MULTIPLICATION IN VECTOR LATTICES

NORMAN M. RICE

1. Introduction. B. Z. Vulih has shown (**13**) how an essentially unique intrinsic multiplication can be defined in a Dedekind complete vector lattice L having a weak order unit. Since this work is available only in Russian, a brief outline is given in § 2 (cf. also the review by E. Hewitt (**4**), and for details, consult (**13**) or (**11**)).

In general, not every pair of elements in L will have a product in L. In § 3 we discuss certain properties which ensure that, in fact, the multiplication will be universally defined, and it turns out that L can always be embedded as an order-dense order ideal in a larger space $L^{\#}$ which has these properties. It is then possible to define multiplication in spaces without a unit.

In § 4 we show that if L has a normal integral ϕ, then ϕ can be extended to a normal integral on a larger space $L_1(\phi)$ in $L^{\#}$, and $L_1(\phi)$ may be regarded as an abstract integral space. In § 5 a very general form of the Radon-Nikodym theorem is proved, and in § 6 this is used to give a relatively simple proof of a theorem of Segal giving a necessary and sufficient condition for the Radon-Nikodym theorem to hold in a measure space.

2. Multiplication in spaces with a unit. Let L be a vector lattice which is Dedekind complete (i.e., every set which is bounded above has a least upper bound) and has a weak order unit $\mathbf{1}$ (i.e., $\inf(\mathbf{1}, x) > 0$, whenever $x > 0$). An element $e \in L$ is called *unitary* if $\inf(e, \mathbf{1} - e) = 0$. (These correspond, roughly, to characteristic functions.) e will always denote a unitary element, and $U(L, \mathbf{1}) = U(L)$ will denote the set of unitary elements.

It is easy to see that any set E of unitary elements is bounded below by 0 and above by $\mathbf{1}$ so that $\sup(E)$ and $\inf(E)$ exist, and it is not hard to show that $\sup(E)$ and $\inf(E)$ are also unitary, so we can conclude (and this will be useful in § 6) that $U(L)$ is a complete Boolean algebra.

For any $x \in L$ the *characteristic element* of x is defined to be

$$s(x) = \sup_n \inf(n\,|x|, \mathbf{1}).$$

$s(x)$ is always a unitary element, $s(x) = 0$ if and only if $x = 0$, $s(ax) = s(x)$ for any real number $a \neq 0$, and $x \perp y$ (i.e., $\inf(|x|, |y|) = 0$) if and only if $s(x) \perp s(y)$. Freudenthal has shown (**3**) that for every $0 \leqq x \in L$ there exists a largest unitary element e such that $e \leqq x$, and that $e = \mathbf{1} - s[(\mathbf{1} - x)_+]$. It follows

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that if we define $k_a(x) = \mathbf{1} - s[(a\mathbf{1} - x)_+]$, then $a \cdot k_a(x) \leqq x$ for any $0 \leqq x \in L$, and $k_a(x)$ is the largest unitary element with this property. Freudenthal showed that it also follows that if $0 < x \in L$, then there exists $0 < e \in U(L)$ and $a > 0$ such that $ae \leqq x$.

Vulih used these results to show that any $0 \leqq x \in L$ can be achieved as the supremum of all the linear combinations of unitary elements that lie below it. For applications in § 6 we shall need a somewhat stronger result.

THEOREM 2.1. *If* $0 \leqq x \in L$, *then* $x = \sup\{rk_r(x) \colon \text{rational } r \geqq 0\}$.

Proof. Since $rk_r(x) \leqq x$ for every r, there exists

$$y = \sup\{rk_r(x)\} \leqq x.$$

Suppose $x - y > 0$; then there exists $e > 0$ and $a > 0$ such that $x - y > 2ae$. Let $b = \sup\{b' \colon b'e \leqq x\} \geqq 2a$, and let r be a rational number such that $b - a \leqq r \leqq b$. Then $0 < re \leqq x$, therefore $e \leqq k_r(x)$, and hence

$$re \leqq rk_r(x) \leqq y.$$

But then $(b + a)e = (b - a)e + 2ae \leqq re + 2ae \leqq y + 2ae \leqq x$, contradicting the maximality of b.

We now define multiplication:

(i) If $e, e' \in U(L)$, the product ee' is defined by $ee' = \inf(e, e')$.

(ii) If $x \geqq 0$ and $y \geqq 0$, the product xy is defined by $xy = \sup(abee' \colon 0 \leqq ae \leqq x,\ 0 \leqq be' \leqq y\}$ if this supremum exists. xy is not defined if the supremum does not exist.

(iii) In general, the product xy is defined by

$$xy = x_+y_+ - x_+y_- - x_-y_+ + x_-y_-$$

if all the products on the right exist.

Note. Vulih's definition of multiplication in (**13**) is formally somewhat different. For $x, y \geqq 0$, if $0 \leqq x' = \sum a_\lambda e_\lambda \leqq x$ and $0 \leqq y' = \sum b_\mu e'_\mu \leqq y$ are two finite sums, he defines $x'y'$ to be $\sum_{\lambda,\mu} a_\lambda b_\mu e_\lambda e'_\mu$, and then defines xy to be $\sup\{x'y' \colon 0 \leqq x' \leqq x, 0 \leqq y' \leqq y\}$ if this supremum exists. He shows, however, that the particular representation of x' as a finite sum does not affect the product $x'y'$, and with this observation it is easy to see that his definition of xy coincides with the one given above; for we may write x' and y' in such a way that they have disjoint summands, so that $\sum a_\lambda b_\mu e_\lambda e'_\mu$ has disjoint summands and hence equals $\sup_{\lambda,\mu}\{a_\lambda b_\mu e_\lambda e'_\mu\}$.

We list below some of the properties of the multiplication.

M(i) $x\mathbf{1}$ always exists and equals x.

M(ii) If xy exists, then yx exists and equals xy.

M(iii) If xy, $(xy)z$, and yz all exist, then $x(yz)$ exists and equals $(xy)z$.

M(iv) If xy and xz exist, then $x(y + z)$ exists and equals $xy + xz$.

M(v) If xy exists and a is real, then $(ax)y$ exists and equals $a(xy)$.

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M(vi) If $x, y \geqq 0$ and xy exists, then $xy \geqq 0$.

M(vii) If xy exists, and $|x'| \leqq |x|$ and $|y'| \leqq |y|$, then $x'y'$ exists.

M(viii) $x \perp y$ if and only if xy exists and equals 0.

It can be shown that any (partially defined) multiplication in L which satisfies the above eight properties must in fact be identical to the Vulih multiplication.

Remark. The uniqueness referred to above depends, of course, on the unit $\mathbf{1}$ (cf. property M(i)). In general, two elements which have a certain product with respect to one unit will have a different product (or none at all) with respect to another unit. However, there is a connecting formula (cf. **11**, Theorem 5.3): let $\mathbf{1}$ and $\mathbf{1}'$ be two units of L; denote the product of x and y with respect to $\mathbf{1}$ by xy, and the product with respect to $\mathbf{1}'$ by $x * y$; if xy and $x * y$ both exist, then $\mathbf{1}'(x * y) = xy$.

Some further properties of the multiplication are the following.

M(ix) If xy exists, then $s(xy) = \inf(s(x), s(y))$.

M(x) If $x \nleqq y$, then there exists $e > 0$ and $a > 0$ such that $xe \geqq ye + ae$.

M(xi) For any element $x \geqq 0$ and any integer $n > 0$ there is a unique positive nth root of x, i.e., a unique $y \geqq 0$ such that y^n exists and equals x.

M(xii) Let $\{x_\alpha\}$ and $\{y_\alpha\}$ be two nets in L indexed by the same directed set. Suppose $(0)\text{-}\lim(x_\alpha) = x$, $(0)\text{-}\lim(y_\alpha) = y$, $x_\alpha y_\alpha$ exists for each α, and there exists $z \in L$ such that $|x_\alpha y_\alpha| \leqq z$ for all α. Then the product xy exists in L, and $(0)\text{-}\lim(x_\alpha y_\alpha) = xy$.

M(xiii) Since $xe + x(\mathbf{1} - e) = x$ with $x(\mathbf{1} - e) \in \{e\}^\perp$ and $xe \in \{e\}^{\perp\perp}$, we see that xe is the component of x in $[e]$, the normal subspace of L generated by e (cf. **2**, Chapter II, § 1.5).

Vulih defines the inverse of an element x to be an element y (if such exists) such that $s(y) = s(x)$ and $xy = s(x)$. He denotes the inverse of x by x^{-1}, and proves, for instance:

I(i) If $x \geqq 0$ and x^{-1} exists, then $x^{-1} \geqq 0$,

I(ii) If $xy = s(x)$, then x^{-1} exists and $x^{-1} = y \cdot s(x)$,

I(iii) Let $x = y + z$, where $y \perp z$. If x^{-1} exists, then y^{-1} and z^{-1} exist, and $x^{-1} = y^{-1} + z^{-1}$. Conversely, if y^{-1} and z^{-1} exist, then x^{-1} exists,

I(iv) If x^{-1} exists, and $|y| \geqq |x|$ and $s(y) = s(x)$, then y^{-1} exists and $|y^{-1}| \leqq |x^{-1}|$.

Remark. Vulih's proof of I(iv) can be considerably simplified by noting the following criterion (cf. **11**, Theorem 4.2): for $x \geqq 0$, let

$$S = \{y \geqq 0 \colon s(y) \leqq s(x), \text{ and } xy \leqq s(x)\};$$

then x^{-1} exists if and only if $\sup(S)$ exists, and in this case $x^{-1} = \sup(S)$.

3. Rings, and extensions to rings. L may fail to be a ring because the multiplication may not be universally defined. Therefore, it is of interest to have conditions which will guarantee that L does indeed become a ring. We list below several properties that a Dedekind complete vector lattice may have; we shall show that they are mutually equivalent, and are sufficient to make the multiplication universally defined.

P_1: There exists a unit $\mathbf{1} \in L$; and, taking unitary elements with respect to any unit, a subset $S \subset L^+$ has a supremum if for every $0 < e \in U(L)$ there exists $0 < e' \leqq e$ and a real number b such that $xe' \leqq be'$ for all $x \in S$.

P_2: A subset $S \subset L^+$ has a supremum if for every $0 < y \in L$ there exists a real number b such that

$$\sup_{x \in S} \inf(by, x) < by.$$

P_3: If the elements of the subset $S \subset L^+$ are mutually disjoint, then $\sup(S)$ exists.

THEOREM 3.1. *In a Dedekind complete vector lattice*, L, P_1, P_2, *and* P_3 *are mutually equivalent.*

Proof. We shall prove $P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow P_1$.

(i) Suppose P_1 holds, and suppose that $S \subset L^+$ is such that for every $0 < y \in L$ there exists b such that

$$\sup_{x \in S} \inf(by, x) < by.$$

In particular, if $e > 0$, there exists b such that

$$\sup_{x \in S} \inf(be, x) < be,$$

and hence by Freudenthal's result (**3**, Theorem 7.4.4) there exists $0 < e' \leqq e$ and $c > 0$ such that

$$\sup_{x} \inf(be, x) \leqq be - ce'.$$

Then it follows that $xe' \leqq be'$ for every $x \in S$; for if $xe' \nleqq be'$, then there exists $0 < e'' \leqq e'$ such that $xe'' \geqq be''$, and then $be'' > (b - c)e'' \geqq \inf(be'', xe'') = be''$, a contradiction. Hence, by P_1, $\sup(S)$ exists, and therefore P_2 holds.

(ii) Suppose P_2 holds, and suppose that $S \subset L^+$ is a set of mutually disjoint elements. For $0 < y \in L$ we want to find b such that

$$\sup_{x \in S} \inf(by, x) < by.$$

If y is disjoint from every $x \in S$, then $b = 1$ will do. Suppose that for some $z \in S$, $y' = \inf(y, z) > 0$. Then there exists b such that $by' \nleqq z$, i.e., $\inf(0, z - by') < 0$, and since y' is disjoint from every other $x \in S$,

$$\sup_{x \in S} \inf(0, x - by') < 0.$$

But then, since $y \geqq y'$,

$$\sup_{x \in S} \inf(0, x - by) < 0, \quad \text{i.e.,} \quad \sup_{x \in S} \inf(by, x) < by.$$

Hence, by P_2, $\sup(S)$ exists and so P_3 holds.

(iii) Suppose that P_3 holds. We first show that L then has a unit. In fact, let $\{x_\alpha\}$ be a collection of positive elements, maximal with respect to the property that its elements are mutually disjoint. By property P_3 it follows immediately that $\mathbf{1} = \sup(x_\alpha)$ exists, and it is clear that $\mathbf{1}$ is a weak order unit (for otherwise there would exist $x > 0$ such that $x \perp x_\alpha$ for all α, and then $\{x_\alpha\}$ could be enlarged).

Now let $S \subset L^+$ be such that for every $0 < e \in U(L)$ there exists $0 < e' \leqq e$ and b such that $xe' \leqq be'$ for all $x \in S$. We shall say (for the moment) that a set E of unitary elements is *admissible* if its elements are mutually disjoint and for each $e \in E$ there exists a_e such that $xe \leqq a_e e$ for every $x \in S$. Let A be the collection of admissible sets. A is inductively ordered by inclusion, so there is a maximal admissible set E_0, and we can see by the assumption on S that $\sup(e\colon e \in E_0) = \mathbf{1}$. Now, since E_0 is admissible, its elements are mutually disjoint, thus by property P_3 there exists

$$y = \sup(a_e e\colon e \in E_0).$$

We can see that y is an upper bound for S; for if not, then there is an $x \in S$ such that $x \nleqq y$, so there exists $e' > 0$ and $b > 0$ such that $xe' \geqq ye' + be'$ (property M(x)). But since $\sup(e\colon e \in E_0) = \mathbf{1}$, there exists $e \in E_0$ such that $e'' = ee' \neq 0$, and then

$$ye'' = ye\cdot e' = a_e e\cdot e' \geqq xe\cdot e' = xe'' \geqq ye'' + be'',$$

a contradiction. Thus, y is an upper bound for S, and therefore, since L is Dedekind complete, $\sup(S)$ exists. Hence P_1 holds.

We will occasionally refer to any of the properties P_1, P_2, P_3 as simply property P.

Next we show that property P is the sort of property we want.

THEOREM 3.2. *If L is a Dedekind complete vector lattice with property* P, *then the multiplication is universally defined.*

Proof. It is sufficient to prove that xy exists for any $x, y \geqq 0$. Let $S = \{abee'\colon 0 \leqq ae \leqq x, 0 \leqq be' \leqq y\}$, and consider any $e_0 > 0$. Now $ce_0 \nleqq x$ for some c, so there exists $0 < e'_0 \leqq e_0$ such that $ce'_0 \geqq xe'_0$. Similarly, $de'_0 \nleqq y$ for some d, so there exists $0 < e''_0 \leqq e'_0$ such that $de''_0 \geqq ye''_0$.

Now suppose $abee' \in S$, i.e., $ae \leqq x$ and $be' \leqq y$. Then

$$(abee')e''_0 = (aee''_0)(be'e''_0) \leqq (xe''_0)(ye''_0) \leqq (ce''_0)(de''_0) = cde''_0.$$

Thus e''_0 and cd are as required in property P_1, thus $\sup(S)$ exists, i.e., xy exists.

Remark. Another property that is sufficient to make the multiplication universally defined is that $\mathbf{1}$ be a *strong unit* (i.e., for every $x \in L$ there should be a real number a such that $|x| \leqq a\mathbf{1}$). This follows immediately from the properties of multiplication M(i) and M(vii). However, these two conditions are independent: for instance, the space of all real sequences has property P but not a strong unit, whereas the space of all bounded sequences has a strong unit but not property P. Hence, none of these conditions is necessary for multiplication to be universally defined with respect to some particular unit. On the other hand, property P is a necessary condition for multiplication to be universally defined with respect to *every* unit in L (cf. **11**, Theorem 7.3). It is also true that every element in L has an inverse if and only if L has property P (cf. **11**, Theorems 7.1 and 7.2).

A. G. Pinsker has shown (see **8**; **9**) how a Dedekind complete vector lattice L may be embedded as an order-dense order ideal in a certain Dedekind complete space $L^\#$ which turns out to have property P. His construction of $L^\#$ is, essentially, to adjoin to L the suprema of sets $S \subset L^+$ satisfying the conditions of property P_2. More precisely (for details, see **8**; **9**; or **11**, § 8): A subset $X \subset L^+$ will be called a *section* if $y \in X$ whenever $0 \leqq y \leqq x \in X$, and if X is closed in the sense that: $\{x_\alpha\} \subset X$ and $x_\alpha \leqq x \in L$ for all α implies $\sup(x_\alpha) \in X$. Let $\bar{L}$ be the collection of sections of L. An order can be defined in $\bar{L}$ by: $X \leqq Y$ if $X \subset Y$; denote $\mathbf{0} = \{0\}$, thus $X \geqq \mathbf{0}$ always. For $a \geqq 0$ we define $aX = \{ax: x \in X\}$, and $X + Y$ is defined by $X + Y = \{x + y: x \in X, y \in Y\}$; these two sets are again sections. We embed $L^+ \to \bar{L}$ by $0 \leqq x \to \{y: 0 \leqq y \leqq x\}$; thus we may consider L^+ a subset of $\bar{L}$.

For $X, Y, Z \in \bar{L}$, it is *not* necessarily true that $X + Z = Y + Z$ implies $X = Y$ (e.g., consider $Z = L^+$). However, this is true if we restrict ourselves to *locally bounded sections*: a section $X \in \bar{L}$ will be called locally bounded if for every $0 < x \in L$ there exists a real number b such that $bx \nleqq X$ (i.e., $bx \notin X$). Let $L^{\#+}$ be the set of locally bounded sections; then for $X, Y, Z \in L^{\#+}$, $X + Z = Y + Z$ implies $X = Y$; and furthermore, for $Y \leqq Z \in L^{\#+}$ there exists a unique element $X \in L^{\#+}$ such that $Y + X = Z$. Thus $L^{\#+}$ is the positive part of a partially ordered linear space $L^\#$; and it turns out that $L^\#$ is a Dedekind complete vector lattice with property P, and that L is embedded in $L^\#$ as an order-dense ideal.

Remarks. 1. $L^\#$ is, in a sense, both a minimal and maximal extension of L. More precisely (cf. **11**, Theorem 8.5): If L has property P, and is an order-dense ideal in an Archimedean vector lattice E, then $L = E$; in particular, $L = L^\#$ if L has property P, and always $L^\# = (L^\#)^\#$. On the other hand, if L is an order-dense ideal in a Dedekind complete vector lattice E with property P, then $L^\# = E$.

2. Nakano, by a different construction, has shown (**7**, Theorem 34.4) how to imbed L in a space with property P_3 (his "universal completion"), which must then (by Remark 1 above) be the same as Pinsker's extension.

3. Vulih refers to this imbedding $L \subset L^\#$, showing that multiplication is universally defined in $L^\#$ and that every element in $L^\#$ has an inverse, but he does not isolate the implicit necessary and sufficient condition (property P).

It is useful to note that now we can easily define multiplication in a Dedekind complete vector lattice L not necessarily having a unit. For $L^{\#}$ has a unit and universal multiplication with respect to it, so we may say: for $x, y \in L$, if xy (which exists in $L^{\#}$) is in L, then the product of x and y is defined and equals xy. It is easy to verify that the multiplication thus defined in L satisfies properties M(ii) to M(viii) and also M(xii).

4. Abstract integral spaces. We now take L to be a Dedekind complete vector lattice, not necessarily having a unit. Let ϕ be a non-negative normal integral on L (i.e., a non-negative linear functional such that if a set $\{x_\alpha\} \subset L$ is directed down to 0, $x_\alpha \downarrow 0$, then $\phi\{x_\alpha\} \downarrow 0$). As usual, x, y, and z will denote elements of L and f, g, and h will denote elements of $L^{\#}$.

We define a new functional $\phi^{\#}$ on $L^{\#+}$ as follows: for $0 \leqq f \in L^{\#+}$, $\phi^{\#}(f) = \sup(\phi(x): x \in L, 0 \leqq x \leqq f)$. $\phi^{\#}(f)$ may equal $+\infty$, but for $0 \leqq x \in L$, $\phi^{\#}(x) = \phi(x)$.

LEMMA 4.1 (cf. **6**, Theorem 30.6 in Note IX). *If* $0 \leqq f_\alpha \uparrow f \in L^{\#+}$, *then* $\phi^{\#}(f) = \sup \phi^{\#}(f_\alpha)$.

Proof. Assume first that $\phi^{\#}(f) < \infty$. Then, given $\epsilon > 0$, there exists $x \in L$ such that $\phi^{\#}(f) \leqq \phi(x) + \epsilon$. Let $x_\alpha = \inf(f_\alpha, x) \leqq f_\alpha$. Then $x_\alpha \in L$ and $x_\alpha \uparrow x$, so $\phi(x_\alpha) \uparrow \phi(x)$. Thus $\sup \phi^{\#}(f_\alpha) + \epsilon \geqq \phi^{\#}(f)$.

If $\phi^{\#}(f) = \infty$, then for any N there exists $x \leqq f$ such that $\phi(x) > N$. Now, $\inf(x, f_\alpha) \uparrow x$, therefore $\sup \phi^{\#}(f_\alpha) \geqq \phi(x) > N$. Hence $\phi^{\#}(f_\alpha) \uparrow \infty$.

LEMMA 4.2. $\phi^{\#}$ *is additive on* $L^{\#+}$.

Proof. Let $f, g \in L^{\#+}$. Every $z \in L^+$ with $z \leqq f + g$ can be written $z = x + y$ with $f \geqq x \in L^+$ and $g \geqq y \in L^+$, and so

$$\begin{aligned} \phi^{\#}(f+g) &= \sup(\phi(x+y): 0 \leqq x \leqq f, 0 \leqq y \leqq g) \\ &= \sup(\phi(x): 0 \leqq x \leqq f) + \sup(\phi(y): 0 \leqq y \leqq g) \\ &= \phi^{\#}(f) + \phi^{\#}(g). \end{aligned}$$

Since $\phi^{\#}$ is an extension of ϕ, we may (when confusion does not result) write ϕ for $\phi^{\#}$. Let us now suppose that ϕ, and hence $\phi^{\#}$, is strictly positive. We define $L_1(\phi, L) = L_1(\phi) = L_1 = \{f \in L^{\#}: \phi(|f|) < \infty\}$. A norm is defined on $L_1(\phi)$ by: $||f||_1 = \phi(|f|)$. (This is a norm rather than a seminorm since ϕ is strictly positive.) ϕ can then be extended to all of $L_1(\phi)$ by defining $\phi(f) = \phi(f_+) - \phi(f_-)$. We note that L is an ideal in $L^{\#}$ and that, by Lemmas 4.1 and 4.2, ϕ (i.e., $\phi^{\#}$) is a strictly positive normal integral on L_1. The next theorem is the key to showing that $L_1(\phi)$ (and later $L_2(\phi)$) is complete.

THEOREM 4.3. *If* $0 \leqq f_\alpha \uparrow \in L_1(\phi)$ *and* $\sup||f_\alpha||_1 < \infty$, *then there exists* $\sup(f_\alpha) \in L_1(\phi)$.

Proof. First we use property P_2 to show that there exists $\sup(f_\alpha) \in L^\#$. Let $0 < g \in L^\#$, and suppose that for every b

$$\sup_\alpha \inf(bg, f_\alpha) = bg.$$

Then

$$b\phi(g) = \phi(bg) = \phi\left(\sup_\alpha \inf(bg, f_\alpha)\right) = \sup_\alpha \phi(\inf(bg, f_\alpha)) \leqq \sup_\alpha \phi(f_\alpha) < \infty.$$

But since $\phi(g) > 0$, this cannot be true for every b, i.e., there must exist b such that

$$\sup_\alpha \inf(bg, f_\alpha) < bg.$$

But then, since $L^\#$ has property P_2, there exists $f = \sup(f_\alpha) \in L^\#$.

Then to show $f \in L_1(\phi)$ we only have to notice that by Lemma 4.1, $\phi(f) = \sup(f_\alpha) < \infty$.

THEOREM 4.4. *$L_1(\phi)$ is complete (in the norm $||\cdot||_1$).*

Proof. Suppose $0 \leqq f_n \uparrow \in L_1$ and $\sup||f_n||_1 < \infty$. Then the theorem above implies that $\sup(f_n)$ exists in L_1. But this is exactly the criterion of Amemiya (**1**) that a normed vector lattice be complete. (Cf. also **6**, Theorem 5.3 in Note II, and Theorem 26.3 in Note VIII.)

More generally, if ϕ is *not* strictly positive, decompose $L = C_\phi \oplus N_\phi$ (where N_ϕ is the null ideal of ϕ and $C_\phi = N_\phi{}^\perp$ is the carrier or support of ϕ; cf. (**6**, pp. 107–108 in Note VIII)). Since L is an order-dense ideal in $L^\#$, this decomposition induces a decomposition $L^\# = C_\phi{}^\# \oplus N_\phi{}^\#$ with ϕ zero on $N_\phi{}^\#$, and $C_\phi{}^\# = N_\phi{}^{\#\perp}$. ϕ is strictly positive on C_ϕ, so we may define $L_1(\phi, L)$ in general to be $L_1(\phi, C_\phi)$. By an abuse of language we shall sometimes say that $f \in L_1(\phi, L)$ if the component of f in $C_\phi{}^\#$ is in $L_1(\phi, C_\phi)$. For ϕ strictly positive we may also define $L_2(\phi, L) = L_2(\phi) = L_2 = \{f \in L^\#: \phi(f^2) < \infty\}$. We can see that L_2 is a linear subspace of $L^\#$, for

$$(f + g)^2 = f^2 + g^2 + 2fg \leqq 2(f^2 + g^2),$$

so $\phi((f + g)^2) \leqq 2(\phi(f^2) + \phi(g^2))$, and hence $f, g \in L_2$ implies $(f + g) \in L_2$. Also, since $fg \leqq \frac{1}{2}(f^2 + g^2)$, then $|\phi(fg)| \leqq \frac{1}{2}(\phi(f^2) + \phi(g^2)) < \infty$, thus we may define in L_2 an inner product $(f, g) = \phi(fg)$ and a norm $||f||_2 = (f, f)^{1/2}$. ($||\cdot||_2$ is a norm rather than a seminorm since $\phi(f^2) = 0$ implies $f^2 = 0$ which implies $f = 0$.)

THEOREM 4.5. *$L_2(\phi)$ is a Hilbert space.*

Proof. We only have to prove that L_2 is complete in the norm $||\cdot||_2$. Suppose $0 \leqq f_n \uparrow \in L_2$ and $\sup||f_n||_2 < \infty$. Then $0 \leqq f_n{}^2 \uparrow \in L_1$ and

$$\sup||f_n{}^2||_1 = \sup||f_n||_2{}^2 < \infty,$$

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thus by Theorem 4.3 there exists $g = \sup(f_n^2) \in L_1$. But

$$f_n \leqq \sup(f_n^2, 1) \leqq \sup(g, 1) \in L^\#,$$

thus there exists $\sup(f_n) \in L^\#$, and by the continuity of the product and uniqueness of the square root (properties M(xii) and M(xi)) we have $\sup(f_n) = g^{1/2} \in L_2$. Thus by Amemiya's theorem (**1**), L_2 is complete.

5. The Radon-Nikodym theorem. Let ϕ be a (non-negative) normal integral on the Dedekind complete vector lattice L, and let ψ be a (non-negative) normal integral on some subspace $E \subset L^\#$. Then ψ is said to be *absolutely continuous with respect to* ϕ if $L_1(\psi) \oplus N_\psi$ is order dense in $L^\#$, and for every $0 \leqq f \in L^\#$, $\phi(f) = 0$ implies $\psi(f) = 0$.

Note. Requiring that $L_1(\psi) \oplus N_\psi$ be dense in $L^\#$ is equivalent to the more usual condition (cf. **14**, p. 134) that ϕ and ψ be initially defined on the same space, for we may regard $(L_1(\phi) \oplus N_\phi) \cap (L_1(\psi) \oplus N_\psi)$ as the initial domain of ϕ and ψ, and this is order dense in $L^\#$.

THEOREM 5.1. *Let ϕ be a normal integral on L, and let $0 \leqq g \in L^\#$. Define ψ on $L^{\#+}$ by $\psi(f) = \phi(fg)$ for all $0 \leqq f \in L^{\#+}$, and then on some subspace $E \subset L$ by $\psi(f) = \psi(f_+) - \psi(f_-)$ whenever $\psi(f_+)$ and $\psi(f_-)$ are finite. Then ψ is a normal integral, absolutely continuous with respect to ϕ.*

Proof. Since ϕ is normal and multiplication is (0)-continuous, ψ is a normal integral on $L_1(\psi) \oplus N_\psi = \{f \in L^\#: \psi(|f|) < \infty\}$. Next, if $\phi(f) = 0$, then $f \in N_\phi^\# = (C_\phi^\#)^\perp$, and hence $fg \in (C_\phi^\#)^\perp = N_\phi^\#$, i.e., $\psi(f) = \phi(fg) = 0$.
Finally, we must show that, given $0 < f \in L^\#$, there exists

$$0 < h \in L_1(\psi) \oplus N_\psi$$

such that $h \leqq f$. But if $f > 0$, then there exists $0 < f_1 \in L_1(\phi) \oplus N_\phi$ with $0 < f_1 \leqq f$, and $e > 0$, $a > 0$, and $0 \leqq b < \infty$ such that $0 < ae \leqq f_1$ and $ge \leqq be$. It follows that $\psi(ae) = \phi(aeg) \leqq \phi(abe) \leqq b\phi(f_1) < \infty$, and hence ae is a suitable element in $L_1(\psi) \oplus N_\psi$.

The main object in this section is to prove a converse to the preceding theorem. First we prove a special case. (Notice that the proof parallels very closely that given in (**14**) for measure spaces. Other classical proofs can also be adapted to this abstract situation.)

THEOREM 5.2. *Let L be a Dedekind complete vector lattice with a unit* **1**. *Let ϕ be a strictly positive normal integral on L, and let $0 \leqq \psi$ be any normal integral on L. Then there exists a unique $0 \leqq g \in L_1(\phi)$ such that $f \in L_1(\psi)$ if and only if $fg \in L_1(\phi)$, and $\psi(f) = \phi(fg)$ for every $f \in L_1(\psi)$.*

Proof. (i) Define ω on L by $\omega = \phi + \psi$. ω is clearly a strictly positive normal integral on L. We must verify that $\omega^\# = \phi^\# + \psi^\#$. For $0 \leqq f \in L^\#$,

$$(\phi + \psi)^{\#}(f) = \sup((\phi + \psi)(x) \colon 0 \leqq x \leqq f) = \sup(\phi(x) + \psi(x) \colon 0 \leqq x \leqq f) \leqq \phi^{\#}(f) + \psi^{\#}(f).$$

On the other hand, $0 \leqq x, y \leqq f$ implies $z = \sup(x, y) \leqq f$; thus,

$$\phi(x) + \psi(y) \leqq (\phi + \psi)(z),$$

therefore

$$\phi^{\#}(f) + \psi^{\#}(f) = \sup(\phi(x) + \psi(y) \colon 0 \leqq x, y \leqq f) \leqq \sup((\phi + \psi)(z) \colon 0 \leqq z \leqq f) = (\phi + \psi)^{\#}(f).$$

Since, then, $\omega^{\#} = \phi^{\#} + \psi^{\#}$, we shall henceforth omit the $\#$ on ϕ, ψ, and ω.

(ii) Consider the Hilbert space $L_2(\omega)$. For $f \in L_2(\omega)$ we have

$$|\psi(f)| \leqq \psi(|f|) \leqq \omega(|f|) = (|f|, \mathbf{1}) \leqq ||f||_2 \, ||\mathbf{1}||_2$$

by the Schwarz inequality. Thus ψ is a bounded linear functional on $L_2(\omega)$, therefore there exists $h \in L_2(\omega)$ such that

$$\psi(f) = (f, h) = \omega(fh) = \phi(fh) + \psi(fh)$$

for all $f \in L_2(\omega)$. Since $L_2(\omega)$ is order-dense in $L^{\#}$, the same equation holds for any $0 \leqq f \in L^{\#}$.

(iii) We prove now several facts about h. First of all, $h \geqq 0$, for, taking $f = s(h_-)$ in the above we have that

$$0 \leqq \psi(s(h_-)) = \omega(s(h_-)h) = \omega(-h_-) \leqq 0,$$

and hence $h_- = 0$.

Secondly, $s[(\mathbf{1} - h)_+] = \mathbf{1}$. For if not, then there exists $e > 0$ such that $e \perp s[(\mathbf{1} - h)_+]$, and then $e - he = (\mathbf{1} - h)e \leqq 0$, i.e., $he \geqq e$. But then $\psi(e) = \omega(eh) \geqq \omega(e) = \phi(e) + \psi(e) \geqq \psi(e)$; hence equality holds throughout, and thus $\phi(e) = 0$, a contradiction since ϕ is strictly positive. Note that it follows immediately from $s[(\mathbf{1} - h)_+] = \mathbf{1}$ that $\mathbf{1} - h = (\mathbf{1} - h)_+ \geqq 0$, i.e., $h \leqq \mathbf{1}$; but this is a weaker statement.

(iv) Now we use the fact that every element in $L^{\#}$ has an inverse to define $g = h(\mathbf{1} - h)^{-1} \in L^{\#}$. Since $\mathbf{1} - h \geqq 0$ and $s(\mathbf{1} - h) = \mathbf{1}$ we have that $(\mathbf{1} - h)^{-1} \geqq 0$ so that $g \geqq 0$ and $(\mathbf{1} - h)^{-1}(\mathbf{1} - h) = \mathbf{1}$.

Consider any $0 \leqq f \in L^{\#}$. Noting that $f(\mathbf{1} - h)^{-1} \in L^{\#+}$ we have that

$$\phi(fg) = \phi(f(\mathbf{1} - h)^{-1}h) = \psi(f(\mathbf{1} - h)^{-1}) - \psi(f(\mathbf{1} - h)^{-1}h) = \psi(f(\mathbf{1} - h)^{-1}(\mathbf{1} - h)) = \psi(f).$$

This equation shows that g is a suitable element in $L^{\#}$. It also shows that $f \in L_1(\psi)$ if and only if $fg \in L_1(\phi)$ and, in particular, taking $f = \mathbf{1}$, it shows that $g \in L_1(\phi)$.

(v) Finally, we show that g is unique. Suppose there also exists g' such that $\psi(f) = \phi(fg')$ for $f \in L_1(\psi)$. Let $e = s[(g - g')_+]$. Then

$$\phi(eg') = \psi(e) = \phi(eg),$$

so $0 = \phi(ge - ge') = \phi((g - g')e) = \phi((g - g')_+)$, and hence $(g - g')_+ = 0$, i.e., $g \leqq g'$. Similarly, $g' \leqq g$, therefore $g' = g$.

THEOREM 5.3 (Radon-Nikodym). *Let L be a Dedekind complete vector lattice, ϕ a (non-negative) normal integral on L, and ψ a (non-negative) normal integral absolutely continuous with respect to ϕ. Then there exist a unit $\mathbf{1} \in L^\#$ and an element $0 \leqq g \in L^\#$ such that $f \in L_1(\psi)$ if and only if $fg \in L_1(\phi)$, and $\psi(f) = \phi(fg)$ for every $f \in L_1(\psi)$. g is unique in the sense that its component in $C_\phi^\#$ is uniquely determined as soon as the unit $\mathbf{1}$ is determined.*

Proof. Write $L = C_\phi \oplus N_\phi$. ϕ is zero on N_ϕ, thus, by absolute continuity, ψ is also zero on N_ϕ, i.e., we may consider ψ simply as a normal integral on C_ϕ. And ϕ is strictly positive on C_ϕ.

Let $\{x_\alpha\}$ be a maximal collection of mutually disjoint positive elements of C_ϕ, and take $\sup(x_\alpha)$ (which exists in $L^\#$ by property P_3) as a unit for $C_\phi^\#$. We have that $C_\phi = \cup \oplus [x_\alpha]$ and $C_\phi^\# = \cup \oplus [x_\alpha]^\#$ (where $\cup \oplus [x_\alpha]$ denotes the smallest normal subspace of L containing all the normal subspaces $[x_\alpha]$).

For each α, $[x_\alpha]$ is a Dedekind complete vector lattice with a unit x_α, and on $[x_\alpha]$, ϕ acts as a strictly positive normal integral. Thus, by Theorem 5.2, there exists a unique $0 \leqq g_\alpha \in [x_\alpha]^\#$ such that $\psi(f_\alpha) = \phi(f_\alpha g_\alpha)$ for every $0 \leqq f_\alpha \in [x_\alpha]^\#$. Let $0 \leqq g = \sup(g_\alpha) \in C_\phi^\#$ (again, g exists since $C_\phi^\#$ has property P_3). For any $0 \leqq f \in C_\phi^\#$ (whose component in $[x_\alpha]$ is f_α) the component of fg in $[x_\alpha]$ is $f_\alpha g_\alpha$, for

$$\begin{aligned}(fg)_\alpha &= fg \cdot x_\alpha \quad \text{(by property M(xiii))}\\ &= (fx_\alpha)(gx_\alpha) = f_\alpha g_\alpha.\end{aligned}$$

But then $\psi(f) = \sum_\alpha \psi(f_\alpha) = \sum_\alpha \phi(f_\alpha g_\alpha) = \sum_\alpha \phi((fg)_\alpha) = \phi(fg)$. The theorem now follows immediately; in particular, the uniqueness of g follows from the uniqueness of each g_α.

Remark. The proof above depends on picking a particular unit for $L^\#$. Actually, however, the formula for a change of units shows that the theorem is true for multiplication with respect to any unit of $L^\#$.

6. Segal's theorem. It is interesting to note that in Theorem 5.3, no condition such as σ-finiteness is required. In this section we use this fact to give a new proof of Segal's theorem (**12**), that the Radon-Nikodym theorem holds in a measure space with no purely infinite sets if and only if the measure algebra is localizable, i.e., complete as a lattice. (The Radon-Nikodym theorem is said to hold in a given measure space (X, S, μ) if for any integral ψ, absolutely continuous with respect to the integral $\int \cdot \, d\mu$, there exists a μ-unique measurable function g such that $\psi(f) = \int fg \, d\mu$ for every ψ-integrable f.)

The proof proceeds essentially as follows: $L_1(X, S, \mu)$ can be embedded in the space of measurable functions M, but it can also be thought of as an abstract vector lattice L and embedded in $L^\#$. It turns out that the Radon-Nikodym theorem holds in (X, S, μ) if and only if M and $L^\#$ are isomorphic, which occurs if and only if the measure algebra of (X, S, μ) is localizable.

In detail: (i) Let (X, S, μ) be a measure space. We may suppose that μ is already extended by the Carathéodory procedure, so that S is the σ-algebra of measurable sets. Let S_0 be the subring of measurable sets with finite measure. We shall *assume* that there are no purely infinite sets, i.e., if E is a measurable set with $\mu(E) > 0$, then there exists a measurable set $K \subset E$ such that $0 < \mu(K) < \infty$. It follows immediately from this that if $F \subset X$ is such that $\mu(F \cap K) = 0$ for all $K \in S_0$, then $F \in S$ and $\mu(F) = 0$.

As usual, two sets $E, F \in S$ are said to be equivalent if $E \triangle F$ is a null set. We shall denote by E^* the equivalence class of sets equivalent to E, and by B the collection of equivalence classes. Then B is a σ-algebra, the mapping $E \to E^*$ is a σ-algebra homomorphism, and μ may be considered as a measure on B by setting $\mu(E^*) = \mu(E)$. The system (B, μ) is the measure algebra of the measure space (X, S, μ).

Let B_0 be the subalgebra of B consisting of those elements which have finite measure. Since X has no purely infinite sets we can see that for any $E^* \in B$, $E^* = \sup(K^*: K^* \in B_0, K^* \leqq E^*)$; indeed, E^* is certainly an upper bound for all such K^*, and if F^* is also an upper bound, then $F^* \geqq E^* \cap K^*$ for all $K^* \in B_0$, so that $(E^* - F^*) \cap K^* = (E^* \cap K^*) - F^* = 0$ for all $K^* \in B_0$, and hence $E^* - F^* = 0$, i.e., $E^* \leqq F^*$ as required. Thus B_0 is order-dense in B.

(ii) Let $L = L_1(X, S, \mu)$ be equivalence classes of integrable functions modulo null functions. Denote by f^* the equivalence class of functions equal to f almost everywhere. L is a σ-Dedekind complete vector lattice with an integral ϕ defined by $\phi(f^*) = \int f\, d\mu$ for $f^* \in L$. ϕ is strictly positive on L, hence L is Dedekind complete (in fact, super-Dedekind complete) and ϕ is a normal integral (cf. **6**, Lemma 27.16 in Note VIII).

(iii) Embed $L \subset L^\#$. For a unit in $L^\#$, let $\mathbf{1} = \sup(e_\alpha)$, where e_α is the element of L determined by the characteristic function of E_α for $E_\alpha \in S_0$. Note that this unit is suitable for use in the Radon-Nikodym theorem. Also recall that $U(L^\#, \mathbf{1})$ is a complete Boolean algebra.

(iv) We want to define a measure-preserving isomorphism ρ of B into $U(L^\#)$. For $E^* \in B_0$ define $\rho(E^*)$ to be the element in L determined by χ_E. We note that $\rho(B_0)$ is order-dense in $U(L^\#)$: indeed, if $0 < e \in U(L^\#)$, then (since L is order-dense in $L^\#$) there exists $x \in L$ such that $0 < x \leqq e$; we may take E to be a measurable set of finite measure which is contained in the support of an integrable function determining x, and then $\rho(E^*) \leqq s(x) \leqq e$, as required.

Since B_0 is order-dense in the Boolean algebra B, and $\rho(B_0)$ is order-dense in the complete Boolean algebra $U(L^\#)$, ρ can be extended uniquely to an (algebraic) isomorphism of B into $U(L^\#)$, and the extension (again denoted by ρ) maps B onto $U(L^\#)$ if and only if B is complete, i.e., if and only if μ

is localizable (cf. **12**, Lemma 3.3.2). It is easy to see that ρ is measure-preserving, i.e., $\phi(\rho(E^*)) = \mu(E^*)$ for all $E^* \in B$; indeed, if $\mu(E^*) < \infty$, then this is true by definition, and if $\mu(E^*) = \infty$, then there are elements $K^* \leqq E^*$ with finite but arbitrarily large measures so that $\phi(\rho(E^*)) \geqq \phi(\rho(K^*)) \uparrow \infty$. We also note that if $e \in U(L^\#)$ is such that $\phi(e) < \infty$, then there exists $E^* \in B_0$ such that $\rho(E^*) = e$; indeed, since $\rho(B_0)$ is dense in $U(L^\#)$ and ϕ is strictly positive, there is a sequence $\{\rho(E^*_n)\}$ such that $\rho(E^*_n) \leqq e$ and $\phi(\rho(E^*_n)) \uparrow \phi(e)$, so that $\rho(E^*_n) \uparrow e$ and hence $\rho(\sup E^*_n) = \sup \rho(E^*_n) = e$.

(v) Let M denote equivalence classes of measurable functions modulo null functions. The map $\rho: B \to U(L^\#)$ induces in a natural way an algebraic isomorphism ρ^* of M into $L^\#$ as follows: for every measurable function $f \geqq 0$ we have $f^* = \sup(a\chi^*_E: 0 \leqq a\chi^*_E \leqq f^*)$. The set

$$\{a\rho(E^*): 0 \leqq a\chi^*_E \leqq f^*\} \subset L^{\#+}$$

satisfies the conditions of property P_2, therefore we may define, for $0 \leqq f^* \in M$, $\rho^*(f^*) = \sup(a\rho(E^*): 0 \leqq a\chi^*_E \leqq f^*) \in L^\#$. In general, we define

$$\rho^*(f^*) = \rho^*(f^*_+) - \rho^*(f^*_-).$$

It is clear that ρ^* is measure-preserving in the sense that, for $0 \leqq f^* \in M$, $\phi(\rho^*(f^*)) = \int f\,d\mu$. In fact, ρ^* is an extension of the identity map of $L \to L$. We can even see that ρ^* maps $L_1(X, S, \mu)$ onto $L_1(\phi, L)$: for, given $0 \leqq f^\# \in L_1(\phi, L)$, we have $f^\# = \sup(r \cdot k_r(f^\#)$: rational $r > 0)$ by Theorem 2.1. But $\phi(k_r(f^\#)) \leqq r^{-1}\phi(f^\#) < \infty$, thus there exists $E^* \in B_0$ such that $\rho(E^*) = k_r(f^\#)$, and hence $\rho^*(r\chi^*_E) = rk_r(f^\#)$. The set $\{r \cdot k_r(f^\#)$: rational $r > 0\}$ is countable, thus there exists $f^* = \sup\{(\rho^*)^{-1}(r \cdot k_r(f^\#))\} \in M$, and $\rho^*(f^*) = f^\#$. In addition, $\int f^* d\mu = \phi(f^\#) < \infty$, therefore $f^* \in L_1(X, S, \mu)$. Thus $L_1(X, S, \mu)$ and $L_1(\phi, L)$ are identical, and, in particular, there is no confusion in saying that one integral is absolutely continuous with respect to another without specifying which space is being considered.

Note that $\rho(X^*) = \mathbf{1}$, so that by the uniqueness of multiplication, ρ^* is also an isomorphism of the multiplicative structure.

(vi) We have, in general, that $\rho^*(M) \subset L^\#$, and we want to show that equality holds if and only if μ is localizable. In one direction this is clear, for if ρ^* maps M onto $L^\#$, then ρ maps B onto $U(L^\#)$, and hence μ is localizable. Conversely, suppose μ is localizable, so that ρ maps B onto $U(L^\#)$. Then for any $e \in U(L^\#)$ there exists $\rho^{-1}(e) = E^* \in B$, thus for any element of the form $ae \in L^\#$ there exists $(\rho^*)^{-1}(ae) = a\chi^*_E \in M$. Now suppose that $0 \leqq f^\# \in L^\#$. Again we have that $f^\# = \sup(r \cdot k_r(f^\#)$: rational $r > 0)$ and the set $\{r \cdot k_r(f^\#)$: rational $r > 0\}$ is countable, thus there exists

$$f^* = \sup((\rho^*)^{-1}(r \cdot k_r(f^*))) \in M,$$

and $\rho^*(f^*) = f^\#$. Thus ρ^* maps M onto $L^\#$ as required.

(vii) Now suppose μ is localizable. Then M is isomorphic to $L^\#$ and hence the Radon-Nikodym theorem holds in M since it holds in $L^\#$. Conversely, suppose the Radon-Nikodym theorem holds in the measure space. For any $0 \leqq g^{\#} \in L^{\#}$ we want to find $g^* \in M$ such that $\rho^*(g^*) = g^{\#}$. To do this, define the normal integral ψ by $\psi(f^{\#}) = \phi(f^{\#}g^{\#})$. ψ is absolutely continuous with respect to ϕ, and thus, the Radon-Nikodym theorem for $L_1(X, S, \mu)$ implies that there exists $g^* \in M$ such that $\psi(f^*) = \phi(f^*g^*)$ for all $0 \leqq f^* \in M$. Then, considering ϕ and ψ as integrals on $L_1(\phi, L)$ again, we have $\psi(f^{\#}) = \phi(f^{\#}\cdot\rho(g^*))$ for all $0 \leqq f^{\#} \in L^{\#}$, and hence, by the uniqueness of the Radon-Nikodym derivative, $\rho^*(g^*) = g^{\#}$ as required. Thus ρ^* maps M onto $L^{\#}$, and hence, by (vi), μ is localizable.

Note. Zaanen (**15**) gives a discussion of Segal's theorem along somewhat different lines. He also shows that, if the measure space has purely infinite sets, then the Radon-Nikodym theorem holds if and only if the contracted measure is localizable.

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Queen's University,
Kingston, Ontario