cdots$$

with the induced Lie bracket, where

$$H_1 = H + [H,H], H_2 = H_1 + [H_1,H],\ldots.$$

In particular, if H is a contact distribution, then the nilpotent group at each point is just the Heisenberg group.

The nilpotent group at x is called the tangent cone to M at x. Note that on the tangent cone there is a left-invariant Carnot-Carathéodory metric, induced from g_H on H_{x_0}. Then, g_H is almost Heisenberg iff the induced Carnot-Caratheodory metric on every tangent cone is isometric to the canonical metric (3.2).

From now on we assume that g_H is almost Heisenberg.

Let U be a neighborhood of x_0 in M. We say that a smooth map $\Theta\colon U\times U\longrightarrow N_n$ is an *admissible coordinate system* if one denotes $\Theta_x = \Theta(x,\cdot)$ for $x\in U$, then

1. Θ_x is a diffeomorphism, and maps x to $0\in N_n$.

2. $T_x\Theta_x$ maps the induced metric on the tangent cone at x_0 to g_{N_H} on N_n (*cf.* (3.2)) isometrically.

3. Θ_x maps the leave of the foliation of $H^\perp$ onto the fibers of the homogeneous fiber bundle $N_n\longrightarrow C^n$.

LEMMA 3.1. *Admissible coordinates system exist.*

PROOF. A similar construction is given in [20]. We will give an intrinsic construction.

We identify the Lie algebra of the tangent cone at $x \in M$ with

$$H_x \oplus H_x^{\perp}.$$

Let $\exp: H_x \oplus H_x^{\perp} \longrightarrow N$ be the exponential map of the Heisenberg group, then exp is a diffeomorphism and maps every affine space $\{\eta\} + H_x^{\perp}$, $\eta \in H_x$, to a fiber of the homogeneous fiber bundle $N \longrightarrow C^n$.

Next we will use the exponential map for the Carnot-Carathéodory metric (*cf.* [6])

$$\exp_x: T_xM \longrightarrow M,$$

where we have identified T_xM with T_x^*M using g. $\exp_x$ is a local diffeomorphism. At $0 \in T_xM$, $T\exp_x$ maps the tangent space to the subspace $H_x^{\perp} \subset T_xM$ to $H_x^{\perp}$. Moreover, at x, the map $T\exp \circ (T\exp_x)^{-1}: T_xM \longrightarrow T_0N$ is an isometry between the Carnot-Carathéodory metrics. Now modify $\exp_x$ to a map Φ_x such that $T\Phi_x$ agrees with $T\exp_x$ at 0, Φ_x maps every affine subspace $\{\eta\} + H_x^{\perp}$ to a leaf of the foliation induced by $H^{\perp}$, and depends smoothly on x. Now define $\Theta_x = \exp \circ (\Phi_x)^{-1}$, then it satisfies the requirement.

REMARK. Under admissible coordinates, the metric g_λ can be written as

$$(\Theta_x^* g_\lambda)(y) = \bar{g}_H(y) \oplus \lambda^2 b(x,y)\,(dt)^2, \quad y \in N, \tag{3.6}$$

where $\Theta_x^* g_\lambda$ denotes the induced metric on N_n,

$$\bar{g}_H = g_{N_H} + O(1)(y).$$

Here g_{N_H} is as in (3.2), and $O(1)(y)$ denotes a quadratic form whose entries are almost homogeneous functions of degree 1.

3.3 *Fundamental solutions.* As a corollary of (3.6), we have

LEMMA 3.2. *If Θ is an admissible coordinate system, then for fixed x,*

$$\triangle_\lambda (f \circ \Theta_x^{-1}) = \bar{\triangle}_{\lambda b(x,0)} f + Lf, \quad f \in C^\infty\big(N_n \times [0,\infty)\big), \quad f \in C^\infty(M),$$

where $b(x,0)$ is the positive function in (3.6), $\bar{\triangle}_{\lambda b(x,0)}$ is the left-invariant Laplacian on N_n, L is an almost homogeneous differential operator of degree 1.

Let $u(x,y,s,a)$ be the fundamental solution of

$$\begin{aligned} &\frac{\partial u}{\partial s} + \triangle_\lambda^x u = 0, \\ &u(x,y,0,a) = \delta(x-y); \end{aligned} \tag{3.7}$$

where $\triangle_\lambda^x$ means that the partial differentiation is only for the x variable.

Suppose M is covered by a finite number of admissible coordinate systems U_i, $M = \bigcup U_i$, Θ_i: $U_i \times U_i \longrightarrow N_n$, ϕ_i a partition of unity of the open cover. Let $c(x) = b^{-1/2}(x, 0)$. Define

$$p_a(x, y, s) = \sum \phi_i \bar{p}_{c(x)a}\big(0, \Theta_i(x, y), s\big), \tag{3.8}$$

where $\bar{p}_{c(x)a}$ is the fundamental solution to the heat equation $\partial/\partial s - \bar{\triangle}_{b(x,0)\lambda}$ on N_n, given by (3.5).

Then, multiplying (3.7) by (3.8), using the integration by parts, we have

(3.9)

$$u_a(x, y, s) = p_a(x, y, s) + \int_0^s \int_M \Big(u_a(x, y_1, s - s_1), \Big(\frac{\partial}{\partial s} + \triangle_\lambda^{y_1}\Big) p_a(y_1, y, s_1)\Big)\, dv(y_1)\, ds_1.$$

Now, for *fixed a*, by the arguments of [15], if $a \neq 0$ (or that of [20] if $a = 0$), the fundamental solution u satisfies (3.9).

Given two functions $f(x, y, s, a)$, $g(x, y, s, a)$, denote

$$(f \dagger g)(x, y, s) = \int_0^s \int_M f(x, y_1, s - s_1) g(y_1, y, s_1)\, dv(y_1)\, ds_1,$$

where $dv(y_1)$ means that the integration is only for the y_1 variables. Define inductively

$$q(x, y, s, a) = \Big(\frac{\partial}{\partial s} + \triangle_\lambda^x\Big) p_a(x, y, s),$$
$$q^k = q \dagger q^{k-1}, \quad k = 2, 3, \ldots .$$

Then, from (3.9), the fundamental solution $u(x, y, s, a)$ can be written as (formally)

$$u_a = p_a + \sum (-1)^k p_a \dagger q^k. \tag{3.10}$$

Again for *fixed a*, by the same arguments in [15] for $a \neq 0$ (or [20] for $a = 0$), one can show that (3.10) is convergent and thus is the fundamental solution. What is more difficult is to obtain a uniform estimate for the series (3.10), which we will do next.

3.4 *Uniform estimates.* In this subsection we will obtain uniform estimates for the series (3.10).

First we will introduce a new dilation. First recall that if $a \neq 0$, the Laplacian can be approximated by an operator with constant coefficients, *i.e.* an invariant operator on the abelian group R^n, and the appropriate dilation is $a \cdot (x) \longrightarrow (ax)$ (*cf.* McKean-Singer [15]); whereas for $a = 0$, the sub-Laplacian can be approximated by a left-invariant operator on the Heisenberg group, and here the appropriate dilation is the Heisenberg dilation on N_n, *cf.* Stanton-Tartakoff [20]. However, to find a uniform estimate both dilations are no longer sufficient, so we will introduce a new dilation depending on a, the limits of which as $a/s \longrightarrow \infty$ and $a/s \longrightarrow 0$ will be the abelian dilation and the Heisenberg dilation respectively. W.o.l.g. we assume that $a < 1, s < 1$.

The new dilation $T_{r,s,t}$: $N_n \longrightarrow N_n$ is defined as

$$T_{r,s,a}(z, t) = \begin{cases} (rz, r^2 t), & a \leq s; \\ (rz, r^{2\theta} t), & a \geq s; \end{cases} \tag{3.11}$$

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where

$$\theta = \frac{1}{2} + \frac{\ln a}{2\ln s}, \quad a \geq s.$$

Note that $1/2 \leq \theta \leq 1$ if $a \geq s$. If $a = 1$, then the dilation is that in Stanton-Tartakoff [20]; while if $a = 0$, then it is that in McKean-Singer [15].

We say that a function $f(y, s, a)$ on $N_n \times \Omega$, considered as a smooth function of y with parameters $a, s \in \Omega$ (may not depend continuously on a, s), is *uniformly fast decreasing with respect to* $(s, a) \in \Omega$ if for every $(\alpha, k) \in Z_+^{2n+1} \times Z_+$, there is a constant C independent of $(s, a) \in \Omega$ such that

$$|\partial_y^\alpha f(y, a)| < C|y|^{-k}.$$

LEMMA 3.3. *The function*

$$g(z, t, s, a) = \begin{cases} \int (2\tau/\operatorname{sh} 2\tau)^n \exp(-a\tau^2/2s) \exp(-it\tau - (\sum_{i=1}^{2n} |z_i|^2)\tau/\operatorname{th} 2\tau)\, d\tau, & a \leq s, \\ \int \exp(-s^{2\beta-1}\tau^2/2) \exp(-i\tau t) \exp(-\sum z^2 s^\beta \tau/\operatorname{th} 2s^\beta\tau)(2s^\beta\tau/\operatorname{sh} 2s^\beta\tau)^n\, d\tau, & a \geq s; \end{cases}$$

where $\beta = 1 - \theta$, *is uniformly fast decreasing with respect to* $(s, a) \in [0, \infty)^2$.

PROOF. We rewrite

$$g(z, t, s, a) = \int \exp(-i\tau t) g_1(\tau, z, s, a)\, d\tau \tag{3.12}$$

where

$$g_1(\tau, z, s, a) = \begin{cases} (\tau/\operatorname{sh} 2\tau)^n \exp(-\tau^2 a/2s) \exp(-(\sum_{i=1}^{2n} |z_i|^2)\tau/\operatorname{th} 2\tau), & a \leq s; \\ \exp(-s^{2\beta-1}\tau^2/2) \exp(-\sum z^2 s^\beta \tau/\operatorname{th} 2s^\beta\tau)(2s^\beta\tau/\operatorname{sh} s^\beta 2\tau)^n, & a \geq s \end{cases}$$

Since $2\beta - 1 \leq 0$, $\exp(-s^{2\beta-1}\tau^2/2)$ is a uniformly fast decreasing function of τ as long as $a \geq s \in [0, 1]$. On the other hand, if $a \leq s$, the function $\big(\tau/\operatorname{sh}(2\tau)\big)^n$ is fast decreasing uniformly with respect to a, s. So g_1 is uniformly fast decreasing with respect to $a, s \in [0, 1]$, *i.e.* for any $l, m > 0$,

$$|\partial_\tau^l g_1(\tau, s, a)| \leq C(1 + |\tau|)^{-m}\Big(1 + \sum |z_i|^2\Big)^{-m}.$$

Using integration by parts in (3.12) repeatedly, we prove the lemma.

Note that the fundmental solution on N_n, $\bar{p}$ can be rewritten as $(c(x) = b^{-1/2}(x, 0))$

$$\bar{p}\big((z, t), s, ac(x)\big) = \begin{cases} (2\pi)^{-n-1} s^{-n-1} (g \circ T_{s^{-1/2},1,1})(z, t), & a \leq s; \\ (2\pi)^{-n-1} s^{-n-1+\beta} (g \circ T_{s^{-1/2},s,ac(x)})(z, t), & a \geq s; \end{cases} \tag{3.13}$$

which inspires the following definition (compare Stanton-Tartakoff [20]).

DEFINITION 3.3. We say a function $f(x, y, s, a)\colon M \times M \times R_+ \times R_+ \longrightarrow R$ is of type (l, m) if there are uniformly fast decreasing functions $g_{1,i}, g_{2,i}, \ldots,$ on N_n and functions b_i with support in U_i respectively such that

(3.14)

$$f(x, y, s, a) = \begin{cases} \sum_i \sum_{j \geq 0} s^{-n-2+(l+j)/2} (g_{j,i} \circ T_{s^{-1/2},1,1} \circ \Theta_i)(x, y) b_j(x, y), & a \leq s; \\ \sum_i \sum_{j \geq 0} s^{-n-1+\beta+(m+j)\theta} (g_{j,i} \circ T_{s^{-1/2},s,ac(x)} \circ \Theta_i)(x, y) b_j(x, y), & a \geq s \end{cases}$$

where $T_{s^{-1/2},s,a}$ is the dilation (3.11) on N_n.

LEMMA 3.4. *$p(x, y, s, a)$ is of type (2, 0).*

PROOF. This follows from Lemma 3.3 and (3.13).

REMARK 1. Let M be the Heisenberg group N_n, Θ_{x_1} the map $x \longrightarrow x - x_1$. If $k(x, y, s, a)$ is of type (l, m), then $\partial k(x, y, s, a)/\partial z_\imath$ (where $x = (z_1, \ldots, z_{2n}, t)$) is of type $(l - 1, m - 1/2\theta)$, $\partial k(x, y, s, a)/\partial t$ is of type $(l - 2, m - 1)$, $z_\imath k(x, y, s, a)$ is of type $(l + 1, m + 1/2\theta)$, $tk(x, y, s, a)$ is of type $(l + 2, m + 1)$. Note that if $a \geq s$, a can be written as

$$a = s^{2\theta - 1} = s^{\theta(2 - 1/\theta)},$$

so $ak(x, y, s, a)$ is of type $(l + 2, m + 2 - 1/\theta)$.

LEMMA 3.5. *If $k(x, y, s, a)$ is of type (l, m), then*

$$|k| \leq \begin{cases} Cs^{-n-2+l/2}, & a \leq s; \\ Cs^{-n-1+\beta+m\theta}, & a \geq s, \end{cases}$$

$$\int |k| \, dv(x) \leq \begin{cases} Cs^{-1+l/2}, & a \leq s \\ Cs^{m\theta}, & a \geq s, \end{cases}$$

where C is independent of s, a.

PROOF. By a direct computation.

LEMMA 3.6. *$(\partial/\partial s + \triangle_\lambda^x)p(x, y, s, a)$ is of type*

$$\Big(1, \min(\frac{1}{2\theta} - 1, 1 - \frac{1}{\theta}, \frac{3}{2\theta} - 2)\Big).$$

PROOF. By Lemma 3.2,

$$\Big(\frac{\partial}{\partial s} - \triangle_\lambda\Big)p = L \cdot k \circ \Theta_x,$$

where k is of type (2, 0), and L is a sum of operators of the form

$$\frac{\partial}{\partial z_\imath}, \; z_\imath \frac{\partial}{\partial t}, \; y_\imath \frac{\partial^2}{\partial z_\imath^2}, \; z_\imath t \frac{\partial}{\partial z_\jmath}, \; z_\imath z_\jmath z_k \frac{\partial}{\partial t^2}, \; t \frac{\partial}{\partial z_\imath^2}, \; at \frac{\partial^2}{\partial t^2}, \; az_\jmath \frac{\partial^2}{\partial t^2}$$

over the coefficients of smooth functions. The action of L on k, $L(k)$ is of type $(1, m)$, where m is the smallest one among the following numbers

$$-\frac{1}{2\theta}, \frac{1}{2\theta} - 1, \; -\frac{1}{2\theta}, \frac{1}{2\theta} - 1, \; \frac{3}{2\theta} - 2, \; 1 - \frac{1}{\theta}, \; -\frac{1}{2\theta}.$$

It turns out that the smallest one is among $1/(2\theta) - 1, 1 - 1/\theta, 3/(2\theta) - 2$.

As a corollary

COROLLARY 3.7. *We have the following estimates*

$$(3.15)\quad |q(x,y,s,a)| = \left|\left(\frac{\partial}{\partial s} + \Delta_\lambda^x\right)p(x,y,s,a)\right| \leq \begin{cases} Cs^{-n-3/2}, & a \leq s, a \neq s; \\ Cs^{-n-3/2+\beta}, & a \geq s \end{cases}$$

$$(3.16)\quad \int_M |q(x,y,s,a)|dv(x) \leq \frac{C}{s^{1/2}},$$

where C is independent of s, a.

PROOF. We will check

$$m\theta > -1/2, \text{ for } m = \frac{1}{2\theta} - 1,\ 1 - \frac{1}{\theta},\ \frac{3}{2\theta} - 2.$$

Now this follows from the inequality $1/2 \leq \theta \leq 1$. Hence q is of type $(1, -1/2\theta)$, so (3.15) and (3.16) follow from Lemma 3.4 and Lemma 3.5.

We will denote $q^k(x,y,s,a)$ by $q_a^k(x,y,s)$. From the above estimates we have

LEMMA 3.8. *(1)*

$$(3.17)\quad \|q_a^k(x,y,s)\|_{L^1(M_x)} \leq \frac{1}{\Gamma(\frac{k}{2})} C^k s^{k/2-1};$$

(2)

$$(3.18)\quad \|q_a^k(x,y,s)\|_{L^\infty(M_x \times M_y)} \leq \begin{cases} A_k s^{-n-k/2-5/2+\beta}, & a \leq s, a \neq s; \\ A_k s^{-n-k/2-5/2}, & a \geq s; \end{cases}$$

where for $k \geq 2n+3$,

$$A_k = \frac{A^k}{\Gamma(k/2 - n - 1)}.$$

Here A is independent of a.

(3)

$$(3.19)\quad \left\| u_a - p_a - \sum (-1)^j_{j \leq k-1} p_a \dagger q_a^j \right\|_{L^\infty(M \times M)} \leq \begin{cases} Cs^{\frac{k}{2}-n-1/2+\beta}, & a \leq s, a \neq s; \\ Cs^{\frac{k}{2}-n-1/2}, & a \geq s; \end{cases}$$

where C is independent of s, a.

PROOF. (3.17) can be proved by the same method as in [20], so we will only prove (3.18) and (3.19).

First we prove (3.18) by induction on k. The case $k = 1$ is given in Corollary 3.7. Suppose (3.18) is true for $k-1$. Now, if $a > s$,

$$(3.20)\quad \begin{aligned} q_a^k(x,y,s) = &\int_0^{s/2} \int_M q(x,y_1,s-s_1) q^{k-1}(y_1,y,s_1)\, ds_1 \\ &+ \int_{s/2}^{s} \int_M q(x,y_1,s-s_1) q^{k-1}(y_1,y,s_1)\, ds_1. \end{aligned}$$

Now

$$\left|\int_0^{s/2}\int_M q(x,y_1,s-s_1,a)q^{k-1}(y_1,y,s_1)\,ds_1\right|$$
$$\leq \int_0^{s/2}\int_M \|q(x,y_1,s-s_1,a)\|_{L^\infty(M\times M)}\|q^{k-1}(y_1,y,s_1,a)\|_{L^1(M_{y_1})}\,ds_1$$
$$\leq A_{k-1}C\int_0^{s/2}(s-s_1)^{-n-3/2+\beta}s_1^{k/2-3/2}\,ds_1$$
$$\leq A_{k-1}Cs^{-n-2+k/2+\beta}\int_0^{1/2}(1-s_1)^{-n-3/2+\beta}s_1^{k/2-3/2}\,ds_1$$
$$\leq A_kCs^{-n-2+k/2+\beta}/2,$$

and

$$\left|\int_{s/2}^{s}\int_M q(x,y_1,s-s_1)q^{k-1}(y_1,y,s_1)\,ds_1\right|$$
$$\leq \int_{s/2}^{s}\int_M \|q(x,y_1,s-s_1)\|_{L^1(M_{y_1})}\|q^{k-1}(y_1,y,s_1)\|_{L^\infty(M\times M)}\,ds_1$$
$$\leq A_{k-1}C\int_{s/2}^{s}s_1^{-n-3/2+\beta}\frac{1}{(s-s_1)^{k/2-3/2}}\,ds_1$$
$$\leq A_{k-1}Cs^{-n-2+k/2+\beta}\int_0^{1/2}(1-s_1)^{-n-3/2+\beta}s_1^{k/2-3/2}\,ds_1$$
$$\leq A_kCs^{-n-2+k/2+\beta}/2,$$

so by (3.20),

$$\left|\int_0^{s}\int_M q(x,y_1,s-s_1)q^{k-1}(y_1,y,s_1)\,ds_1\right|\leq CA_ks^{-n-2+k/2+\beta}.$$

Similarly we can prove (3.18) for $a\leq s$. Now we prove (3.19) for $s\leq a$.

$$\left\|p_a\dagger\sum_{j\geq k-1}(-1)^jq^j\right\|_{L^\infty(M\times M)}=\left\|\int_0^s p_a(x,y_1,s-s_1)\sum_{j\geq k}q^j(y_1,y,s_1)\,dv_{y_1}\,ds_1\right\|_{L^\infty(M\times M)},$$

so

$$\left\|\int_0^{s/2}p_a(x,y_1,s-s_1)\sum_{j\geq k}q^j(y_1,y,s_1)\,dv_{y_1}\,ds_1\right\|_{L^\infty(M\times M)}$$
$$\leq\int_0^{s/2}\|p_a(x,y_1,s-s_1)\|_{L^\infty(M\times M)}\sum_{j\geq k}\|q^j(y_1,y,s_1)\|_{L^1(M_{y_1})}\,dv_{y_1}\,ds_1$$
$$\leq\int_0^{s/2}C(s-s_1)^{-n-2+k+\beta}\sum_{j\geq k}s_1^{j/2-1}\frac{1}{\Gamma(\frac{j}{2})}s_1^{j/2-1}$$
$$\leq Cs^{k/2-n-1/2+\beta},$$

and

$$\left\|\int_{s/2}^{s}p_a(x,y_1,s-s_1)\sum_{j\geq k}q^j(y_1,y,s_1)\,dv_{y_1}\,ds_1\right\|_{L^\infty(M\times M)}$$
$$\leq\int_{s/2}^{s}\|p_a(x,y_1,s-s_1)\|_{L^1(M_{y_1})}\sum_{j\geq k}\|q^j(y_1,y,s_1)\|_{L^\infty(M\times M)}\,dv_{y_1}\,ds_1$$
$$\leq Cs^{k/2-n-1/2+\beta}.$$

So (3.19) follows in this case. Similarly we can prove (3.19) for $s > a$.

PROOF OF THEOREM 1. This follows directly from Lemma 3.8.

3.5 *Open problem.* Let μ_λ be the Wiener measure associated with g_λ. What is the asymptotic behavior of μ_λ as $\lambda \longrightarrow \infty$? One might conjecture that the following is true: let $\Omega(x_0, \cdot)$ (resp. $\Omega_H(x_0, \cdot)$) be the space of continuous paths (resp. horizontal paths) starting from x_0, then

$$\mu_\lambda\big(\Omega(x_0, \cdot) - \Omega_H(x_0, \cdot)\big) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty,$$

in a weak sense.

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