Canad. Math. Bull. Vol. **50** (1), 2007 pp. 113–122

Hermitian Harmonic Maps into Convex Balls

ZhenYang Li and Xi Zhang

Abstract. In this paper, we consider Hermitian harmonic maps from Hermitian manifolds into convex balls. We prove that there exist no non-trivial Hermitian harmonic maps from closed Hermitian manifolds into convex balls, and we use the heat flow method to solve the Dirichlet problem for Hermitian harmonic maps when the domain is a compact Hermitian manifold with non-empty boundary.

1 Introduction

Let (M, h) be a Hermitian manifold with Hermitian metric $(h_{\alpha\bar{\beta}})$, and let (N, g) be a Riemannian manifold with metric (g_{ij}) and Christoffel symbols Γ^i_{jk}. A Hermitian harmonic map $u\colon M \to N$ satisfies the following elliptic system

$$h^{\alpha\bar{\beta}} \frac{\partial^2 u^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + h^{\alpha\bar{\beta}} \Gamma^i_{jk} \frac{\partial u^j}{\partial z^\alpha} \frac{\partial u^k}{\partial z^{\bar{\beta}}} = 0. \tag{1.1}$$

This system is more appropriate to Hermitian geometry than the harmonic map system since it is compatible with the holomorphic structure of the domain manifold, in the sense that holomorphic maps are Hermitian harmonic maps when target manifolds are Kähler. Since (1.1) does not have a divergence structure nor a variational structure, it is analytically more difficult than a harmonic system. It was first studied by Jost and Yau [5], and was applied to study the rigidity of compact Hermitian manifolds. Jost and Yau [5] consider the existence problems of Hermitian harmonic maps under the assumption that the target manifold N is nonpositively curved. Chen [1] also studied the situation that the target manifold has non-empty boundary. In this paper, we consider the case where the target manifolds are convex balls.

Let N be a complete Riemannian manifold with sectional curvature bounded above by a positive constant k, and $B_R(O)$ be a geodesic ball of radius R with center at fixed point $O \in N$. If $R < \frac{\pi}{2\sqrt{k}}$, and $B_R(O)$ lies in the cut locus of O, then the geodesic ball $B_R(O)$ will be called by a convex ball.

Now let us fix some notation. Assume that N is a Riemannian manifold. On N we always choose the Levi–Civita connection which is compatible with the Riemannian structure. On M we now choose the connection $\widetilde{\nabla}$ such that it is compatible with the holomorphic structure on M and torsion free. We denote the standard Beltrami–Laplacian by $\triangle$ and the Laplacian of the holomorphic torsion free connection by $\widetilde{\triangle}$,

Received by the editors December 15, 2004.
The second author was supported by the NSF in China, grant no. 10201028
AMS subject classification: 58E15, 53C07.
Keywords: Hermitian harmonic map, Hermitian manifold, convex ball.

respectively. Then one can define $\nabla du(X,Y)$ by $\nabla du(X,Y) = \nabla_Y du(X) - du(\widetilde{\nabla}_X Y)$ for any smooth map u from M to N. The torsion free assumption makes the above defined $\nabla du(\cdot\,,\cdot)$ symmetric. And it is natural to define the tension fields of the map u as

(1.2)
$$\sigma(u) = \sigma(u)^i \frac{\partial}{\partial u^i} = h^{\alpha\bar{\beta}} \nabla du\Big(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^{\bar{\beta}}}\Big) = h^{\alpha\bar{\beta}} \Big(\frac{\partial^2 u^i}{\partial z^\alpha \partial z^{\bar{\beta}}} + \Gamma^i_{jk} \frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar{\beta}}}\Big) \frac{\partial}{\partial u^i}$$

If u is a function on M, then

$$\widetilde{\triangle} u = h^{\alpha\bar{\beta}} \frac{\partial^2 u}{\partial z^\alpha \partial z^{\bar{\beta}}}, \tag{1.3}$$

$$\widetilde{\triangle} u - \frac{1}{2}\triangle u = \langle V, \nabla u\rangle, \tag{1.4}$$

where V is a well-defined vector field on M and ∇ is the Levi–Civita connection on M.

After giving some preliminaries in Section 2, we will discuss the case where domain manifolds are compact. First we prove that there exist no non-trivial Hermitian harmonic maps from closed Hermitian manifolds into convex balls. In fact, we obtain the following theorem:

Theorem 1.1 *Let M be a closed (compact, without boundary) Hermitian manifold, and N a complete Riemannian manifold with sectional curvature bounded above by a positive constant k. Let $u\colon M \to N$ be a Hermitian harmonic map with image $u(M) \subset B_R(O)$. If $R < \frac{\pi}{2\sqrt{k}}$, and $B_R(O)$ lies in the cut locus of O, then u must be a constant map.*

Secondly, we consider the case that the domain manifold has a non-empty smooth boundary. We use the heat flow method to prove the solubility of Dirichlet problem for Hermitian harmonic maps. We obtain:

Theorem 1.2 *Let M be a compact Hermitian manifold with non-empty smooth boundary ∂M, and N be a complete Riemannian manifold with sectional curvature bounded above by a positive constant k. Let $\phi\colon M \to N$ be a smooth map, and the image $\phi(M) \subset B_R(O)$. If*

$$R < \frac{\arccos \frac{2\sqrt{5}}{5}}{\sqrt{k}}$$

and $B_R(O)$ lies in the cut locus of O, then there must exist one unique Hermitian harmonic map u such that $u|_{\partial M} = \phi|_{\partial M}$.

Remark We hope that the condition $R < \frac{\arccos \frac{2\sqrt{5}}{5}}{\sqrt{k}}$ can be weakened to $R < \frac{\pi}{2\sqrt{k}}$, but for technical reasons, in this paper we can only solve the Dirichlet problem for Hermitian harmonic maps under this stronger condition.

Similarly to harmonic maps, we can consider Hermitian harmonic maps from complete Hermitian manifolds into convex balls. The existence of Hermitian harmonic maps from complete Hermitian manifolds into Riemannian manifolds with non-positive curvature had been investigated by Lei Ni [6]. Grunau and Kühnel [2] introduced an invertibility condition on the holomorphic Laplace operator between suitable chosen function spaces. Their proof shows that the solubility of the Poisson equation with respect to the holomorphic Laplace operator ensures the existence of Hermitian harmonic maps. So, it is natural to prove the existence of Hermitian harmonic maps from complete Hermitian manifolds satisfying the same conditions as in [2, 6] into convex balls by using Theorem 1.2 and the compact exhaustion method. The proof is similar to the one given in [2, 6]; hence we omit it here.

2 Preliminary Results

We will solve (1.1) by the method of heat flow. Let $u\colon M \times \mathbf{R} \to N$. We consider the following parabolic system:

$$\begin{aligned} &h^{\alpha\bar\beta}\frac{\partial^2 u^i}{\partial z^\alpha \partial z^{\bar\beta}} + h^{\alpha\bar\beta}\Gamma^i_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial z^{\bar\beta}} = \frac{\partial u^i}{\partial t} \\ &u(z,0) = \phi(z) \text{ for } z \in M \\ &u(z,t) = \phi(z) \text{ for } z \in \partial M, 0 \le t \le \infty \end{aligned} \tag{2.1}$$

where ϕ is a smooth map from M to N such that $\phi(M)$ is contained in the convex ball $B_R(O)$. By linearizing and using results about linear parabolic systems and the implicit function theorem, it follows in a standard manner that (2.1) has short time existence.

In the following computation we need a Hessian comparison theorem [3]. On the product $N \times N$ we introduce the Riemannian metric

$$\langle X_1 \oplus X_2, Y_1 \oplus Y_2\rangle := \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle$$

for $X_i, Y_i \in T_{y_i}N, y_i \in N, i = 1, 2$.

The distance function on N, $\rho\colon N \times N \to \mathbf{R}$ is of class C^2 on $B_R(O) \times B_R(O)$ outside the diagonal. So we have:

Lemma 2.1 ([3, 4]) *For $y = (y_1, y_2) \in B_R(O) \times B_R(O), V \in T_y(N \times N)$, assume*

$$Q = (1 - \cos(\sqrt{k}\rho(y_1, y_2)))/k\colon B_R(O) \times B_R(O) \to \mathbf{R}.$$

Then the Hessian of Q admits the following estimates:

$$\nabla^2 Q(V,V) \ge \begin{cases} |V|^2 & \text{if } y_1 = y_2, \\ \frac{\langle \nabla Q(y), V\rangle^2}{2Q(y)} - kQ(y)|V|^2 & \text{if } y_1 \ne y_2, \end{cases}$$

and

$$\nabla^2 Q(V,V) \ge (1 - kQ(y))|U|^2$$

if V has the special form $U \oplus 0$ or $0 \oplus U$.

Multiplying the metric tensor by a suitable constant we may assume the upper bound of the sectional curvature of N to be 1 throughout the rest of this paper. We set $f(x,t) = Q(u(x,t), O)$ and

$$e(u) = h^{\alpha\bar{\beta}} g_{ij} \frac{\partial u^i}{\partial z^\alpha} \frac{\partial u^j}{\partial z^{\bar{\beta}}}.$$

First, we will prove that the image of (2.1) contained in the convex ball $B_R(O)$ ($R < \frac{\pi}{2}$) under the flow such that we can use Lemma 2.1 for any time t. We have:

Lemma 2.2 *Assume $u(z,t)$ is a solution of* (2.1), *then $\rho(u(z,t), O) < R$ for any $(z,t) \in M \times \mathbf{R}$.*

Proof Suppose not, so we can assume that at some point $(z_0, t_0) \in M \times \mathbf{R}$, $\rho(u(z_0, t_0), O)$ is equal to R for the first time, so we have:

$$\frac{\partial}{\partial t} f|_{(z_0,t_0)} \geq 0, \quad \nabla f|_{(z_0,t_0)} = 0, \quad \triangle f|_{(z_0,t_0)} \leq 0.$$

On the other hand, from (2.1) and Lemma 2.1, we compute at (z_0, t_0)

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) f = h^{\alpha\bar{\beta}} \nabla^2 Q\Big(\frac{\partial u}{\partial z^\alpha} \oplus 0, \frac{\partial u}{\partial z^{\bar{\beta}}} \oplus 0\Big) \geq e(u) \cos R > 0. \tag{2.2}$$

So we have a contradiction. ∎

Next we will give estimates for $|u_t|^2 := |\frac{\partial u}{\partial t}|^2 = g_{ij} \frac{\partial u^i}{\partial t} \frac{\partial u^j}{\partial t}$ and $e(u)$. By the assumption of the curvature, using [5, (4) and (7)], we have

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) |u_t|^2 \geq |\nabla u_t|^2 - 2|u_t|^2 e(u), \tag{2.3}$$

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) e(u) \geq \frac{1}{2} |\nabla^2 u|^2 - 2e(u)(e(u) + c), \tag{2.4}$$

where c is a positive constant depending on the upper bound of both first and second derivatives of domain metric.

Let $b = 1 - \cos R_0, 0 < R < R_0 < \frac{\pi}{2}$. Hence $(b - f)$ is a positive bounded function. By (2.3) we can obtain

$$\begin{aligned}
\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) \frac{|u_t|^2}{(b-f)^2} &= \frac{1}{(b-f)^4} [(b-f)^2 \Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) |u_t|^2 \\
&\quad + 2|u_t|^2 (b-f) \cdot \Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) f \\
&\quad + 2(b-f) \langle \nabla |u_t|^2, \nabla f \rangle + 3|u_t|^2 |\nabla f|^2] \\
&\geq \frac{1}{(b-f)^4} \Big[\big(|\nabla u_t|^2 - 2|u_t|^2 e(u)\big) (b-f)^2 \\
&\quad + 2(1-f) e(u) |u_t|^2 (b-f) \\
&\quad + 2(b-f) \langle \nabla |u_t|^2, \nabla f \rangle + 3|u_t|^2 |\nabla f|^2 \Big].
\end{aligned}$$

So we get

(2.5)
$$\Big(\widetilde{\triangle}-\frac{\partial}{\partial t}\Big)\frac{|u_t|^2}{(b-f)^2}\geq\frac{1}{(b-f)^4}\Big[|\nabla u_t|^2(b-f)^2+2(1-b)(b-f)e(u)|u_t|^2 + 2(b-f)<\nabla|u_t|^2,\nabla f>+3|u_t|^2|\nabla f|^2\Big]$$

and

$$\nabla|u_t|^2=(b-f)^2\nabla\frac{|u_t|^2}{(b-f)^2}-2\frac{1}{b-f}|u_t|^2\nabla f. \tag{2.6}$$

By the Schwarz inequality and (2.6) we have

$$\Big(\widetilde{\triangle}-\frac{\partial}{\partial t}\Big)\frac{|u_t|^2}{(b-f)^2}\geq\frac{1}{b-f}\Big\langle\nabla\frac{|u_t|^2}{(b-f)^2},\nabla f\Big\rangle \tag{2.7}$$

from the maximum principle, $|u_t|$ is uniformly bounded.

Similarly we also have

$$\Big(\widetilde{\triangle}-\frac{\partial}{\partial t}\Big)\frac{e(u)+c}{(b-f)^2}\geq\frac{1}{b-f}\Big\langle\nabla\frac{e(u)+c}{(b-f)^2},\nabla f\Big\rangle \tag{2.8}$$

where c is the same constant in (2.4).

To get a global C^1 estimate from the formula (2.8), we know that it is sufficient to prove the boundary C^1 estimate. First, we need the following:

Lemma 2.3 ([5]) *There exist $\delta_0>0$ and $R_0>0$ with the following property: If u is a solution of (2.1) for $0\leq t\leq T$ and if for some t_0, $0<t_0\leq T$, $u(B(x_0,R),t_0)\subset B_\delta(p), x_0\in M, B(x_0,R)\subset M, 0<\delta\leq\delta_0$, for some $R, 0<R\leq R_0, p\in N$, $(B(q,r):=\{q'\in M: d(q,q'))<r\})$, then*

$$|\nabla u(x_0,t_0)|\leq\frac{c\delta}{R}\quad(\nabla \text{ denotes the spatial gradient})$$

where δ_0, R_0 and c depend on the geometry of M and N and on $\sup_{B(x_0,R)}|u_t(x,t_0)|$.

Lemma 2.4 *Let u be a solution of (2.1) for $0\leq t<T$, and the radius of convex ball $R<\frac{\pi}{4}$. Then there exits $0<t_0<T$ such that*

$$|\nabla u|(z,t)\leq c\quad(\nabla \text{ denotes the spatial gradient})$$

for $z\in\partial M, 0<t_0\leq t<T$, where c is independent of t.

https://doi.org/10.4153/CMB-2007-011-1 Published online by Cambridge University Press

Proof Lemma 2.3 has given interior gradient bound, and it consequently suffices to show if $d(z_1, z_0) = r, z_1 \in \partial M, d(z_2, z_0) \le r$, we have

$$\rho(u(z_1, t), u(z_2, t)) \le c_1 r \tag{2.9}$$

for some constant c_1 independent of t.

By Lemma 2.1 we know that for any $z_1, z_2 \in M, u(z_1, t), u(z_2, t)$ can be joined by a unique geodesic arc. We continue the geodesic arc from $u(z_2, t)$ to $u(z_1, t)$ beyond $u(z_1, t)$ until we reach a distance τ from $u(z_1, t)$ such that the corresponding point denoted by q is contained in $B_{\frac{\pi}{4}}(O)$. We consider $L(z, t) = 1 - \cos \rho(u(z, t), q)$. Then by the choice of q, we have

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) L(z, t) \ge 0 \tag{2.10}$$

We then solve the following linear parabolic problem:

$$\begin{aligned} &\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) H(z, t) = 0 \\ &H \colon M \times \mathbf{R} \to \mathbf{R} \\ &H(z, 0) = L(z, 0) \quad \text{for } z \in M \\ &H(z, t) = L(z, t) \quad \text{for } z \in \partial M, 0 \le t \le \infty \end{aligned} \tag{2.11}$$

since L has smooth boundary values, so does H.

The maximum principle implies

$$L(z, t) \le H(z, t) \quad \text{for } t \ge 0, z \in M. \tag{2.12}$$

Now by (2.11) and (2.12)

$$\begin{aligned} \rho(u(z_1, t), u(z_2, t)) &= \rho(u(z_2, t), q) - \rho(u(z_1, t), q) \quad \text{by the choice of q} \\ &\le \frac{1}{\sin \tau}(L(z_2, t) - L(z_1, t)) \\ &\le \frac{1}{\sin \tau}(H(z_2, t) - H(z_1, t)), \end{aligned} \tag{2.13}$$

since $z_1 \in \partial M$. From the theory of linear parabolic equations, the solution of (2.11) has a boundary Lipschitz bound. So we get (2.9). ■

Using (2.8) together with Lemma (2.4), we know $e(u)$ is also uniformly bounded, so we obtain the long time existence of the solution of the equation (2.1).

Let $u_1(x, t)$ and $u_2(x, t)$ be two sequences of maps from M into the convex ball $B_R(O)$. Let

$$\begin{aligned} \psi(z, t) &= Q(u_1(z, t), u_2(z, t)), \\ f_i(z, t) &= Q(u_i, O), \\ \varphi(z, t) &= \sum_{i=1}^{2} \omega(f_i(z, t)). \end{aligned} \tag{2.14}$$

https://doi.org/10.4153/CMB-2007-011-1 Published online by Cambridge University Press

Using Lemma 2.1, one can check that when $u_1(z,t) \neq u_2(z,t)$,

(2.15)
$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)\psi \geq \frac{|\nabla\psi|^2}{4\psi} - \psi\sum_{i=1}^{2} e(u_i) + dQ\Big(\Big(\sigma(u_1) - \frac{\partial u_1}{\partial t}\Big) \oplus \Big(\sigma(u_2) - \frac{\partial u_2}{\partial t}\Big)\Big),$$

when $u_1(z,t) = u_2(z,t)$,

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)\psi \geq \sum_{i=1}^{2} e(u_i) + dQ\Big(\Big(\sigma(u_1) - \frac{\partial u_1}{\partial t}\Big) \oplus \Big(\sigma(u_2) - \frac{\partial u_2}{\partial t}\Big)\Big), \tag{2.16}$$

and

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)\varphi \geq \sum_{i=1}^{2}\Big(\frac{1}{2}\omega''(f_i)|\nabla f_i|^2 + \omega'(f_i)(1-f_i)e(u_i) + \omega'(f_i)dQ\Big(\Big(\sigma(u_i) - \frac{\partial u_i}{\partial t}\Big) \oplus 0\Big)\Big). \tag{2.17}$$

Assume that $u_1(z,t)$ and $u_2(z,t)$ both satisfy (2.1), and set function $\omega(s) = -\ln(1-s)$ in (2.14). By formulas (2.15), (2.16), (2.17), and the fact $\omega'' = \omega'^2$, one can check that when $u_1(z,t) \neq u_2(z,t)$,

(2.18)
$$\begin{aligned}
e^{-\varphi}\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)e^{\varphi}\psi &= \Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)\psi + \psi\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)\varphi + \langle\nabla\varphi, \nabla\psi\rangle + \frac{1}{2}\psi|\nabla\varphi|^2 \\
&\geq \frac{|\nabla\psi|^2}{4\psi} - \psi\sum_{i=1}^{2} e(u_i) + \frac{1}{2}\psi\sum_{i=1}^{2}\omega''|\nabla f_i|^2 \\
&\quad + \psi\sum_{i=1}^{2}\omega'(1-f_i)e(u_i) + \langle\nabla\varphi, \nabla\psi\rangle + \frac{1}{2}\psi|\nabla\varphi|^2 \\
&\geq \frac{|\nabla\psi|^2}{4\psi} + \frac{1}{2}\psi\sum_{i=1}^{2}\omega''|\nabla\psi_i|^2 + \frac{1}{2}\langle\nabla\varphi, \nabla\psi\rangle \\
&\quad + \frac{1}{2}e^{-\varphi}\langle\nabla\varphi, \nabla(e^{\varphi}\psi)\rangle \\
&\geq \frac{1}{2}\psi\sum_{i=1}^{2}\omega''|\nabla\psi_i|^2 - \frac{1}{4}\psi|\nabla\varphi|^2 + \frac{1}{2}e^{-\varphi}\langle\nabla\varphi, \nabla(e^{\varphi}\psi)\rangle \\
&\geq \frac{1}{2}e^{-\varphi}\langle\nabla\varphi, \nabla(e^{\varphi}\psi)\rangle.
\end{aligned}$$

When $u_1(z,t) = u_2(z,t)$, we have $\psi = \nabla\psi = 0$. So

$$e^{-\varphi}\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big)e^{\varphi}\psi \geq \sum_{i=1}^{2} e(u_i) \geq 0. \tag{2.19}$$

https://doi.org/10.4153/CMB-2007-011-1 Published online by Cambridge University Press

From (2.18) and (2.19), the maximum principle implies the uniqueness of the solution of the equation (2.1). So we already have:

Proposition 2.5 *If $\phi(M) \subset B_R(O)$, $R < \frac{\pi}{4}$ and $B_R(O)$ lies in the cut locus of O, then the evolution equation* (2.1) *has a unique solution which exits for $0 \le t < \infty$.*

3 Hermitian Harmonic Maps From Compact Manifolds Into Convex Balls

First, we consider a Hermitian harmonic map from a closed manifold (compact without boundary) to convex ball $B_R(O)$.

Proof of Theorem 1.1 Let $u\colon M \to N$ be a Hermitian harmonic map, that is, u satisfies (1.1), and $u(M) \subset B_R(O)$, $R < \frac{\pi}{2}$. Putting $f(z) = Q(u(z), O)$, we have

$$\widetilde{\triangle} f(z) \ge e(u) \cos\rho(u(z), O) \ge 0 \tag{3.1}$$

Since M is closed, the function $f(z)$ must be constant, hence $\rho(u(z), O)$ is a constant. We conclude that $u(M) \subset S_r(O)$, where $S_r(O)$ denotes a geodesic sphere of radius r with center at O, and there must be a point $z_0 \in M$ such that $u(z_0) \in S_r(O)$. We join $u(z_0)$ with O by a geodesic arc. On this geodesic arc, we can choose a point O' different from O such that we can find another geodesic ball $B_{r'}(O')$ satisfying $B_r(O) \subset B_{r'}(O') \subset B_R(O)$. Setting $F'(z,t) = 1 - \cos\rho(u(z), O')$, we also have $\widetilde{\triangle} F'(z,t) \ge 0$. Using the maximum principle again, we have $\rho(u(z), O') = r - \rho(O, O') = r''$, for any $z \in M$, *i.e.*, $u(M) \subset S_{r''}(O')$. But it is easy to see that there is only one point in $S_r(O) \cap S_{r''}(O')$, so $u(M) = u(z_0)$. ■

Proof of Theorem 1.2 First of all, we will show that if $\phi(M) \subset B_R(O)$,

$$\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) \frac{|u_t|^2}{(b-f)^2} \ge 0 \tag{3.2}$$

where $R = \arccos\frac{2\sqrt{5}}{5} < \frac{\pi}{4}$, $R_0 = \arccos\frac{\sqrt{5}}{5}$, and $b = 1 - \cos R_0$. In fact, from (2.5) we have

(3.3)
$$\begin{aligned}\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) \frac{|u_t|^2}{(b-f)^2} &\ge \frac{1}{(b-f)^4}\Big\{ \big[\big|\nabla u_t\big|(b-f) - 2|u_t||\nabla f|\big]^2 \\ &\qquad + 2(1-b)(b-f)|u_t|^2 e(u) - |u_t|^2|\nabla f|^2 \Big\} \\ &\ge 2\frac{1}{(b-f)^4}|u_t|^2 e(u)[\cos R_0(\cos\rho - \cos R_0) - \sin^2\rho]\end{aligned}$$

To get (3.2), it suffices to prove that $\cos R_0(\cos\rho(u,O) - \cos R_0) - \sin^2\rho(u,O) \ge 0$, *i.e.*,

$$\cos\rho(u,O) \ge \sqrt{\frac{5}{4}\cos^2 R_0 + 1} - \frac{1}{2}\cos R_0 \tag{3.4}$$

And it is easy to see that when $R_0 = \arccos\frac{\sqrt{5}}{5}$, the right side of (3.4) reaches its minimum $\frac{2\sqrt{5}}{5}$. Therefore we have proved (3.2), when $\rho(u(z), O) \leq R = \arccos\frac{2\sqrt{5}}{5}$.

Now, we solve the following Dirichlet problem on M [7, Ch. 5, Proposition 1.8]

$$\begin{aligned} \widetilde{\triangle} v(z) &= -\frac{|u_t|^2}{(b-f)^2}\Big|_{t=0} \\ v(z)|_{\partial M} &= 0 \end{aligned} \tag{3.5}$$

Setting $w(z,t) = \int_0^t \frac{|u_t|^2}{(b-f)^2}(z,s)\,ds - v(z)$, by (3.2), we can see that

$$\begin{aligned} &\Big(\widetilde{\triangle} - \frac{\partial}{\partial t}\Big) w(z,t) \geq 0 \\ &w(z,t)|_{\partial M} = 0 \\ &w(z,0) = -v(z), \quad z \in M \end{aligned} \tag{3.6}$$

Again by the maximum principle we have

$$\int_0^t \frac{|u_t|^2}{(b-f)^2} \leq \sup_{z\in M} |v(z)| < \infty \tag{3.7}$$

Hence, we conclude that there exists a sequence $\{t_i\}$ such that

$$\lim_{t_i\to\infty} u_t := \lim_{t_i\to\infty} \frac{\partial u}{\partial t} = 0.$$

Then the standard elliptic regularity implies that there exists a subsequence $u(z, t_i)$ converging to a Hermitian harmonic map as i goes to ∞.

Assume that $u_1(z)$ and $u_2(z)$ both are Hermitian harmonic maps satisfying the same boundary condition. Similar to (2.18) and (2.19), it is easy to obtain

$$e^{-\varphi}\widetilde{\triangle} e^{\varphi}\psi \geq \frac{1}{2} e^{-\varphi}\langle \nabla\varphi, \nabla(e^{\varphi}\psi)\rangle. \tag{3.8}$$

or, when $u_1(z) = u_2(z)$,

$$e^{-\varphi}\widetilde{\triangle} e^{\varphi}\psi \geq 0, \tag{3.9}$$

where ψ, φ is defined in (2.14).

Then the maximum principle implies the uniqueness of Hermitian harmonic map. ∎

References

[1] J. Y. Chen, *A boundary value problem for Hermitian harmonic maps and applications.* Proc. Amer. Math. Soc. **124**(1996), no. 9, 2853–2862.

[2] H. C. Grunau and M. Kühnel, *On the existence of Hermitian-harmonic maps from complete Hermitian to complete Riemaniann manifolds.* Math. Z. **249**(2005), no. 2, 297–327.

[3] W. Jäger and H. Kaul, *Uniqueness and stability of harmonic maps and their Jacobi fields.* Manuscripta Math. **28**(1979), no. 1-3, 269–291.

[4] J. Jost, *Harmonic Mappings Between Riemannian Manifolds.* Proceedings of the Centre for Mathematic Analysis 4, Australian National University, Canberra, 1984.

[5] J. Jost and S. T. Yau, *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry.* Acta Math. **170**(1993), 221–254.

[6] L. Ni, *Hermitian harmonic maps from complete Hermitian manifolds to complete Riemannian manifolds.* Math. Z. **232**(1999), no. 2, 331–355.

[7] M. E. Taylor, *Partial Differential Equations. I. Basic Theory.* Applied Mathematical Sciences 115, Springer-Verlag, New York, 1996.

School of Mathematical Sciences
Peking University
Beijing 100871
P. R. China
e-mail: lzymath@sina.com

Department of Mathematics
Zhejiang University
148 Tianmushan Road
Hangzhou 310028, Zhejiang
P. R. China
e-mail: xizhang@zju.edu.cn