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FINITE GROUPS WHICH ADMIT A FIXED-POINT-FREE AUTOMORPHISM GROUP ISOMORPHIC TO S_3

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Abstract

Let G be a finite group of even order coprime to 3. If G admits a fixed-point-free automorphism group isomorphic to the symmetric group on three letters, then we prove that G is soluble.

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A number of authors (for example, [6], [7], [8]) have shown that finite groups admitting certain fixed-point-free abelian automorphism groups are soluble. In this paper we show that a finite group G which admits a fixed-point-free automorphism group isomorphic to S_3 (the symmetric group on 3 letters) is soluble if $|G|$ is even and coprime to 3. A similar result for groups of odd order coprime to 3 has been proved by B. Dolman [1].

The result proved here is a consequence of Glauberman's characterization of simple groups of order coprime to 3 [3]. However the proof given in this paper uses fairly elementary methods and (of course) relies on the fixed-point-free automorphism group.

Throughout the paper we put

$$\Sigma = \langle \sigma , \pi | \sigma^3 = \pi^2 = 1 , \pi\sigma\pi = \sigma^{-1} \rangle \cong S_3.$$

Our notation will in general follow Gorenstein's book [4]. In particular, if P is a p-group, $J(P) = \langle A | A \subseteq P , A$ is abelian of maximal order$\rangle$. In addition, $J_e(P) = \langle E | E \subset P , E$ is elementary abelian of maximal order$\rangle$. The theorem proved in this paper is as follows:

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THEOREM. *Let G be a finite group of even order coprime to* 3. *Suppose G admits a fixed-point-free group of automorphisms $\Sigma \cong S_3$. Then G is soluble and either*

(i) *G has a normal 2-complement, or*

(ii) *$G = (T \times O(G)) \cdot C_G(\sigma)$, where $T = O_2(G)$ is a Sylow 2-subgroup of G.*

1. Preliminary results

PROPOSITION 1 (Burnside, [5, (10.15)]). *If the finite group X admits a fixed-point free automorphism of order* 3 *then X is nilpotent of class at most* 2.

PROPOSITION 2 (Dolman [1]). *Let G be a finite group, $(|G|, 3) = 1$. Suppose G admits a fixed-point-free group of automorphisms $\Sigma \cong S_3$. Then G contains a unique Σ-invariant Sylow p-subgroup for all primes p that divide $|G|$. Further, any Σ-invariant p-subgroup is contained in this unique Σ-invariant Sylow p-subgroup.*

PROOF. Let $\mathscr{S} = \{P | P$ is a σ-invariant Sylow p-subgroup of $G\}$. By [4, Theorem 6.2.2], $\mathscr{S} \neq \varnothing$ and if $P, Q \in \mathscr{S}$ then P and Q are conjugate by some element in $C_G(\sigma)$. Clearly $C_G(\sigma)$ has odd order as π is fixed-point-free on $C_G(\sigma)$. Thus $|\mathscr{S}| = |C_G(\sigma) : N_G(P) \cap C_G(\sigma)|$ is odd $(P \in \mathscr{S})$. As π permutes the subgroups of $\mathscr{S}$, π fixes a subgroup of $\mathscr{S}$.

Suppose P, Q are both Σ-invariant Sylow p-subgroups of G such that $P = Q^x$ for some $x \in C_G(\sigma)$. Thus

$$Q^x = P = \pi(P) = \pi(Q^x) = \pi(Q)^{x^{-1}} = Q^{x^{-1}}$$

whence $x^2 \in N_G(Q)$. Thus $x \in N_G(Q)$ as $x \in C_G(\sigma)$ which has odd order. The last part now follows from the fact that the normalizer of a Σ-invariant p-group is also Σ-invariant.

If the finite group G has order coprime to 3, $\mathrm{SL}(2, p)$ cannot be involved in G. Hence two consequences of Glauberman's ZJ-theorem apply for primes $p \geq 5$.

PROPOSITION 3 [2, Corollaries 2.1, 2.2]. *Let p be an odd prime which divides G, G a finite group of order coprime to* 3. *Let S be a Sylow p-subgroup of G and $N = N_G(Z(J(S)))$. Then*

(i) $G/O^p(G) \cong N/O^p(N)$;

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(ii) *two subsets of* S *are conjugate in* G *if and only if they are conjugate in* N.

The structure of soluble groups of odd order admitting a fixed-point-free group of automorphisms isomorphic to S_3 has been determined by E. Shult.

PROPOSITION 4 [10, Corollary 2.1]. *Let* H *be a soluble group of order coprime to* 6 *which admits a fixed-point-free group of automorphisms isomorphic to* S_3. *Then* H' *is nilpotent.*

The following result (also due to Shult), which plays a key role in the proof of Proposition 4, is used (in this paper) to study soluble groups of even order admitting Σ fixed-point-free.

PROPOSITION 5 [9, Theorem 3.1]. *Let* p *be an odd prime and* H *the semi-direct product of the normal subgroup* K, *of order coprime to* p, *and* $\langle\rho\rangle$ *of order* p. *Suppose* H *acts faithfully on the elementary abelian* q*-group* V, *where* $(q, |H|) = 1$. *If* $C_V(\rho) = 1$ *then* $[\langle\rho\rangle, K] = 1$ *unless* K *has a non-abelian Sylow* 2*-subgroup and* p *is a Fermat prime.*

PROPOSITION 6. *Let* Σ *act fixed-point-free on the finite group* G *of order coprime to* 3. *Then* $C_G(\sigma)$ *is abelian of odd order and for any subgroup* $X \subseteq C_G(\sigma)$ *we have* $N_G(X) = C_G(X)$. *In particular, if* $C_P(\sigma) = P$ *for any Sylow* p*-subgroup* P *of* G, *then* G *has a normal* p*-complement.*

PROOF. Since π acts fixed-point-free on $C_G(\sigma)$, $C_G(\sigma)$ is inverted by π and is therefore abelian of odd order. Let $N = [N_G(X), \sigma]$, so $[N, \sigma] = N$ as $(|N|, 3) = 1$.

By the Three Subgroups Lemma [4, Lemma 2.2.3], $[N, \sigma, X] \subseteq [X, N, \sigma] \cdot [\sigma, X, N] = 1$, whence $N \subseteq C_G(X)$. As $X \subseteq C_G(\sigma)$, $C_G(\sigma) \subseteq C_G(X)$ and therefore $N_G(X) = (C(\sigma) \cap N_G(X)) \cdot N \subseteq C_G(X)$ as required. The final statement follows from Burnside's Transfer Theorem [4, Theorem 7.4.3].

PROPOSITION 7. *Suppose* Σ *acts fixed-point free on the group* $H = V \cdot U$ *where* V *is elementary abelian of order* p^n, $p \geq 5$, U *is a* Σ*-invariant four group and* $V = [V, U]$. *If* $\langle u\rangle = C_U(\pi)$ *then* π *inverts* $C_V(u)$, $V = C_V(u) \times C_V(u^\sigma) \times C_V(u^{\sigma^2})$ *and* $C_V(\sigma) = \{vv^\sigma v^{\sigma^2} | v \in C_V(u)\}$. *In particular,* $|C_V(\sigma)| = |C_V(u)| = p^{n/3}$.

PROOF. As $[V, U] = V$, $C_V(U) = 1$ so the decomposition of V follows from [4, Theorem 5.3.16]. The three factors have the same order as σ per-

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mutes the 3 involutions in U. As $[\pi, u] = 1$, π normalizes $C_V(u)$. Since π inverts $C_V(\sigma)$, π must invert $C_V(u)$ also.

2. Groups of even order

We begin by determining the structure of a soluble group of even order coprime to 3 which admits Σ as a fixed-point-free automorphism group.

PROPOSITION 8. *Let M be a soluble group of even order coprime to* 3 *which has a fixed-point-free group of automorphisms $\Sigma \cong S_3$. Then either*

(i) *M has a normal* 2*-complement, or*

(ii) *if T is a Sylow* 2*-subgroup of M, $T \triangleleft M$ and $M = (T \times O(M)) \cdot C_M(\sigma)$.*

PROOF. Let S be a Σ-invariant Sylow 2-subgroup of $O_{2',2}(M)$ so that $M = N_M(S) \cdot O(M)$. Since σ is fixed-point-free on S, $C_M(\sigma)$ covers $N_M(S)/C_M(S) \cdot S$ by Proposition 5. Proposition 6 and the fact that $C_M(S) \subseteq O_{2',2}(M)$ [4, Theorem 5.3.3] yield that $S = T$.

Suppose now that (i) does not hold, so that $M \neq T \cdot O(M)$. We must show that $T = O_2(M)$. As Σ is fixed-point-free on $M/O_2(M)$ we may assume $O_2(M) = 1$.

Let $x \in C_M(\sigma) \cap N_M(T) - O(M)$ with $x^p \in O(M)$ for some odd prime p. Suppose first that $[x, Z(T)] = 1$. Note that $[x, T] \triangleleft T$ and $[x, T] \neq 1$ because $x \notin O(M)$ and $C_M(T) \subseteq T \cdot O(M)$. Hence there exists a Σ-invariant four group $E \subseteq [x, T] \cap Z(T)$. As $O_2(M) = 1$, there exists a prime q with $Q = O_q(M)$ and $[E, Q] \neq 1$. Let $V = [Q/\Phi(Q), E] \neq 1$ (where $\Phi(Q)$ is the Frattini subgroup of Q). As $[x, E] = 1$ and $C_V(x) \supseteq C_V(\sigma)$, it follows from Proposition 7 that $[x, V] = 1$. Now $E \triangleleft T$ so V is T-invariant and $V = [V, E] = [V, T]$. The three subgroups lemma yields $[T, x, V] = 1$, which contradicts $E \subseteq [T, x]$.

We may now suppose that $[x, Z(T)] \neq 1$. Let $F \subseteq \Omega_1(Z(T))$ be a minimal $\Sigma\langle x\rangle$-invariant subgroup with $[F, x] = F$. As $O_2(M) = 1$ there exists a prime q with $[Q, F] \neq 1$ where $Q = O_q(M)$. Let V be a minimal $\Sigma\langle x\rangle F$-invariant subgroup of $[F, Q/\Phi(Q)] \neq 1$.

If W is a minimal F-invariant subgroup of V then W has $|\langle\sigma\rangle||\langle x\rangle| = 3|\langle x\rangle|$ conjugates under the action of $\langle\sigma\rangle \times \langle x\rangle$ by [4, Theorem 3.4.3]. This implies however that there exists $w \in C_V(\sigma) - C_V(x)$, against the fact that $C_G(\sigma) \subseteq C_G(x)$. This completes the proof of the proposition.

PROPOSITION 9. *Suppose the dihedral group $D = \langle \pi, x | x^p = \pi^2 = 1, \pi x \pi = x^{-1}, p$ an odd prime$\rangle$ acts on the* 2*-group T of order 2^n with $C_T(x) = 1$.*

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For any chief factor $V = S/R$ *of* TD *contained in* T *we have* $C_V(\pi) = C_S(\pi)R/R$. *Further,* $C_T(\pi) = 2^{n/2}$.

PROOF. By a result of Suzuki [4, page 328], any involution in πR inverts an element of odd order in DR. As $C_T(x) = 1$, it follows from Sylow's theorem that all involutions in πR are conjugate in DR and hence in $\langle \pi, R\rangle$.

Now let $s \in S - R$ with $[\pi, s] \in R$. Then $s^{-1}\pi s = \pi r$ for some $r \in R$. We know that $\pi \underset{R}{\sim} \pi r$ so there exists $t \in R$ with $st \in C_S(\pi)$. Thus $C_T(\pi)$ covers $C_V(\pi)$ as asserted.

The final conclusion follows by induction on the length of a chief series for DT and the fact that $|C_V(\pi)| = 2^{k/2}$ if $|V| = 2^k$ (note that V is elementary and $C_V(x) = 1$).

We conclude this section with a result on finite groups with Sylow 2-subgroups of class at most 2.

PROPOSITION 10. *Suppose the finite group* G *has Sylow* 2*-subgroup* T *of class at most* 2. *If* $N_G(Z(T)) = N_G(T) = T \cdot C_G(T)$ *then either*

(i) G *has a normal* 2*-complement, or*

(ii) T *contains a normal subgroup* S *with* T/S *cyclic and* $N_G(S)/C_G(S)S$ *has a non-trivial normal* 2*-complement. Further if* $(|G|, 3) = 1$ *then* $J_e(T) \subseteq S$.

PROOF. Let $Z = Z(T)$. We have that $N_G(Z) = N_G(T) = T \times O(C_G(T))$. If Z is weakly closed in T then Grun's theorem [4, Theorem 7.5.2] states that $N_G(Z)' \cap T = G' \cap T$. Thus $T' = T \cap O^2(G) \cdot G'$ and as $T' \subseteq Z$, the Frattini argument yields $N_G(T') = C_G(T')$. It follows that $O^2(G) \cdot G'$ has a normal 2-complement by Burnside's transfer theorem, and (i) holds.

We now assume that Z is not weakly closed in T and choose S of maximal order such that

$$Z \neq Z \cdot Z^g \subseteq S = T \cap T^g \quad \text{for } g \in G - N_G(T).$$

As $\langle Z, Z^g\rangle \subseteq S$ and $T' \subseteq Z$ we have $S \triangleleft \langle T, T^g\rangle$. Put $N = N_G(S)$ and $C = C_G(S) \cdot S$ and note that $C_G(S) = Z(S) \times O(C_G(T))$. If $h \in N - N_G(T)$ then $Z \neq Z \cdot Z^h \subseteq S \subseteq T \cap T^h$. The maximality of $|S|$ forces $T \cap T^h = S$. We use the bar convention for N/C and we have that $\overline{T}$ is an abelian T.I. Sylow 2-subgroup of $\overline{N}$. Now $\overline{N}$ has one class of involutions [4, Theorem 9.1.4] and by Burnside's Lemma [4, Theorem 7.1.1] all involutions of $\overline{T}$ are conjugate in $N_{\overline{N}}(\overline{T})$. As $N_{\overline{N}}(\overline{T}) = \overline{T}$ it follows that $\overline{T}$ is cyclic and $\overline{N}$ has a non-trivial normal 2-complement. Finally, if $\langle \bar{t}\rangle = \Omega_1(\overline{T})$, $\bar{t}$ inverts an element $\bar{r}$ of or odd order at least 5 (if 3 does not divide $|\overline{N}|$). As $Z \subseteq Z(S)$,

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$C(Z(S)) = S \times O(C_G(T)) = C$. Thus $|\Omega_1(Z(S)) : C(t) \cap \Omega_1(Z(S))| \geq 4$ and $J_e(T) \subseteq S$.

3. Proof of the theorem

For the rest of the paper, G will denote a finite group of even order coprime to 3, and Σ a group of fixed-point-free automorphisms of G. Further we let G be a minimal counterexample to the theorem. If G is soluble, the theorem follows from Proposition 8. Thus G is a non-soluble group and therefore all proper Σ-invariant subgroups of G are soluble.

LEMMA 1. *The group G is simple.*

PROOF. If $N \lhd G$ and N is Σ-invariant, Σ is fixed-point-free on G/N. Thus as G is a minimal counterexample, so $N = 1$ and $G = G_1 \times G_2 \times \cdots \times G_k$, the G_i non-abelian simple groups which are transitively permuted by Σ. If σ normalizes G_i for some i, then $C(\sigma) \cap G_i \neq 1$ by Proposition 1. As π inverts $C_G(\sigma)$, π normalizes G_i so $G = G_i$ as required. If σ permutes G_1, G_2, G_3 then $C(\sigma) \cap G_1 \times G_2 \times G_3 \cong G_1$ is non-abelian. This contradicts Proposition 6, and the lemma is proved.

NOTATION. T will denote the (unique) Σ-invariant Sylow 2-subgroup of G and $M = N_G(T)$. Also $Z = \Omega_1(Z(T))$.

By Proposition 8, M is a maximal Σ-invariant subgroup of G and $N_G(Z) = M$ also.

The theorem will be proved by determining the structure of M and using this to deduce that $C_G(\pi)$ has a normal 2-complement.

LEMMA 2. (i) *We have $M = (T \times O(M)) \cdot C_M(\sigma)$ and $T \times O(M) \neq M$.*

(ii) *If H is a maximal Σ-invariant subgroup of G, $H \neq M$, then H has a normal 2-complement.*

(iii) *If U is any Σ-invariant four group in T then $C_G(U) \subseteq M$.*

PROOF. (i) By Proposition 8, $M = (T \times O(M)) \cdot C_M(\sigma)$. As T has class at most 2 (Proposition 1), G is simple (Lemma 1) and $M = N_G(J_e(T))$ (M is maximal Σ-invariant), Proposition 10 yields that $M \neq T \cdot C_G(T) = T \times O(M)$. (Note that $C_G(T) = Z(T) \times O(C_G(T))$ by Burnside's transfer theorem. As $T \lhd M$, $O(M) = O(C_G(T))$ and so $T \cdot C_G(T) = T \times O(M)$.)

(ii) This follows from Proposition 8.

(iii) Suppose that $C_G(U) \subseteq H \neq M$ where H is a maximal Σ-invariant subgroup of G. Let $R = H \cap T \supseteq C_T(U)$ so that $H = R \cdot O(H)$ (by

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(ii)). If $U \not\subseteq Z$, take A to be maximal abelian in R with $UZ(T) \subseteq A$. Then $A \triangleleft T$ and in any case there exists $A \in SCN(T)$ with $A \leq R$. By a result of Thompson [4, Theorem 8.5.2], $O(H) \cap M \subseteq O(M)$ and so $H \cap M = R \times (O(M) \cap H)$. Since $N_G(Z) = M$ and $C_{O(H)}(U) \not\subseteq M$ we have $U \neq Z$ and also $[Z, C_{O(H)}(U)] \neq 1$. Hence there exists $1 \neq y \in C_G(\sigma) \cap [Z, C_{O(H)}(U)]$ (by Proposition 1).

As $C_M(\sigma) \not\subseteq T \times O(M)$ and $C_G(\sigma) \subseteq C_G(y)$ we have that $C_G(y) \not\subseteq H$. By definition, $y \notin M$ so $C_G(y) \not\subseteq M$ either.

Let L be a maximal Σ-invariant subgroup of G containing $C_G(y)$. As $[U, C_M(\sigma)] \neq 1$ and $U \subseteq L$ we have $1 \neq [L \cap T, C_M(\sigma)] \subseteq L \cap T$. This contradicts (ii); namely that L has a normal 2-complement. The lemma is proved.

NOTATION. Let $\mathscr{P} = \{p | p \text{ prime}, p \text{ divides } |M : T \times O(M)|\}$. For $p \in \mathscr{P}$, P_1 denotes the Σ-invariant Sylow p-subgroup of M; $P_0 = P_1 \cap O(M)$; P denotes the Σ-invariant Sylow p-subgroup of G. Note that $\mathscr{P} \neq \varnothing$ (Lemma 2(i)) and $P_1 \subseteq P$ (Proposition 2).

LEMMA 3. *Let* $p \in \mathscr{P}$. *Then*

(i) $P_1 = P_0 \cdot C_{P_1}(\sigma)$ *is abelian*;

(ii) P_1 *is not a Sylow p-subgroup of* G; *that is,* $P_1 \neq P$;

(iii) *if* $P_0 \neq 1$ *then* $Z(P)$ *is cyclic and* $Z(P)^{\#} \subseteq C_{P_1}(\sigma) - P_0$.

PROOF. (i) If $P_1' \neq 1$ then $N_G(P_1') \subseteq M$ (as $T \subseteq N_G(P_1')$ and $[P_1, T] \neq 1$). Thus $P_1 = P$, a Sylow p-subgroup of G. Now $N_M(Z(J(P)))' \cap P \subseteq P_0 \neq P$, so by Proposition 3(i) and Lemma 1, $N_G(Z(J(P))) = N \not\subseteq M$. Since $[T \cap N, P] = 1$ (Lemma 2(ii)) and $O(N)' \subseteq F(N)$ (Proposition 4), we must have $P = O_p(N)$ by Proposition 3(i). However $1 \neq P \triangleleft \langle M, N \rangle = G$ against Lemma 1. Thus P_1 is abelian and $P_1 = P_0 C_{P_1}(\sigma)$ by Proposition 8(ii).

(ii) If $P_0 = 1$, the assertion follows from Proposition 6. Suppose $P_1 = P$, and $P_0 \neq 1$. Using the same argument as in (i), we see that Proposition 3(i) forces $P = O_p(H)$, where H is the maximal Σ-invariant subgroup of G containing $N_G(Z(J(P)))$. As $C_G(P_0) \subseteq M$ we have that $F(O(H)) \subseteq M$ and therefore $O(H) \cap M \triangleleft O(H)$ (Proposition 4). Since $[T \cap H, P] = 1$, a transfer theorem [4, Theorem 7.4.4] implies that $[O(H), P] = P$; in particular $O(H) \not\subseteq M$. Now $[O(H) \cap M, P] \triangleleft \langle O(H), T\rangle = G$ so that $N_M(P) = C_M(P)$.

Let $P_\pi = C_P(\pi)$ and note that $P_\pi \subset P_0 \subseteq O(M)$. By the Frattini argument $N_G(P_\pi) = C_G(P_\pi) \cdot (N_G(T) \cap N_G(P_\pi)) = C_G(P_\pi) \cdot N_M(P_\pi)$. As $P \subseteq C_M(P_\pi)$ we have, in the same way, that $N_G(P_\pi) = C_G(P_\pi) \cdot C_M(P_\pi) \cdot N_M(P) = C_G(P_\pi) \cdot N_M(P) = C_G(P_\pi) \cdot C_M(P) = C_G(P_\pi)$.

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From above, $[O(H), P] = P$, whereas $[C_{O(H)}(\sigma), P] \subseteq P_0$, because $C_G(\sigma)$ is abelian and $P = P_0 C_M(\sigma)$. We apply the bar convention to $O(H)/C_{O(H)}(P)$. Therefore we have a Σ-invariant subgroup $\overline{X} \cong Z_q \times Z_q$ for some prime $q \neq p$ with $[\overline{X}, \sigma] = \overline{X}$. There exists $x \in C_{O(H)}(\pi)$ with $\overline{x} \in \overline{X}$ and clearly $x \in N_H(P_\pi)$. We complete the proof by showing that $[x, P_\pi] \neq 1$.

Let $\tilde{P}$ be a minimal $\Sigma\overline{X}$-invariant subgroup of $[P, \overline{X}] \neq 1$. Suppose that $C_{\tilde{P}}(\overline{x}) \neq 1$. Then $\tilde{P} = C_{\tilde{P}}(\overline{x}) \times C_{\tilde{P}}(\overline{x}^{\sigma}) \times C_{\tilde{P}}(\overline{x}^{\sigma^2})$. Since π inverts an element $\overline{x}_1$ in $\overline{X}$ and $\overline{x}_1$ is fixed-point-free on $C_{\tilde{P}}(\overline{x})$, there exists $y \in C_{\tilde{P}}(\overline{x}) \cap C(\pi)$ (clearly π normalizes $C_{\tilde{P}}(\overline{x})$). However $1 \neq y y^{\sigma} y^{\sigma^2} \in C_{\tilde{P}}(\sigma)$ and as π inverts $C_G(\sigma)$, $y^{\pi} = y^{-1}$, a contradiction. Thus $C_{\tilde{P}}(\overline{x}) = C_{\tilde{P}}(x) = 1$. Since $[\tilde{P}, \overline{X}] = \tilde{P}$, $[\tilde{P}, \sigma] \neq 1$ so $C_{\tilde{P}}(\pi) \neq 1$. We have that $[x, P_\pi \cap \tilde{P}] \neq 1$ as required.

(iii) Let $P^* \neq 1$ be any Σ-invariant subgroup of P_0. As $[T, P_1] \subseteq N_G(P^*)$ and $1 \neq [T, P_1] \subseteq T$, Lemma 2(ii) forces $N_G(P^*) \subseteq M$. From $P_1 \neq P$ it follows that $Z(P)^{\#} \subseteq P_1 - P_0$ and as $Z(P)$ is Σ-invariant, $Z(P) \subseteq C_P(\sigma)$. Suppose that $\Omega_1(Z(P)) \supseteq \langle a_0, b_0 \rangle$. Without loss we may assume that $[C_T(a_0), b_0] \neq 1$. Now $C_G(a_0) \supseteq P$ so $C_G(a_0) \subseteq H \neq M$, H a maximal Σ-invariant subgroup of G. However $1 \neq [C_T(a_0), b_0] \subseteq C_T(a_0)$ means that H does not have a normal 2-complement. This contradiction (of Lemma 2(ii)) completes the proof of (iii).

Notation. For $p \in \mathscr{P}$ let $\Omega_1(Z(P)) = \langle a_0 \rangle$ if $P_0 \neq 1$ and if $P_0 = 1$ take a_0 to be an element of order p in P_1.

Lemma 4. *For $p \in \mathscr{P}$ we have $P_1 = \langle a \rangle \times P_0$, for some element $a \in C_{P_1}(\sigma)$ with $\Omega_1(\langle a \rangle) = \langle a_0 \rangle$.*

Proof. Since $Z(P) \cap P_0 = 1$ and $P_1 = P_0 C_{P_1}(\sigma)$, it is enough to show that P_1/P_0 is cyclic. Suppose to the contrary; so we may choose $b \in C_{P_1}(\sigma) - P_0$, $b^p \in P_0$ and $C_T(b) \neq 1$. The argument given in the proof of Lemma 3(iii) may be repeated to prove that $C_T(a_0) = 1$ and $C_G(\sigma) \subseteq C_G(b) \subseteq M$. In particular, $[\sigma, P_1] = [\sigma, P_0] \neq 1$ by Proposition 6. Thus $\Omega_1(P_1) \supseteq \langle a_0, b_0, Y \rangle$ with $\langle b_0 \rangle = \Omega_1(\langle b \rangle)$ and Y a Σ-invariant subgroup of type (p, p) with $[Y, \sigma] = Y$.

The following remark will be used in the proof:

(*) If P^* is any Σ-invariant subgroup of P_0 then $N_G(P^*) \subseteq M$; and if $d \in C_{P_1}(\sigma) - Z(P)$ then $C_P(d) = P_1$.

(Recall $1 \neq [T, P_1] \subseteq T$. As $\langle T, P_1 \rangle \subseteq C_G(P^*)$, Lemma 2(ii) yields $N_G(P^*) \subseteq M$. If $\langle d_0 \rangle = \Omega_1(\langle d \rangle)$ then for some $x \in \langle d_0, a_0 \rangle$, $C_T(x) \neq 1$. As $1 \neq [C_T(x), a_0] \subseteq C_T(x)$, the same argument yields that $C_G(x) \subseteq M$.

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Thus $C_P(d_0) = C_P(x) = P_1 = P \cap M$ as required.)

Let $R = N_P(P_1) \neq P_1$ and note that $\Omega_1(Z(R)) = \langle a_0 \rangle$. Therefore if $y \in \Omega_1(Z_2(R))$, y has at most p conjugates in R. Since $\Omega_1(P_1) \supseteq \langle a_0, b_0, Y \rangle$ it follows from $(*)$ that $\Omega_1(Z_2(R)) \subseteq P_1$. Now σ is fixed-point-free on R/P_1 so $|R : P_1| \geq p^2$. We conclude that $C(\sigma) \cap \Omega_1(Z_2(R)) = \langle a_0 \rangle$ and $|R : P_1| = p^2$. For $x \in R - P_1$, $|C(x) \cap \langle Y, a_0, b_0 \rangle| \leq p^2$ so P_1 is the unique abelian subgroup of R of its order. Hence P_1 char R so $R = P$ and P_1 char P.

Let H be a maximal Σ-invariant subgroup (of G) containing $N_G(Z(J(P)))$. Since $\langle a_0 \rangle = \Omega_1(Z(P))$, $C_H(a_0)$ covers $H/O(H)$ by the Frattini argument. Thus $|H|$ is odd as $C_T(a_0) = 1$. It now follows from Propositions 3(i) and 4 that $P = O_p(H)$. Clearly $O_{p'}(H) \subseteq M$ as $C_G(P_0) \subseteq M$ (by $(*)$).

Suppose that $H = P(H \cap M) = PN_M(P_1)$; that is, $N_M(P_1)$ covers $H/F(H) \neq 1$. Let q divide $|H : C_H(P_1)|$ and let $\tilde{Q}$ be the Σ-invariant Sylow q-subgroup of H. We have $\tilde{Q} \subseteq N_M(P_1)$ whence $\tilde{Q} \subseteq Q_1$, the Σ-invariant Sylow q-subgroup of M. If $Q_1 \subseteq O(M)$, $1 \neq [N_M(Q_1), T] \subseteq T$, so $N_G(Q_1) \subseteq M$ by Lemma 2(ii). It follows that $Q_1 \triangleleft M$ (by Propositions 3(i) and 4). However this forces $[\tilde{Q}, P_1] = 1$ which contradicts the choice of q. We have shown that $q \in \mathscr{P}$ and so Q_1 is abelian. If Q is the Σ-invariant Sylow q-subgroup of G, there exists $c \in C_{Q_1}(\sigma)$ with $C_Q(c) \neq Q_1$ by Lemma 3 (if $Q_0 = O(M) \cap Q_1 \neq 1$) or Proposition 7 (if $Q_0 = 1$).

As $\langle a_0 \rangle = \Omega_1(Z(P))$, $\langle a_0 \rangle \triangleleft H$, so $\langle a_0 \rangle \subseteq Z(H)$ since $H/C_H(a_0)$ must be cyclic and $a_0 \in C_G(\sigma)$ which is abelian. Now $c \in C_G(\sigma) \subset C_G(a_0) = H$ so $c \in \tilde{Q}$. Let $\tilde{P} = C_P(c)$ and let L be a maximal Σ-invariant subgroup of G containing $C_G(c)$ (note that $M \neq L \neq H$). Since $H = N_G(Z(J(P)))$, Proposition 3(ii) yields that $\langle a_0 \rangle$ is weakly closed in P and hence in $\tilde{P}$. It follows from $C_T(a_0) = 1$ and the Frattini argument that $|L|$ is odd. Let $P_2 = P \cap L$, a Sylow p-subgroup of L. If $P_2 \nsubseteq P_1$, P_2 is non-abelian (as $b \in P_2$). However $a_0 \in P_2' \subseteq F(L)$ whence $\langle a_0 \rangle \subseteq Z(L)$, a contradiction. Hence $P_2 \subseteq P_1$ and P_2 is abelian. Now $\langle a_0 \rangle$ weakly closed in P_2 forces $N_G(P_2) \subseteq H = P(H \cap M)$. Thus $P_3 = N_L(P_2)' \cap P_2 \subseteq P_0$, as $N_L(P_2) \subseteq H \cap M$. By Proposition 4, $P_3 \subseteq F(L)$ so $P_3 \triangleleft O_{p'}(L)N_L(P_2) = L$ whence $P_3 = 1$ by $(*)$. Burnsides' Transfer Theorem yields that L has a normal p-complement. Thus $[P_2, \tilde{Q}] = 1$ so that $\tilde{P} \subseteq P_2 \subset P_1$ (by the choice of q). If Q_2 is the Σ-invariant Sylow q-subgroup of L, $Q_2 \neq Q_1$ (by the choice of c) and so there exists $d \in \langle b_0, a_0 \rangle$ with $C_{Q_2}(d) \neq Q_1$.

Let F be a maximal Σ-invariant subgroup of G containing $C_G(d)$ (note that $H \neq F \neq M$). As $C_P(d) = P_1$ and P is non-abelian, arguing as above we conclude that P_1 is a Sylow p-subgroup of F, $P^* = N_F(P_1)' \cap P_1 \triangleleft F$ and $P^* \subseteq P_0$. Since $\tilde{P} \subset P_1$, $1 \neq [c, P_1] \subseteq P^*$ which contradicts $(*)$.

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We have proved that $H \neq P(H \cap M)$. Use the bar convention for $H/C_H(P_1)$. Since $C_G(\sigma) \subseteq H \cap M$, there exists $\overline{X} \cong Z_q \times Z_q$, Σ-invariant with $[\overline{X}, \sigma] = \overline{X}$. Put $\Omega = \Omega_1(Z_2(P)) \subseteq P_1$. Recall that $C_\Omega(\sigma) = \langle a_0 \rangle$ and $1 \neq [\Omega, \sigma] \subseteq P_0$. If $[\overline{X}, \Omega] \neq 1$, σ has a fixed point on $[\overline{X}, \Omega]$, which contradicts $C_\Omega(\sigma) = \langle a_0 \rangle$. Thus $[\Omega, \overline{X}] = 1$ so that $C_{P_0}(\overline{X}) \neq 1$. Thus the centralizer of the Σ-invariant subgroup $C_{P_0}(\overline{X})$ does not lie in M, which contradicts $(*)$. This completes the proof of the lemma.

LEMMA 5. *If $p \in \mathscr{P}$ then $C_Z(a) = 1$. Further, if $z \in Z$, then $C_M(z) = T \times O(M) = C_G(z)$.*

PROOF. *Suppose $C_Z(a) \neq 1$ and let $U = \langle u, u^\sigma \rangle$ be a Σ-invariant four* group in $C_Z(a)$ with $u \in C_Z(\pi)$. Suppose that $P_0 = O(M) \cap P_1 \neq 1$. Let L be a maximal Σ-invariant subgroup of G containing $N_G(P_1)$, P_2 a Sylow p-subgroup of L and $R = N_{P_2}(P_1)$. Using the same argument as in the previous lemma we get that $|R : P_1| = p^2$. As L has a normal 2-complement we may assume that ΣU normalizes R/P_1. By Proposition 7, $[U, R] \subseteq P_1$, which contradicts Lemma 2(iii). Hence $P_0 = 1$ and $P_1 = \langle a \rangle$.

Let $A = C_P(a)$. As U acts on $N_P(A)/A$ and $C_P(U) = \langle a \rangle$, σ has a nontrivial fixed point on $N_P(A)/A$ if $N_P(A) \neq A$. However $a \in C_G(\sigma)$ is abelian, so we conclude that $A = P$; that is, $\langle a \rangle \subseteq Z(P)$.

Let $N = N_G(Z(J(P)))$. We will show that $N \subseteq C_G(a)$. As $N \nsubseteq M$, a maximal Σ-invariant subgroup of G which contains N has a normal 2-complement (Lemma 2(ii)). Thus N has a normal 2-complement, and so $N = O(N) \cdot (T \cap N)$. Now $[T \cap N, \langle a \rangle] \subseteq T \cap Z(J(P)) = 1$ and $T \cap N \subseteq C_T(a)$. A maximal Σ-invariant subgroup of G containing $C_G(a)$ also has a normal 2-complement. Hence $C_G(a) = O(C_G(a)) \cdot C_T(a)$. The Frattini argument yields that $C_T(a) \subseteq N$. In particular $U \subseteq T \cap N = C_T(a)$. By Proposition 4, $O(N)' \subseteq F(O(N)) \subseteq C_N(a)$, whence $C_{O(N)}(a) \triangleleft O(N)$. Also, Lemma 2(iii) yields $C_{O(N)}(U) \subseteq M \cap O(N) \subseteq C_N(a)$. Therefore, if $O(N)/C_{O(N)}(a) \neq 1$, σ has a non-trivial fixed point on $O(N)/C_{O(N)}(a)$. This contradicts $C_G(\sigma) \subseteq C_G(a)$ and we have shown that $N \subseteq C_G(a)$. It follows now from Proposition 3(ii) that $\langle a \rangle$ is weakly closed in P with respect to G.

Since $P \neq \langle a \rangle$ and $u \sim u^\sigma \sim u^{\sigma^2}$ in $N_G(P) \cdot \Sigma$, $C_P(U) \neq \langle a \rangle$. Now $C_P(U) = P \cap M = \langle a \rangle$ so π inverts $C_{P/\langle a \rangle}(u)$ by Proposition 7. Hence π inverts $C_P(u)$ (as $a \in C_P(\sigma)$) and in particular, $C_P(u)$ is abelian. The fact that $\langle a \rangle$ is weakly closed in $C_P(u)$ means that $C_P(u)$ is a Sylow p-subgroup of $C_G(u)$. Further, $N_G(\langle a \rangle) = C_G(a)$ as σ must centralize the cyclic group $N_G(\langle a \rangle)/C_G(a)$. The transfer theorem [4, Theorem 7.4.4] gives

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$a \notin O^p(C_G(u))$. Since $T \subseteq C_G(u)$, the Frattini argument yields $C_G(u) = O^p(C_G(u)) \cdot \langle a \rangle$.

For any $q \in \mathscr{P} - \{p\}$, let Q_1 be the Σ-invariant Sylow q-subgroup of M, $Q_1 = \langle d \rangle \times Q_0$, $d \in C_M(\sigma)$, $Q_0 = O(M) \cap Q_1$. Suppose that $[d, C_T(a)] \neq 1$. As $d \in C_G(a)$, this implies that the maximal Σ-invariant subgroup of G containing $C_G(a)$ does not have a normal 2-complement. Thus $C_T(d) = C_T(a)$ and $d \in C_G(u)$. The same argument as above yields that $C_G(u) = O^q(C_G(u))\langle d \rangle$. As $P_0 = 1$, M contains an abelian Hall $\mathscr{P}$-subgroup $B \subseteq C_M(\sigma)$, and $M = (T \times O(M)) \cdot B$. Also $C_G(u)$ has a normal subgroup Y with $Y \cap M = T \times O(M)$ and $C_G(u) = Y \cdot B$.

Since $N_Y(Z(T)) = N_Y(J_e(T)) = N_Y(T) = T \times O(M)$, Proposition 10 yields that Y has a normal 2-complement. As $\langle a \rangle$ is a Sylow p-subgroup of M, the Frattini argument yields that $\langle a \rangle T \subseteq N_Y(\tilde{P})$ for some Sylow p-subgroup $\tilde{P}$ of $O(Y)$. The weak closure of $\langle a \rangle$ in P forces $[\langle a \rangle, \tilde{P}] = 1$. However $1 \neq [\langle a \rangle, T]$ must centralize $\tilde{P} \neq 1$, a contradiction of Lemma 2(iii). We have proved that $C_Z(a) = 1$.

It remains to show that $C_G(z) = C_M(z)$ for $z \in Z$. As $C_M(z) = T \times O(M)$ and $N_G(J_e(T)) = M$, Proposition 10 yields that $C_G(z)$ has a normal 2-complement K. We claim that $|Z| \geq 64$. Indeed $\langle \sigma \rangle \times \langle a_0 \rangle$ acts fixed-point-free on Z. If $|Z| = 16$ then a_0 has order 5. However π inverts $\langle \sigma, a_0 \rangle$ whereas $GL(4, 2) \cong A_8$ has no dihedral group of order 30. Thus $|Z| \geq 64$, and $|C_Z(\pi)| \geq 8$. If $K \neq O(M)$ there exists a four group $\langle t, u \rangle \subseteq C_Z(\pi)$ with $C_K(\langle t, u \rangle) \not\subseteq O(M)$. As $\langle t^\sigma, u^\sigma \rangle$ acts on this group we may assume that $C_K(\langle t, t^\sigma \rangle) \not\subseteq O(M)$. However $\langle t, t^\sigma \rangle$ is Σ-invariant (recall that $t \in C_Z(\pi)$). This contradicts Lemma 2(iii). Thus $C_G(z) = T \times O(M)$ and the lemma is proved.

LEMMA 6. *The subgroup $C_G(\pi)$ has a normal 2-complement.*

PROOF. We begin with two remarks. First, if $1 \neq X \subseteq Z(T)$ then $N_G(X) \subseteq N_G(T)C_G(X) \subseteq M$ (by the Frattini argument and Lemma 5). Second, if $T_\pi = C_T(\pi)$, we have that T_π is a Sylow 2-subgroup of $C = C_G(\pi)$. (If not, $T_\pi \subset T^g$ for some $\langle \pi \rangle$-invariant Sylow 2-subgroup T^g of G with $T^g \cap C$ a Sylow 2-subgroup of C. As there exists $u \in T_\pi \cap Z$, $O(M) = O(M^g) = 1$. Further, all involutions in πT are conjugate in T and so, π, π^g are conjugate in $N_G(T^g) = M^g$. That is, there exists $h \in M^g$ with $\pi = \pi^{gh}$. Therefore $T_\pi^{gh} = C \cap T^{gh} = C \cap T^g$ and so T_π is (conjugate to) a Sylow 2-subgroup of C.)

Now suppose that C does not have a normal 2-complement. We note that $N_G(Z(T_\pi)) = N_G(T_\pi) \cdot C_G(Z(T_\pi)) \subseteq N_G(T_\pi) \cdot T$ as $C_G(Z(T_\pi)) \subseteq T \times O(M)$ by Lemma 5. Therefore we have $N_C(T_\pi) = N_C(Z(T_\pi))$. It follows from

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Proposition 10 that either $N_C(T_\pi) \neq C_C(T_\pi) \cdot T_\pi$ or there exists $S_\pi \triangleleft T_\pi$, T_π/S_π cyclic and $N_C(S_\pi)/S_\pi C_C(S_\pi)$ has a non-trivial normal 2-complement. Let S denote either T_π or S_π and let x be an element of odd order in $N_C(S) - M$. (We can choose $x \notin M$ as $C_M(\pi) = T_\pi \times C_{O(M)}(\pi)$.)

As x normalizes $S' \subseteq Z(T)$, the first remark (at the beginning of the proof) yields that S is abelian. Since $Z \cap S \neq 1$, $C_G(S) = C_T(S) \times O(M)$. Let $A = C_T(S)$. As x normalizes $O(C_G(S)) = O(M)$, $O(M) = 1$ and $C_G(S) = A \supseteq S$.

We have $x \in N_G(A)$ and therefore, arguing as for S, we get that

$$(O_2(N_G(A)))' = 1.$$

Thus $A = O_2(N_G(A)) = C_G(A)$. Set $N = N_G(A)$ and apply the bar convention to $\overline{N} = N/A$. If $N(\overline{T}) \neq C(\overline{T})$ there exists d of order q, for some $q \in \mathscr{P}$ with $d \in C_M(\sigma)$ and $[\overline{d}, \overline{T}] \neq 1$. Since $[d, T] \cap C_T(d) = 1$ (recall that $C_Z(d) = 1$), $\langle d, \pi \rangle$ acting on $[d, T]$ satisfies the assumptions of Proposition 9. Therefore $C_{[d,T]}(\pi)$ covers $C_{[d,T]/[d,T]\cap A}(\pi)$. As this latter group is isomorphic to $C_{[\overline{d},\overline{T}]}(\pi)$ and d has order at least 5, $\overline{C_{[d,T]}(\pi)} \subseteq \overline{T}_\pi$ is non-cyclic. This contradicts the fact that T_π/S is cyclic (recall that $S \subseteq A$). We conclude that $N(\overline{T}) = C(\overline{T})$ and so $\overline{N}$ has a normal 2-complement $\overline{K} \neq 1$.

If $\overline{T}$ is not cyclic, let $\langle \bar{t}_1, \bar{t}_2 \rangle$ be a four group in $\overline{T}$. We may assume that $[\bar{t}_1, C_{\overline{K}}(\bar{t}_2)] = \overline{K}_0 \neq 1$. Now $[\bar{t}_2, A] = A_0 \subseteq Z(T)$ so $[\bar{t}_1, A_0] = 1$. Hence $[\overline{K}_0, A_0] = 1$, which contradicts Lemma 5. If $\overline{T}$ is cyclic, let $\langle \bar{t} \rangle = \Omega_1(\overline{T})$. Clearly $|[\bar{t}, \Omega_1(A)]| \geq 4$ as $\bar{t}$ inverts an element of odd order in $\overline{K}$. This forces $J_e(T) \subseteq A$ and $N \subseteq M$, a contradiction. The lemma is proved.

We are now in a position to complete the proof of the theorem. By Lemma 2(i), $\mathscr{P} \neq \varnothing$, so let $p \in \mathscr{P}$. By Proposition 6, $P_\pi = C_P(\pi) \neq 1$. As a is fixed-point-free on Z (recall that $P_1 = \langle a \rangle \times P_0$, P_1 a Sylow p-subgroup of M), there exists a four group $\langle u_1, u_2 \rangle \subseteq C_Z(\pi)$. Lemma 6 shows that $\langle u_1, u_2 \rangle$ normalizes a Sylow p-subgroup $\tilde{P}$ of $C = C_G(\pi)$. Since $C_G(z) = T \times O(M)$ for $z \in Z^\#$, it follows that $\tilde{P} \subseteq O(M)$. The same argument as in the (second) remark at the beginning of the proof of Lemma 6 yields that $C \cap P_1 = C \cap P_0$ is a Sylow p-subgroup of $C_M(\pi)$. As $\tilde{P} \subseteq O(M)$, $C \cap P_0$ is a Sylow p-subgroup of C. Thus $[\sigma, P] = [\sigma, P_0]$ (if $[\sigma, P] \nsubseteq P_1$ then $P_\pi \nsubseteq P_1$). If H is a maximal Σ-invariant subgroup of G containing $N_G([\sigma, P])$, then $H \supseteq \langle T, P \rangle$ so $H \neq M$ (Lemma 3(ii)). As $p \in \mathscr{P}$ we have $1 \neq [P_1, T] \subseteq T$. Thus H does not have a normal 2-complement, against Lemma 2(ii). This contradiction completes the proof of the theorem.

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