A NON-CYCLIC ONE-RELATOR GROUP ALL OF WHOSE FINITE QUOTIENTS ARE CYCLIC

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(Received 17 June 1969)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. B. Preston

Let G be a group on two generators a and b subject to the single defining relation $a = [a, a^b]$:

$$G = (a, b; a = [a, a^b]).$$

As usual $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$ if x and y are elements of a group. The object of this note is to show that *every finite quotient of G is cyclic*. This implies that every normal subgroup of G contains the derived group G'. But by Magnus' theory of groups with a single defining relation $G' \neq 1$ ([1], § 4.4). So G is not residually finite. This underlines the fact that groups with a single defining relation need not be residually finite (cf. [2]).

In order to prove that G has the described properties let us put

$$a_i = b^{-i}ab^i.$$

Then the normal closure N of a in G is generated by the elements $\cdots, a_{-1}, a_0, a_1, \cdots$ subject to the defining relations

$$a_i = [a_i, a_{i+1}] \qquad (i = 0, \pm 1, \cdots).$$

Thus

$$a_i^2 = a_{i+1}^{-1}a_ia_{i+1} \qquad (i = 0, \pm 1, \cdots).$$

Now suppose that K is a normal subgroup of G of finite index. Put

$$x = aK, y = a^bK.$$

We shall show that $x = 1$ which implies $N(= G') \leqq K$ as desired. For suppose $x \neq 1$. Then x and y are of order $n > 1$, say. Since $x^y = x^2$ we find

$$x = x^1 = x^{y^n} = x^{2^n}.$$

This implies $x^{2^n-1} = 1$ and n divides 2^n-1. But it is easy to see that the smallest prime divisor of n is less than the smallest prime divisor of 2^n-1 (G. Higman [3]). This completes the proof.

References

[1] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory* (Interscience Publishers, 1966).

[2] G. Baumslag and D. Solitar, 'Some two-generator one-relator nonhopfian groups', *Bull. Amer. Math. Soc.* 68 (1962), 199—201.

[3] G. Higman, 'A finitely generated infinite simple group', *J. London Math. Soc.* 26 (1951), 61—64.

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https://doi.org/10.1017/S1446788700007783 Published online by Cambridge University Press